Gérard Vergnaud in action

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We have contemplated what we could add to the description of the legacy of Gérard Vergnaud, and have realized that we have at least one experience that is probably not shared by many researchers in mathematics education. We were driven by Gérard Vergnaud in his car.

It was the gravitational force of proportional reasoning that drew us into the car. Vergnaud was invited to Sweden by Bengt Johansson in 1989 to speak at the international seminar at the mathematics teacher education department in Gothenburg. Bengt and Gérard shared an interest in proportional reasoning. Thirty years later, when Ola learned about Linda's interest in proportional reasoning, he set up a meeting between Linda and Bengt. At some point during the conversation Bengt asked if she had read Vergnaud. From her handbag, Linda picked up a heavily read version of *The theory of conceptual fields* but also acknowledged it was a tough paper to digest. Bengt mentioned in passing that it would in fact be best to meet up with Vergnaud himself to discuss, and that he could set up such a meeting. Then the conversation took off in different directions.

After the meeting, we intensified our studies of Vergnaud's work. One day, after a new and particularly long discussion on *The theory of conceptual fields*, Linda turned to Ola and asked if Bengt was serious about the possibility of meeting Vergnaud. He was. A few days later, we had a two-day meeting set up at Vergnaud's house in Paris, scheduled for Maundy thursday. And so, we ended up being subjected to the experience of traveling along as backseat passengers to Vergnaud. We will not describe the ride in detail. It suffices to say that after a while Linda whispered to Ola: "My kids are 17 and 19, they will manage, but you have small kids you should get out of the car!" It was an exhilarating ride.

We could not prepare for the car ride but came to Paris well prepared for the academic work. We had spent months reading all the papers by Vergnaud that we could find in English. We arrived at Charles de Gaulle

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directly from Greece, where we had attended a conference. In all honesty, we did not attend many sessions but instead continued to prepare ourselves by reading and discussing Vergnaud's work. Those schemes, what are they really? Why the term predicative, in predicate knowledge? How many situations for multiplication are there and how many might a typical grown-up hold?

At the end of the first day of discussions, Gérard cooked a wonderful stew. From his attic, he brought Champagne, which came with a story of being recovered from a sunken ship in the Barents sea. He brought down empirical material illustrating students' work when we asked about the background for the concrete claims concerning some of the particular conceptual fields he wrote about. Some material concerned unpublished research. Gérard talked about the nature of schemes and corrected us every time we happened to say schema. In fact, he often inserted the french word schéme into the English conversation. The vigor with which he pronounced the difference perhaps made us understand more about the nature of schemes than any explicit descriptions could. Gérard knew exactly what a scheme was. This was evident by the way he talked about schemes. How can you know if someone knows? It is not by controlling whether the person can give a definition. It is realizing the variety of perspectives the person can apply to the concept. The variance of metaphors. The multitude of examples. The diversity of applications. In his writings, Vergnaud paid much more attention to describing schemes, than ever did Kant or Piaget, from where he picked the concept up. But the actual nature of schemes cannot be described in full despite that you can understand them on an operational level. Schemes are not schemas.

Maybe it is the depth of Vergnaud's knowledge of the relationships of thinking and acting combined with the difficulty of making such knowledge explicit that explains the consistency of Vergnaud's work over time. It does not only form a research program of its own, it is also very repetitive. The main concepts of his theory are treated again and again in a network of articles spanning over decades, treated with remarkable consistency but still always portraying slightly different narratives. Gérard was well aware: "I keep writing the same thing over and over again", he told us. "I will do it until people understand".

I

Vergnaud served as a communicator of his scientific discoveries and inventions to an international audience. In this duty, he assumed the role of the essayist. Vergnaud was a co-founder of the *International group for psychology in mathematics education*. In the second PME meeting in

1978 he presented the paper *The acquisition of arithmetical concepts*, a text later re-published in Educational Studies in Mathematics. It is somewhat remarkable how much of his scientific contributions are present already in this early paper. Important aspects of his analyses of the additive and multiplicative conceptual fields are there. There are pretty strong indications of his theory for representations, which keeps lurking in the background in many of Vergnaud's papers until it is finally designated an article of its own in the Journal of Mathematical Behaviour in 1998. The paper from PME 2 contains the basics of the theory of conceptual fields too, albeit the term is not coined yet. The strong focus of the important relationship between operative knowledge and symbolic knowledge, that he later calls predicative knowledge, is there. The concepts of theorems- and concepts-in-action too. When you know what to look for, there are even hints of the view of concepts as consisting of situations, invariants, and representations, the view that later becomes such a backbone of the full-fledged theory of mathematical conceptualization that Vergnaud launched.

In the original PME 2 paper, only in two footnotes Vergnaud refers to some data and hints at methods behind the research. But other than that, Vergnaud is telling us a story about a reasonable way to think about mathematical conceptualization rather than reporting research results in a traditional manner. The article reads more like a retrospective summary of a research program. Maybe this is also what it is, because at the time Vergnaud had published in French on similar subjects for over a decade. It would be easy to think that the writing style was just contemporary. After all, our field has moved towards more rigor over the decades. But Vergnaud's most important texts have continued to be written and published in the same style for four decades, regardless of whether they have been book chapters, conference contributions, or journal articles. *A classification of cognitive tasks and operations of thought involved in addition and subtraction problems* from 1982, *Multiplicative structures* from 1988, *The nature of mathematical concepts* from 1997, *A comprehensive theory of representations* from 1998, *The theory of conceptual fields* from 2009. A small but representative sample. They all share a similar style of presentation.

Vergnaud is certainly a giant of our field. And like other giants there might be some privileges in terms of the artistic freedom of writing style one is allowed to adopt and still be published. But in our view, few hold so consistently to a particular style as Vergnaud. Style may be a personal thing. But if the stylist is a communicator of scientific discoveries and inventions, the style can be analyzed.

"The most incomprehensible thing about the universe is that it is comprehensible", Einstein wrote in 1936. This says something about the comprehending mind, not only something about the universe. But conceptualizing the universe may very well be easier than conceptualizing the conceptualizing mind, and the latter was the endeavor of Vergnaud. Vergnaud built on both Piaget and Vygotsky. Science is reduction, as the introductory sentence of *The theory of conceptual fields* reads, but Vergnaud realized that a theory for what he tried to capture cannot be reduced too much. At least, it had to contain main elements from both the giants of psychology to properly account for both the operative form of knowledge and knowledge expressed in the different semiotic systems, like natural language or symbol systems. Vergnaud also adds components of his own, like the concept of situations. And the theory should be didactically useful on a medium and long-term basis. A theory encompassing all that is a complex animal, maybe even impossible to pin down in full detail. But to us, reading Vergnaud's work from the present back to the 1970s, then forward in time, and back again to the late seventies work, we think he already knew his way around the theory he was about to build. He had operative knowledge of it, to use his own terminology, and could apply it to things like addition and subtraction and multiplication. Obviously, he is communicating through words, sentences, diagrams, and symbols, which in Vergnaud's terminology is in predicative form. But still, his writing is a form of evolving storytelling that frames the theory he built by explaining what the theory is doing. We think it is Vergnaud's practical mastery of the phenomena he wrote about in conjunction with the difficulty of explicitly pinning the main concepts of the theory down, that puts "nest" in the style[1]. This is also why it pays off to go back and forth among the decades of Vergnaud's writings when you want to understand his theory. You cannot just operationalize a few explicit descriptions. You need to learn it in the operative sense yourself.

II

Vergnaud's life's work was dedicated to a theory for learning and progression of knowledge in mathematics and elsewhere. Together, his *Theory of conceptual fields* and *Theory of representation* constitute an organized network of ideas, notions, distinctions, terms, and claims that carries the legacy of both Piaget's and Vygotsky's work. In contrast to the well-established idea of a hypothetical learning trajectory, on which the students move towards new insights and higher capacity to handle mathematics, Vergnaud developed his theories from the perspective that growth in knowledge is best described as the growth of conceptual fields, which is inherently a network rather than a path. Even his description of the concept of concept shows the dedication to this view. From a teacher's point of view, the theory of conceptual fields provides a foundation for observing and eliciting learning as well as for planning learning on a medium to long term basis. From a researcher's view, the comprehensive theory provides an explanation of observed facts and phenomena within the domain of mathematics education, as well as a tool for prediction and guidance for the present and future action and behavior. All together Vernauds theories of conceptual fields and representation provide a structured set of lenses through which mathematics education, at all levels, can be investigated. We cannot describe it all, only go on a brief journey through Vergnaud's most important contributions.

From Piaget, Vergnaud picks up the idea of schemes and puts it at the center of his theory of knowledge and development. Vergnaud shares Piaget's view of knowledge as adaptation of schemes, in the process of assimilation and accommodation, as well as the overall Piagetian conception that action plays a main part in development. Elaborating on Piaget's three categories of schemes, behavioral, symbolic, and operational, Vergnaud defines a scheme as *the invariant organization of activity for a certain class of situations*. Here we need to stop and ponder upon two notions, those of invariants and situations. The notion of situation is a great contribution to the field of mathematics education research by the French didactics. The way Vergnaud constitutes situations theoretically aligns with the tradition of Steffe, von Glasersfeld, Richards, and Cobb. Situations and schemes are inseparably related, and how a situation is understood is hence a personal matter. Throughout his career, Vergnuad however made methodological and analytical choices that stem from his interest in didactics, that is, the need to beforehand carefully review the choice of the situations and how they shall be sequenced to support the conceptualizing process of students on a medium and long term basis. This involves creating a library of situation types chosen to support the development of desirable schemes related to some set of concepts. To be able to describe situations in this a-priori manner, the concept of situations has to be transferred from the realm of the individually cognitive to the realm of the didactical. There is no other way of doing this than to describe situations as if they exist independent of a particular individual holding them. It is this choice that makes Vergnaud's work so relevant for teaching, rather than just interesting for understanding learning or development.

Vergnaud builds his theories on the basis that it is possible to group situations into classes where the same scheme applies. Classes of situations are identified based on the invariants (theorems and concepts) applicable to the situation. To Vergnaud all situations that require the same set of invariants to be handled belong to the same class. Invariants and situations are dual constructs, as Vergnaud says. Without this theoretical lens, it may be difficult to discover that mathematical tasks, that look very different on the surface, may well belong to the same class of situations, but also to separate situations that can be described by the exact same mathematical expression, but still from a psychological point of view stem from quite different situations.

But for Vergnaud, situations were insufficient without the notion of representation, where mathematics comes to life in semiotics and mathematical symbols. Here it is in place to point out that Vergnaud rejects the idea that mathematics is a language. That idea is wrong, in Verganaud's opinion, but he acknowledges that understanding and wording mathematical sentences play a significant role in the difficulties students encounter, and in conceptualization. Without words and symbols, experience cannot be shared with others. Vergnaud was heavily inspired by Vygotsky in his theory building, in particular concerning the role of semiotic systems for conceptualization. Vergnaud used an analogy for the role of language: Like musical notations shape symphonies, numerical and algebraic notations shape the processes of conceptualization and reasoning in mathematics.

"Cognitively, grasp of just one concept is the sound of one hand clapping". This is how the philosopher of language, Robert Brandom explains what could have been a metaphor for Vergnaud's research program too. The view of concepts depending on other concepts led Vergnaud to develop the *Theory of conceptual fields*, as a structured set of lenses to provide an explanation of why mathematically simple concepts are psychologically complex. A conceptual field consists of a set of different concepts, tied together, and a set of different situations where the concepts apply. According to Vergnaud a variety of situations are necessary to give a concept meaning. Conversely, a class of situations cannot be analyzed with one concept alone. Rather, several related concepts are required to understand any situation. Conceptual fields consist of such clusters of situations and concepts. The learning of different properties of the same concept develops over several years. Everyone with experience from teaching and learning mathematics knows that it is true that a mathematical definition is not enough to extract the properties of a concept. Hence, if you want to analyze how mathematical concepts develop in individuals' minds, concepts need to be considered from a psychological perspective.

Vergnaud's definition of a concept may cause confusion for his readers. A concept, C, is defined as a triple of three sets, $C = (S,I,R)$; S: the set of situations that make the concept useful and meaningful; I: the set of operational invariants, concepts-in-action, and theorems-in-action, that can be used by individuals to deal with these situations *R*: the set of symbolic representations, linguistic, graphic, or gestural that can be used to represent invariants, situations, and procedures. The attentive reader now notes that Vergnaud uses concepts (in-action) to define concepts. How does this work out? Concepts to define concepts? Well, let us not forget that according to Vergnaud, a concept cannot exist independently of other concepts. Knowledge can only be achieved through the use of concepts that give life to the concept in focus. A finesse in Vergnaud's terminology on theorems and concepts is the suffix in-action. Here he wants to emphasize that people know many things that they cannot express in words, only in action, the operational form of knowledge which in the best of worlds over time will be complemented by predicative knowledge.

III

Seducing the teacher. Reasoning proportionally. Guiding the pigs to the slaughterhouse. Considering symmetry a mapping in the class labeled isometries. The breadth of human behaviors that Vergnaud uses to explain the view of knowledge his theories aim to explicate is vast. Science is reduction. But Vergnaud refuses to reduce away phenomena that matter just for the sake of simplifying. He considers totalities. A scheme is a totality that cannot be broken apart. A concept is a totality of situations, invariants, and representations. Vergnaud did more than most to explain how knowledge is manifested in action, but at the same time explained the fundamental role of symbolic knowledge. Vergnaud described totalities by identifying theory components, just like the triplet of sets making up concepts. It is in this way he uses theory to make it possible to analytically identify and analyze components of totalities that in practice are not separable into components.

It is probably the insistence of not simplifying too much that can make Vergnaud's theoretical work somewhat overwhelming. His work is heavily referenced in our field. When we reviewed the hundred most cited articles that cite Vergnaud's main work, we found that most refer to him when discussing some particular content, like addition or multiplication, and the work Vergnaud did on those particular conceptual fields. He is also often referred to in general terms when some author wants to convey the idea that concepts come in clusters. There is work that uses some of his main notions but we find little work that builds on his theories as such. This, we think, is a great opportunity missed. Few if any theories in mathematics education have the potential to elicit mathematical conceptualization in-depth as Vergnaud's work.

"I keep writing the same things over and over again and I will do it until people understand". Now the writing has stopped, but the words and sentences and the ideas they represent are with us and can still take us on exhilarating rides whenever we bring Gérard Vergnaud in action.

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Note

1 This metafor reads more poetically in Swedish. We picked it up from the Swedish essayist Sara Danius. In fact, most of the structure of our text is copied from Danius' essay ”Sven Lindqvist in action”.

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