Creative and conceptual challenges in mathematical problem solving

JONAS JÄDER

Mathematical problem solving has proven to be valuable for students' learning. Yet, problem solving is often referred to and discussed without distinguishing between creative and conceptual aspects. The purpose of this study is to advance understanding of mathematical problem solving through the analysis of students' work with mathematical problems in terms of the creative and conceptual challenges they encounter. It is proposed that the characteristics of creative challenges can be used to obtain a more detailed description of the problem-solving process. Furthermore, the characteristics of conceptual challenges can provide a basis for discussion of what conceptual understanding may entail. Moreover, the analytic framework developed in this study is proven to be of assistance in the visualization of challenges and may be useful in future efforts to investigate challenges and students' problem solving in mathematics.

Students' individual work with task solving constitutes an essential part of the teaching and learning of mathematics in classrooms around the world (Schmidt et al., 2001). With an intended learning goal, a task may be designed, selected and introduced by a teacher to support the students' learning (Simon & Tzur, 2004). By focusing on students' engagement with tasks it is possible to better understand the opportunities to learn in relation to intended learning goals and specific task designs (Johnson et al., 2017). One way of viewing mathematical competency is to consider the ability to act on a mathematical task, as well as the ability to understand the mathematics in that specific situation (Niss, 2003b) and also to recognize the importance of how these aspects, acting and understanding, work together (Maciejewski, 2017). An understanding of the relation between a task and a student's competency can not only be utilized to design tasks and even standards but also to exemplify different

Jonas Jäder
Mid Sweden University

Jäder, J. (2022). Creative and conceptual challenges in mathematical problem solving. *Nordic Studies in Mathematics Education*, 27 (3), 49–68.

competencies and provide metacognitive support for teachers and students in the discussion and development of education (Niss, 2003a).

A distinction can be made between tasks of routine character, where a student has access to a solution method and mathematical problems, where a student needs to construct a solution method (Lithner, 2008; Schoenfeld, 1985a). Mathematical problem solving is the act of working on a task that is a mathematical problem for the individual. Mathematical problem solving and the challenges induced by this action has proven valuable in the teaching and learning of mathematics (Hiebert & Grouws, 2007; Jonsson et al., 2014). Included in problem solving may be creative as well as conceptual challenges (Lithner, 2017). However, problem solving is often described without making a distinction between creative and conceptual aspects, which may affect discussion of students' opportunities to learn. By characterizing the challenges students meet when solving mathematical problems, the purpose of this study is to contribute to a better understanding of mathematical problem solving.

Challenges in mathematical problem solving

A certain degree of challenge is related to mathematical problem solving (Schoenfeld, 1985a). Earlier analyses of students' problem solving have indicated that there are at least two types of challenges which differentiate a mathematical problem from a task of routine character: the "creative" and the "conceptual" challenge (Lithner, 2017). There are several accounts where struggle, persistence and uncertainty are reported as important aspects of students' learning (Hiebert & Grouws, 2007; Sullivan et al., 2015; Zaslavsky, 2005). The relation between what a student is required to know to complete a task and what the student actually knows, can create a certain amount of uncertainty, which in turn motivates students to advance their understanding of, and their experiences with specific aspects of mathematics (Zaslavsky, 2005). A challenging task stimulates students to be persistent which is a valuable factor in learning mathematics (Sullivan et al., 2015).

In this study, focus is on the challenges that students encounter when constructing solutions to mathematical problems, where challenges are seen as being something productive (Hiebert & Grouws, 2007; Schoenfeld, 2017). Three types of distinct yet interrelated uncertainties have been identified in students' work with tasks; competing claims, unknown path or questionable conclusion and non-readily verifiable outcome (Zaslavsky, 2006). Making a distinction between the different uncertainties makes it possible to reflect on what approach students take to a task. These ideas may give guidance when exploring possible learning outcomes and for task design (Zaslavsky, 2005).

Conceptual understanding

Central to mathematics and to mathematics education are mathematical concepts. In this study *concept* refers to a non-mental, abstract meaning of a term (Wedman, 2020) which may include references to other concepts, to concept properties and to mathematical objects such as numbers, variables, functions, diagrams and other conceptually enriched representations, as well as transformations of objects, such as a procedure for a calculation (Hiebert & Lefevre, 1986; Lithner, 2008; Terwel et al., 2009). Thus, a conceptual understanding is made up of connections between, for example different concepts, representations and procedures, and can be said to be rich in relationships (Hiebert & Lefevre, 1986).

Tall and Vinner (1981) describe a difference between a person's concept image and the way the concept is formally defined, for example by a teacher or in a textbook. A person's *concept image* is a mental picture of the cognitive structure associated with a specific concept (Tall & Vinner, 1981). The concept image develops over time and with new experiences (Niss, 2006). The portion of the concept image that a student activates in a specific situation is called the *evoked concept image* (Tall & Vinner, 1981). Even when a formal concept definition is available, students tend to rely on their concept image, developed by personal experiences with mathematical concepts (Niss, 2006). It is essential that the student develops a concept image that integrates a reconstructed, personal concept definition.

Purpose of the study

The purpose of this study is to contribute to a better understanding of mathematical problem solving by distinguishing and characterizing the creative and conceptual challenges students encounter when solving mathematical problems. Such an understanding may support constructive discussions of students' opportunities to learn through mathematical problem solving. This is done through the development of an analytic framework, consisting of definitions and a method of analysis, which makes it possible to capture the challenges in students' work with mathematical problems. The research question that this study seeks to answer is:

- What characterizes creative and conceptual challenges in mathematical problem solving?

Analytic framework – definition of challenges

An analytic framework was developed based on the idea that mathematical problem solving includes creative and conceptual challenges (Lithner, 2017). To be able to capture these challenges in students' work, the concept of discrepancy between a student's prior knowledge and experiences and what the student was required to do to overcome the challenges was used. In their theoretical definitions, the two challenges are intended to be non-overlapping and to concern distinctly different aspects. However, in actual problem solving, different aspects of students' difficulties are likely to be interrelated (Zaslavsky, 2005) and it may be utopian to find definitions that are strictly distinct. The challenges in a task are related to the student solving the task, in the sense that a challenge depends on his or her prior experiences and understanding.

Creative challenge

Creative challenge is defined as a discrepancy between what the student has experienced previously (and in some way has access to) and what is required by a conceivable method. A creative challenge can be seen as a requirement to construct a solution method (new to the student), or to select and modify a familiar method in a new situation, where the choice of method is not obvious (Lithner, 2017). This can relate to the construction of an overall solution method, as well as a sub-method.

Conceptual challenge

The term *adequate concept image* is introduced to supplement Tall and Vinner's (1981) framework. An adequate concept image, in similarity with an evoked concept image, refers to a mental picture. However, an adequate concept image, contrary to an evoked concept image, is something desirable and necessary to solve a specific task rather than something possessed by an individual. Conceptual challenge is defined as a discrepancy between a student's evoked concept image and an adequate concept image. A conceptual challenge is a requirement to make the necessary considerations and proper use of the mathematical concepts, to solve the task.

Method

A case study (Creswell, 2008) was conducted where focus was on students' mathematical problem solving and more specifically on the challenges encountered by students. As the act of problem solving depends on the relation between student and task the selection of both of these were essential to create a situation where students engaged in mathematical problem solving and where the challenges could be identified by the researcher. Nevertheless, the main objective of the study was not to describe students or tasks but the conceptual and creative challenges in students' mathematical problem solving. The study has been carried out in accordance with the guidelines for good research practice (Vetenskapsrådet, 2017) and the guidelines of the author's affiliation (Dalarna University, 2022).

Selection of students

The selection of students was based on two main factors. Firstly, that the student was interested in and had the capacity to communicate his or her solutions and underlying reasons for choices during problem solving, both in writing and orally. Secondly that the student was taking mathematics in upper secondary school (normally at 15–16 years of age in Sweden) at the time of the study. Twelve students from one school, who all gave informed consent to participate, were selected on recommendation of three mathematics teachers of the school. All students were in their first or second year of upper secondary school and attended theoretical programs of study with an intermediate or high intensity of mathematics.

Selection of tasks

The objective of the selection process was to find tasks that were likely to be mathematical problems for the selected students, and at the same time could be expected to be used in mathematics classrooms. The mathematical problems were all collected from resources used in secondary school, such as textbooks and other collections of tasks and national tests, with the intention to create a familiar situation to the students. Mathematical problems at different expected levels of difficulty were selected and sequenced in a progression from easier to more difficult. The reason for this was to be able to, at some point during the problem-solving session, let each of the students meet a mathematical problem that was appropriately challenging. It has been shown in previous research that a student's problem-solving approach depends on his or her capacity, which can be linked to the level of difficulty of the task (Dìaz-Obando et al., 2003, Jäder et al., 2017). During the selection of tasks, mathematical problems that demanded unnecessarily technical manipulations which would raise the cognitive load, were not included (Russo & Hopkins, 2017). Seven mathematical problems were selected.

Task solving sessions and follow-up interviews

The task-solving sessions took place at the students' school. The students were filmed one at a time, using a think out loud protocol, which has proven to be a reasonable method to obtain verbal data from students' problem-solving sessions (Schoenfeld, 1985b). It is argued that students' cognition is best seen if they are asked to work by themselves, but also that students may feel more comfortable thinking out loud together with a peer. However, the disadvantage of having students work individually rather than in pairs was assumed to be decreased by the creation of a familiar situation to the students through the choice of location and the task selection (Schoenfeld, 1985b). The author attended all sessions, only interrupting when clarification of the students' thoughts and reasons was deemed necessary. The students were instructed to write down the full implementation of each of the tasks, including any explorations. Further, they were asked to not erase anything from their papers, but instead lightly cross out what they did not want to include in the final presentations of their solutions. Eight of the twelve students attempted all seven tasks, while four of the students only had time for five or six tasks, leaving out the tasks expected to be the most difficult. A student's task solution in this study implies the written solution as well as what could be seen and heard during the task-solving session.

As suggested by Schoenfeld (1985b), a second perspective on the challenges encountered by the students was collected. Filmed, follow-up interviews were conducted where the written student solutions as well as statements from the solving sessions were used as a starting point. The interviews focused on any similarities and differences between a students' previous experiences and understanding and what was necessary to solve the task or overcome a challenge.

Analytic framework – procedure for analysis

Each student's solution to each of the tasks was divided into subtasks and described focusing on the arguments used for an overall solution method, for strategy choices in relation to each of the subtasks, for the implementation of these and for the conclusions drawn for each subtask as well as the final conclusion (Lithner, 2008). A qualitative content analysis (Robson, 2011) was used, where initially all instances during the task-solving session where the student either showed hesitation or evidence of struggle were indicated and considered as a possible challenge. Such an instance could also be highlighted by the interview. The procedure for analysis relied on the operationalization of the definitions of the two challenges: creative and conceptual.

Creative challenge: The student's argument for their strategy choice was used as an indication of whether the method was familiar or not, thus indicating the need to construct a solution method. All steps of a solution method (fully or partly implemented) where the student in his or her solution made other considerations than the strictly algorithmic ones and had to (partly) construct a solution method, were considered as a creative discrepancy between a conceivable method and the student's prior experiences of the same or similar methods. Any situation where a student hesitated or showed other signs of struggle during the implementation of the strategy were considered to be a possible challenge. The interviews gave further details as to the reasons for any disruptions to or deviation from an algorithmic implementation.

Conceptual challenge: The students' actions and ways of expressing themselves were assumed to be explained partly by his or her concept image (Bingolbali & Monaghan, 2008). What was possibly lacking in this image initially for the student to be able to overcome the challenge was described as a conceptual discrepancy between an adequate concept image and the evoked concept image.

Other challenges: Challenges other than creative and conceptual ones were described to be able to say something about the overall challenges in mathematical problem solving, even though these might not relate specifically to problem solving but rather to task solving in general.

Using the descriptions of the challenges gathered from all student solutions, a search for common features was conducted and key descriptors were attached to each challenge using a thematic coding approach (Robson, 2011).

Examples of analysis and challenges

The analysis is exemplified below using two tasks and one student's solution to each of the tasks, supplemented by statements from the interviews. Together with brief descriptions of two additional tasks and solutions, this also serves as a way to describe the major characteristics of creative and conceptual challenges encountered and thus the outcome of the thematic coding of all twelve students' solutions to all seven tasks.

Task 1

The task solution and student challenges presented here are similar to those of seven of the twelve students. These students experienced challenges related to making use of non-explicit information in the task and to the use of their understanding of the concept of diameter.

Find the ratio of the area of the shaded figure to that of the rectangle.

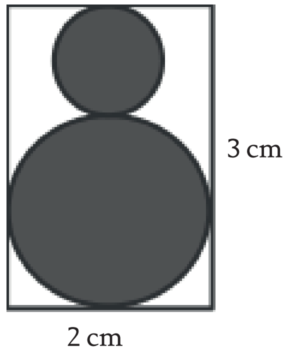

Figure 1. *Example task 1 (translated from Carlsson et al., 2017, p. 94)*

Description of one student's task solution

The student argues for an overall solution method showing an understanding of what needs to be calculated to reach a final conclusion. The student's solution can be seen as involving several subtasks, using several sub-methods. The picture in the task shows the length of the base of the rectangle, being the same length as the diameter of the greater circle. This seems evident to the student, who has no problems calculating the area of the greater circle using the well-known formula. Then the student realizes that she does not know the diameter of the smaller circle and therefore estimates this (to 1 cm) to be able to calculate the area. She goes on to calculate the area of the rectangle and then the requested ratio. However, when being asked by the author to verify the value of the diameter of the smaller circle the student realizes that approximation does not suffice here and that another method is required. The student hesitates and explores the information given in the task to find a way to calculate the diameter of the smaller circle. The student finds it necessary to consider further aspects of the given information and turns the paper in different directions. She hesitates before realizing that it is possible to use the property that the diameter of a circle is any straight line of a circle going through the center of the circle, and that the base of the rectangle, being the diameter of the greater circle, can also be used as a vertical measurement. Once this information is extracted, the student executes the rest of the required calculations without hesitation or struggle, reaching a correct final conclusion.

Creative challenge

Students' prior experiences and pre-knowledge: The overall solution idea of the task, as well as the first subtask (extracting a measurement using a parallel transfer to calculate the area of a circle) are familiar to the students. It is also likely that the students have solved tasks including adding or subtracting two (or more) measurements to get an unknown measurement.

Discrepancy between a conceivable method and prior experiences: These students quickly realize that they need to discover the diameter (or radius) of the smaller circle. However, they do not know that to reach this goal, they must translate the measurement of the base of the rectangle to the diameter of the greater circle (which is done without hesitation or struggle), rotate this horizontal measurement to a vertical one and use this together with the height of the rectangle to set up an equation with the diameter of the smaller circle being d, $2 + d = 3$ and solve it. Students struggle during this process, since what measurements to extract from the figure and how to use them is not obvious. The combination of several steps, including the translation of the base measurement as being the diameter of the greater circle, the rotation of the measurement of the diameter of the greater circle, and the translation of this measurement to be the difference between the height of the rectangle and the smaller circles diameter and finally solving the formulated equation seems to generate a challenge for the students. During the interviews, students specifically mention that the rotation of the horizontal measurement to a vertical one made them struggle. They need to construct a method to calculate the diameter, including to, in an explorative way, extract, derive and use non-explicit information relevant for the calculations, from the task. As part of the solution process, extracting information concerning the smaller circle's diameter can be seen as guidance for a strategy choice as it makes calculating the area possible, as well as being part of a strategy implementation. The discrepancy between prior experiences and a conceivable method is that in this case it is not obvious that the base of the rectangle can be used to calculate the diameter of the smaller circle.

Conceptual challenge

Evoked concept image: The evoked concept image seems to be that the diameter of a circle is a straight line from one side of the circle to the other. Further, it seems that the evoked concept image is that a diameter is a line with a certain direction and length, where the direction of the line corresponds to the way the measurement is presented.

Discrepancy between adequate and evoked concept image: In the first part of the solution attempt, the students do not use the idea that the diameter of the greater circle at the same time as being a horizontal measurement can be a vertical one. At that point they seem to not evoke the property of the concept of diameter that says that a diameter of a circle is a measure of any line through the center of the circle, and that the length of this line is the same, independent of the direction. The students' work indicates that diameters represented vertically are not as familiar and that this is a somewhat new situation for them. The approach taken later and statements from the interviews indicate that although this property of diameters is part of the concept image, it is not evoked initially. However, it becomes necessary to consider it in a more flexible manner to solve the task.

Summary

The analysis shows that the creative challenge includes the extraction, derivation and use of the non-explicit information from the task necessary to make a strategy choice which will lead to a solution. The conceptual challenge is characterized by a necessity to use a property of the concept diameter in a somewhat new situation requiring a more flexible use. No other challenges were identified.

Task 2

The student work presented below is representative of half of the twelve students in terms of creative and conceptual challenges, even though there are differences in the way they approached these challenges. The challenges experienced by these students in this task concern firstly, the relation between speed, distance and time where there are two unknown speeds and secondly, to construct a solution method from two, to the students, seemingly separated sub methods.

> Graham and Colin leave the same place at the same time and drive in opposite directions. Colin drives 10 km/h faster than Graham does. After 2 h, they are 200 km apart. How fast is each man driving?

Figure 2. *Example task 2 (translated and adapted to Swedish for the students from Canton et al., 2007, p.181)*

Description of one student's task solution:

The student assumes the two speeds to be x and $x+10$, but struggles to set up a correct equation using the information about both distance and time in a correct manner. An issue is how to include the two hours in the solution. He realizes the need to consider both distance and time to find the two speeds and explores ways to do this. The attempt initially consists of an equation including considerations of only speed and distance ($2x+10=200$), equating the two quantities. The root of the equation is later divided by 2 to take account of the time as well. The student notices that this is not a correct answer. The student hesitates and struggles to construct an overall solution idea where time, as well as distance and both Graham and Colin's speed are considered in a correct way. In an attempt to include the two hours in his solution he divides the distance travelled by two to get 100 km. He continues to hesitate before concluding that this means that the total speed (Colin's and Graham's added together) is 100 km/h (and that this speed is in the context of the task constant over one as well as two hours). Finally, in a straightforward manner, he sets up the equation $2x+10=100$ and solves it without hesitation.

Creative challenge

Students' prior experiences and pre-knowledge: The students have experience of solving tasks asking for an unknown such as speed, distance or time, while knowing the other two quantities using a well-known formula. They are familiar with setting up and solving an equation using a well-known formula and with drawing a conclusion from this.

Discrepancy between a conceivable method and prior experiences: (1) Initially the discrepancy is the lack of experience in combining two relations ($s=d/t$ and $s_a+s_b=s_{tot}$, where s is speed, d is distance, and t is time) algebraically. The challenge thus lies in the construction of an algebraic representation. This creative challenge relates to a conceptual one of connecting the concept of speed to distance and time and under the given circumstances also connecting this to an algebraic representation. The creative challenge may not have existed if the conceptual challenge had been eliminated. (2) Later, the students attempt to avoid the initial challenges (conceptual and creative) and instead try to consider only two quantities at a time (first distance and time: 200 km in 2 hours, means 100 km in 1 hour, and then the two speeds in relation to the total speed: 100 km/h). This is a creative discrepancy, as it is not obvious and seems to be unfamiliar to the students what relations to use and in what order. They thus need to construct a new overall solution idea, combining the method for calculating speed with a method that considers two speeds in relation to each other. They need to merge the methods of subtasks to each other step by step in an explorative manner. In this process some parts are fairly straight forward, while others need more consideration.

Conceptual challenge

Evoked concept image: The students show an understanding of the basic relations between speed, distance and time by the known formula $s = d/t$. They know how to calculate one of the quantities when the other two are known and try to use this, for example, when attempting to set up an equation. The students' evoked concept image also includes that the speed with which two bodies are separated (given that they have opposite directions) is the sum of the two speeds, $s_a + s_b = s_{tot}$ (and similarly that the distance apart can be calculated $d_{tot} = d_a + d_b$). The work of the students also indicates that they find it possible to represent the two values for speed as x and $x + 10$ respectively.

Discrepancy between adequate and evoked concept image: (1) What these students struggle with initially is the relation between the three quantities considered together and algebraic representations of these quantities, as the usual formula does not work to find the two unknown values for speed. The relation $s = d/t$ has to be used together with another relation, such as $s_a + s_b = s_{tot}$, in this situation with two unknowns. The understanding of these relations seems to depend on partly separated concept images. (2) The method eventually used seems to generate a conceptual challenge as these students hesitate before concluding that the distance traveled in one hour (100 km) implies that the speed is 100 km/h. Although familiar with calculating speed using a formula where both time and distance is made evident, the students seem to struggle with the more unfamiliar situation, where the focus is on the distance traveled (although this is referred to in relation to one hour). This requires a more flexible use of the concept image than the one initially evoked.

Summary

Students encountering these challenges do so in two stages. There is initially a challenge related to representing the relations between the three quantities algebraically, which can be seen as both creative and conceptual. This may be connected to the conceptual challenge since the students' concept images are separated from each other and needs to be connected to solve the task. However, as the students do not overcome these challenges, they change approach and encounter somewhat new conceptual and creative challenges. It seems as though the new challenges emerge from a need to close the gap between the evoked and the adequate concept image and to avoid the initial creative challenge. They need to construct a solution method through the merging of subtasks. Furthermore, the students struggle with an unfamiliar situation requiring them to make more flexible use of their concept image. No other challenges were identified.

Task 3

To some degree, this task generated a creative challenge for all students. They struggled to extract the necessary information from the task in terms of the length of the two indentions together being equal to ”a”.

Which of the following expressions correspond to the perimeter of the figure?

a+b	2a+2b	3a+2b
3a+3b	4a+2b	

Justify your answer.

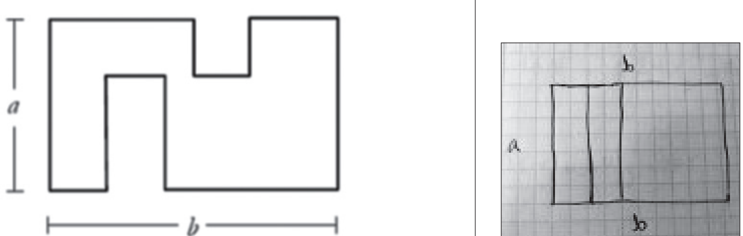

Figure 3. *Task (translated from National test, course A, 2010, part I, Skolverket, 2010, p. 3) and excerpt from student solution (right)*

For six of the twelve students the task seems to present a new situation. What seems to be unfamiliar is that the figure is something other than a rectangle, and that the requested quantity is the perimeter rather than the area. Initially the students have an idea of perimeter as being the total length around a figure and also being constant even if rearranged. However, when the students rearrange the figure and move the indentions to adjust to each other (as shown in figure 3), a new figure appears with some sides inside the figure. The students struggle as it is not clear to them how to interpret ”around the figure” in relation to this new figure, and they have to reconsider and adjust their current concept image.

Task 4

Task 4 required the students to consider a sum of money shared by three friends. The task inflicted struggle on all students as they were uncertain of an overall solution idea. While working on the task the students take different approaches and meet new creative challenges. However, of major interest in relation to this task are two conceptual challenges. For seven of the students the connection between procedural and conceptual understanding is a conceptual challenge as they lack sufficient understanding to reconstruct a procedure. These students struggle to calculate 1/3 of 2/3's. They mention having done this procedurally before but have forgotten how to do it, and now they have to reconstruct the procedure using their understanding of rational numbers and of multiplication. A few students try to grasp how to perform the calculation by using grids to represent the numbers, and to try to understand the effect of the numerator and denominator in the calculations. As the calculations were to be done iteratively, the grid needed to include for example 3 x 9 dots. Thus, being able to connect a representation to the specific properties of a concept necessary to consider when solving the task caused yet another conceptual challenge to these students. However, this challenge seems to also connect to a creative challenge concerning the actual construction of the representation.

Results

The results give support to the usefulness of the analytic framework as a tool when striving to deepen understanding of students' mathematical problem solving. It was possible to distinguish and characterize creative and conceptual challenges using the framework.

Characteristics of creative challenges

Creative challenges could be linked to at least two different characteristics:

1. The merging of two somewhat familiar individual sub-methods (e.g. task 2).
2. To extract, derive and use implicit information (e.g. task 1, 3).

Characteristics of conceptual challenges

Three different characteristics were found in relation to conceptual challenges:

1 A situation that is new to the student, in term of, for example how the concept is presented or what properties are made visible, demands a more flexible use of their concept image (e.g. task 1, 2) or a reconsidered and adjusted concept image (e.g. task 3).

2 A connection between procedural and conceptual understanding (e.g. task 4).

3 A connection between two different concepts, thus combining two seemingly separate concept images (e.g. task 2).

Combined creative and conceptual challenges

There are instances where the distinction between creative and conceptual challenges is not as clear. There seems to be a conceptual challenge in considering the properties of the concepts involved related to the creative challenge of constructing representations (e.g. task 2, 4).

The interplay between challenges is not within the scope of this study. Nevertheless, the two examples show that the creative challenge may be the major one in some cases (task 2) and the conceptual one in other cases (task 4).

Discussion

The analysis implemented shows that it is possible to distinguish and characterize the creative and conceptual challenges experienced by the students. Acknowledging the importance of students' prior knowledge as well as a learning goal (Simon & Tzur, 2004), the notion of discrepancies supports a description of two states of knowledge, one which a student has at present and one adequate. The definitions of challenges and the method of analysis presented, may function as a basis for further development, for discussions on the conditions for students' problem solving, and as part of a basis for the formulation of principles for task selection and (re)design.

In conclusion the results of this study give support to at least two main claims:

1 Knowledge of the characteristics of creative challenges can support a more nuanced description of a problem-solving process, which may be used when analyzing students' work.

2 Knowledge of the characteristics of conceptual challenges can be used to support a discussion of what conceptual understanding may entail, and support conscious choices when designing

education and mathematical problems to develop students' conceptual understanding.

Creative challenges to describe a problem-solving process

Mathematical problem solving has been described as a process with several phases (Lithner, 2008; Schoenfeld, 1985a). It often comprises the recognition of a problem, some kind of argumentation and the performance followed by a conclusion. In relation to creative challenges however, it might be valuable to also consider the extraction of explicit information as a phase of its own in this process. It can be seen as a part of the overall recognition phase as well as an important aspect of a strategy choice for, and implementation of a sub-method. By also breaking down "strategy implementation" and the preceding "strategy choice" into sub-categories such as construction of an overall solution idea (merging of sub-tasks) and construction of representations, it might be possible to further nuance the idea of mathematical problem solving as something broad and including several aspects of mathematical work. These results might be useful for teachers and researcher when analyzing students' problem solving.

Conceptual challenges to develop conceptual understanding

Mathematical problems have been emphasized in relation to mathematical creativity (e.g. Silver, 1997). However, it seems as if the new experiences that these problems offer may also accent the presence of conceptual challenges and thus of the potential development of conceptual understanding. It has been shown that a conceptual challenge may occur as a student's prior experiences with a concept, and thus their concept image, seems to contravene the formal definition of the same concept (Tall & Vinner, 1981). Initial contradictions need to be resolved by reconsidering and adjusting the concept image, and by changing students' mental pictures to fit with the new information. This new experience gives students the possibility to reconstruct their concept images (Niss, 2006). Even with a seemingly sufficient concept image, an evoked concept image has to align with the conceptual requirements of a task. The evocation of a concept image may be connected to the specific situation, and what is evoked in one situation, might not be the same as in another. It is possible to regard this as a lack of experience with both these specific situations and with the evocation of a concept image in an unfamiliar situation. This may be related to the prototype approach as described by Wedman (2020), and students' willingness to make conceptual connections based on how familiar or typical the situation or object presented is. By using students' current knowledge as a vantage point (Simon & Tzur, 2004), and considering students' actions (Johnson et al., 2017), a task design process may thus include considerations of familiar concepts used in an unfamiliar situation or using a non-typical object of some kind.

Further, the results of this study strengthen the idea of conceptual understanding as being rich in relationships (Hiebert & Lefevre, 1986). The relationships observed in this study are specifically the connections between a concept and a procedure or between a concept and a representation as well as between two concepts. Procedural understanding can be distinguished from conceptual understanding (Hiebert & Lefevre, 1986), but as the results of this study show, it is also reasonable to explore the opportunities to develop procedural and conceptual understanding in close association. Procedural work often relates to tasks of routine character (Jonsson et al., 2014). However, it seems that there are aspects of conceptual understanding involving procedures that can be distinguished through mathematical problem solving and conceptual challenges. It might be possible to say that the procedure is an arithmetic concept in itself (Wedman, 2020). By doing this we recognize the importance of the mathematical foundation of these procedures.

Making connections between concepts and representations is another aspect of conceptual understanding worth considering (Hiebert & Lefevre, 1986), not the least in relation to creative challenges and mathematical problem solving. Mathematical work with representations seems to offer a way to trigger conceptual as well as creative challenges. Connecting creative with conceptual challenges may have advantages, as the creative work supports overcoming conceptual challenge, and thus closes the gap between what students already know and what they are required to understand in relation to the task (Terwel et al., 2009), possibly extending their concept images. Representations in this sense are concrete and can be linked to the more abstract, mental representations that build a concept image (Kilpatrick et al., 2001). A representation may be seen as a necessary component of a new solution method (Terwel et al., 2009). At the same time however, using one or several representations may increase the opportunities to develop a conceptual understanding through the properties of, and connection between, representations (Olsson, 2019). This interrelatedness is emphasized by Maciejewski (2017), suggesting that a conceptual understanding depends on both an understanding of the concept, a sufficient concept image and on how this can be applied, in for example a solution method. This is in line with Niss' (2003b) description of mathematical competency as including both an aspect of action and one more related to an understanding of a concept.

References

Bingolbali, E. & Monaghan, J. (2008). Concept image revisited. *Educational Studies in Mathematics*, 68 (1), 19–35.

Canton, B.J., Etienne, S., Huyck, H., Santarelli, J. & Stewart, K. (2007). *Foundations of Mathematics 10*. McGraw-Hill Ryerson.

Carlsson, S., Hake, K. & Öberg, B. (2010). *Matte Direkt 8*. Sanoma utbildning.

Creswell, J. W. (2008). *Educational research: planning, conducting, and evaluating quantitative and qualitative research*. Pearson Education.

Dalarna University (May 12, 2022). *Ethics in research projects*. https://www.du.se/en/medarbetarwebb/forska-och-utbilda/forskningsstod/ethics-in-research-projects/

Dìaz-Obando, E., Plasencia-Cruz, I. & Solano-Alvarado, A. (2003). The impact of beliefs in student's learning: an investigation with students of two different contexts. *International Journal of Mathematical Education in Science and Technology*, 34 (2), 161–173.

Hiebert, J. & Grouws, D. (2007). The effects of classroom mathematics teaching on students learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: a project of the national council of teachers of mathematics* (pp. 371–404). Information Age Pub.

Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: an introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: the case of mathematics* (pp. 1–27). Erlbaum.

Johnson, H. L., Coles, A. & Clarke, D. (2017). Mathematical tasks and the student: navigating "tension of intentions" between designers, teachers, and students. *ZDM*, 49 (6), 813–822.

Jonsson, B., Norqvist, M., Liljekvist, Y. & Lithner, J. (2014). Learning mathematics through algorithmic and creative reasoning. *The Journal of Mathematical Behavior*, 36, 20–32.

Jäder, J., Sidenvall, J. & Sumpter, L. (2017). Students' mathematical reasoning and beliefs in non-routine task solving. *International Journal of Science and Mathematics Education*, 15 (4), 759–776.

Kilpatrick, J., Swafford, J. & Findell, B. (2001). *Adding it up: helping children learn mathematics*. National Academy Press.

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics*, 67 (3), 255–276.

Lithner, J. (2017). Principles for designing mathematical tasks that enhance imitative and creative reasoning. *ZDM*, 49(6), 937–949.

Maciejewski, W. (2017). Mathematial knowledge as memories of mathematics. In B. Kaur, W. K. Ho, T. L. Toh & B. H. Choy (Eds.), *Proceedings of the 41st conference of the international group for the psychology of mathematics education* (Vol 3, pp. 209–216). PME.

Niss, M. (2003a). The Danish KOM project and possible consequences for teacher education. In R. Strässer, G. Brandell, B. Grevholm & O. Helenius (Eds.), *Educating for the future: proceedings of an international symposium on mathematics teacher education* (pp. 179–190). Kungliga Vetenskapsakademin.
Niss, M. (2003b). The mathematical competencies and the learning of mathematics: the Danish KOM project. In A. Gagtsis & S.G. Papastavridis (Eds.), *Third Mediterranean conference on mathematical education* (pp. 115–124). The Hellenic mathematical society.
Niss, M. (2006). The structure of mathematics and its influence on the learning process. In J. Maasz & W. Schloeglmann (Eds.), *New mathematics education research and practice* (pp. 51–62). Sense Publishers.
Olsson, J. (2019). Relations between task design and students' utilization of GeoGebra. *Digital Experiences in Mathematics Education*, 5 (3), 223–251.
Robson, C. (2011). *Real world research*. Wiley.
Russo, J. & Hopkins, S. (2017). Student reflections on learning with challenging tasks: "I think the worksheets were just for practice, and the challenges were for maths". *Mathematics Education Research Journal*, 29(3), 283–311.
Schmidt, W. H., McKnight, C. C., Houang, R. T., Wang, H., Wiley, D. E. et al. (2001). *Why schools matter: a cross-national comparison of curriculum and learning*. The Jossey-Bass education series.
Schoenfeld, A. H. (1985a). *Mathematical problem solving*. Academic Press.
Schoenfeld, A. H. (1985b). Making sense of "out loud" problem-solving protocols. *The Journal of Mathematical Behavior*, 4 (2), 171–191.
Schoenfeld, A. H. (2017). On learning and assessment. *Assessment in Education: Principles, Policy & Practice*, 24 (3), 369–378.
Silver, E. A. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM*, 29(3), 75–80.
Simon, M. A. & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: an elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, 6 (2), 91–104.
Skolverket (2010). *Nationellt kursprov i matematik, kurs A, våren 2010, del I* [National test in Mathematics]. Skolverket.
Sullivan, P., Askew, M., Cheeseman, J., Clarke, D., Mornane, A. et al. (2015). Supporting teachers in structuring mathematics lessons involving challenging tasks. *Journal of Mathematics Teacher Education*, 18 (2), 123–140.
Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*,12 (2), 151–169.

Terwel, J., Oers, B. van, Dijk, I. van & Eeden, P. van den (2009). Are representations to be provided or generated in primary mathematics education? *Effects on transfer. Educational Research and Evaluation*, 15 (1), 25–44.

Vetenskapsrådet. (2017). *God forskningssed*. Vetenskapsrådet.

Wedman, L. (2020). *The concept concept in mathematics education: a concept analysis* [Ph. D. thesis]. Acta Universitatis Gothoburgensis.

Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics. *Educational Studies in Mathematics*, 60 (3), 297–321.

Jonas Jäder

Jonas Jäder is Senior Lecturer in mathematics education at Mid Sweden University. His main research interest is tasks and task design with a focus on mathematical problems and on students' development of conceptual understanding.

jonas.jader@miun.se