Students' reasoning and feedback from a teacher

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The present study investigates how to support students' creative reasoning when they need assistance in solving non-routine tasks. Two groups of 11–12-year-old students solved the same tasks, one group receiving feedback directed at the task solution and the other feedback directed at their thinking processes. The results showed that students who received feedback directed at their thinking processes expressed reasoning based on their attempts to solve tasks while the other group often repeated the researcher's suggestions for solutions. However, there were some instances in which feedback on task level entailed students engaging in creative reasoning.

A wide range of research has stated that teaching in which the teacher explains and shows students how to solve certain kinds of tasks, after which the students solve similar tasks on their own, limits their opportunities to learn (Hiebert, 2003; Boaler, 2002). In a Swedish study observing 200 mathematics classrooms, it was found that students had many opportunities to learn facts and calculating procedures, but that sessions in which students were to learn through constructing solutions were rare (Boesen et al., 2014). If teaching aims to support students in constructing solutions to mathematical tasks, the way the tasks are designed and how the teachers interact with the students are crucial factors (Franke et al., 2007). For example, how feedback is prepared and delivered is an important part of interactions.

Hattie and Timperley (2007) suggest two distinct levels of formative feedback, the task and process levels, both said to be helpful to students working with non-routine tasks. Feedback on task level addresses the way the task at hand may be solved, while feedback on process level focuses on

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supporting students' thinking associated with solving the task. Feedback on task level can be interpreted as the teacher explaining and contributing through offering appropriate solution methods, while feedback on process level means that the teacher challenges students to explain their thinking and justifications.

A way to explore students' thinking is to study their reasoning. Lithner (2008) characterizes reasoning associated with solving mathematical tasks as either *algorithmic reasoning* (AR) or *creative mathematical reasoning* (CMR). Students who engage in AR try to recall a memorized procedure that is intended to solve a task. Engaging in CMR means that students create or recreate an original solution method, for which they formulate arguments anchored in mathematics (Lithner, 2008). Studies have shown that students who engage in CMR when solving tasks understand their solutions and learn mathematics better compared to those who engage in AR (Olsson, 2019; & Olsson & Granberg, 2019; Jonsson et al., 2014). Furthermore, in the study by Jonsson et al. (2014), compared to AR students there was a stronger correlation between CMR students' success in practice and their results on a post-test. In Olsson and Granberg's (2019) study, CMR students' outperforming of AR students on a post-test was only valid for students who had successfully solved tasks in practice; non-successful CMR students did not score differently from non-successful AR students. In these studies, the students received no feedback from a teacher. Therefore, it is interesting to investigate whether appropriate teacher interactions could help students who are not successful in solving tasks aiming at engaging them in CMR. On the assumption that attempting to solve a non-routine task engages students in CMR, a question raised from previous studies is whether it is possible to encourage non-successful students to engage in and proceed with their CMR, or if they need instructions in how to solve the task. This study's interest is whether students can translate feedback into CMR, and asking them to explain their solutions is regarded as a possible way to gather information about their reasoning. The aim is to investigate how established concepts of feedback on process and task level could guide teachers' support for students' engagement in CMR when solving non-routine tasks. The research question is: *What is the character of students' reasoning before receiving feedback and how can feedback on either process or task level support students' CMR?*

While earlier studies addressing CMR and AR have not involved teacher feedback to students, this study constitutes a starting point in gathering knowledge about feedback appropriate for supporting CMR.

Background

Hiebert (2003) points out that research has shown that students can learn what they have opportunities to learn. Research from different parts of the Western world reports that the most common way of teaching still involves emphasizing computational procedures (Hiebert, 2003; Boesen et al., 2014; Blomhøj, 2016). Most commonly, the teacher demonstrates these procedures and the students use them to solve tasks. Little attention is given to students' understanding of mathematical concepts. The absence of opportunities to solve challenging problems and engage in mathematical reasoning and justification may be the reason behind inadequate skills among students (Hiebert, 2003).

Teacher interactions and feedback

Teaching in which students are expected to construct solutions for tasks on their own means that the teacher has other roles, compared to teaching built on instructing students in how to solve tasks. Brousseau (1997) suggests that students must take responsibility for constructing (at least parts of) the solution. If they do not know how to proceed, the teacher should not reveal how to solve the task but rather adjust the instructions in a way that the students can continue constructing the solution. According to Brousseau (1997), learning takes place when constructing meaning from mathematics. That is, when students use mathematics to solve a problem, they must consider the way in which the mathematics contribute to the solution. For example, a task could be formulated as follows: "Construct a rule for how to create two linear functions which have perpendicular corresponding graphs". From this instruction, it is reasonable that constructing the solution entails exploring negative and positive x-coefficients and their relations to slope, observing that the constant term is irrelevant to the rule, determining that one x-coefficient is the negative inverse of the second, etc. If we compare this example with teaching in which the teacher explains the rule that the coefficients $m_1 \times m_2$ equal -1, after which the students are to use the rule to solve a couple of tasks, it is reasonable to assume that the students in the first example have greater motivation to understand the mathematical content. Furthermore, drawing on Brousseau (1997), if a teacher reveals the rule when students are struggling, it is no longer necessary for them to understand the mathematical content.

However, there is research claiming that students who receive explicit instructions for how to solve tasks are more likely to learn than are students who learn from discovery-based teaching (Mayer, 2004; Kirschner et al., 2006). These studies claim that when students receive minimal instructions or guidance this places heavy demands on their working memory resources, which in turn affects their ability to maintain information online and to store and retrieve information from their long-term memory. On the other hand, there is also research suggesting that teaching in which students have the responsibility to construct solutions does not necessarily mean that they have no guidance. Hmelo-Silver et al. (2007) claim that there are other ways of facilitating students' learning that do not include providing the solution methods. For instance: frames and learning goals for tasks may be explicitly clarified; classroom work may be organized for student collaboration; or the teacher may provide knowledge when students ask for it or pose questions to help them focus on productive parts of their reasoning. A benefit from teaching approaches in which students have the responsibility to construct solutions is that the teacher may ask them to explain, engaging them in justifying their solutions (Ball et al., 2008). Teaching in which students solve problems and are challenged to justify their solutions has often been found to promote engagement in and positive attitudes towards mathematics, as well as knowledge beyond performing procedures (Boaler, 2002; Hiebert, 2003).

The way the teacher organizes learning activities affects the students' possibilities for learning (Hiebert, 2003). Boaler (1998) compared students taught through traditional methods – with the teacher explaining mathematics and demonstrating procedures and the students then working in textbooks using these procedures to solve tasks – with students taught through discussions and project-oriented tasks. It was found that students from the two approaches performed equally well on procedural tasks, but that those from project-oriented classrooms were better at using mathematics in situations outside the classroom. Typical of the project-oriented teaching was that students were encouraged to develop their own ideas, formulate problems, and make use of mathematics. Such a teaching approach is a challenge for the teacher, as students may come up with non-standard solutions that are not familiar to the teacher (Ball et al., 2008). In a traditional teaching approach, the teacher has prepared one or a few efficient solution methods which are communicated to the students, either before they work with the tasks or when a student fails to solve a task. This is not possible in learning situations in which students are to develop their own ideas (Dyer & Sherin, 2016). Ball et al. (2008) claim that in teaching focusing on students' thinking, the teacher must efficiently and fluently perform error analysis on students' failures. A skilful teacher is able to size up the source of a mathematical error and make the student aware of how to adjust her thinking. This resonates well with Hattie and Timperley's (2007) definition of process-oriented feedback, which is given in dialogue focusing on supporting the thinking process that results in the solution of a task. However, both forms are presented as appropriate for learning and useful in supporting students' independent work with tasks. Hattie (2012) suggests that feedback on task level is for novices building up surface understanding, while feedback on process level is for students who are already at a proficient level. It is claimed that feedback on task level is the most common (Airasian, 1997; Hattie, 2012), which may be interesting in light of reports of students often only reaching surface mathematical knowledge (Hiebert, 2003).

Earlier studies

Earlier studies have shown that students who are successful in constructing solution methods learn mathematics better compared to students who are given a solution method (Jonsson et al., 2014; Norqvist, 2018; Olsson & Granberg, 2019). However, earlier studies do not agree as to whether students who fail need instruction in how to solve the task (Kirschner et al., 2006) or whether the teacher should encourage them to evolve their attempts to solve it (Brousseau, 1997; Ball et al., 2008). In this study, we aim to add knowledge of how to achieve the latter.

Framework

The research question focuses on two components, feedback and reasoning. Feedback will be prepared as being on either task or process level (Hattie & Timperley, 2007), and will be given to students when they ask for help. The students' reasoning will be the unit of analysis, characterized as either AR or CMR (Lithner, 2008). On the issue of reasoning, Lithner's (2008) framework for investigating imitative and creative mathematical reasoning includes distinctions between different characters of reasoning that are appropriate for this study, while Hattie and Timperley's (2007) definitions of feedback on the task and process levels will be modified in order to distinguish them from each other.

Feedback on task and process levels

Feedback on task level means that someone provides students with information about misunderstandings, and suggests alternative and/or more effective strategies and concrete help for processing and understanding the task (Hattie & Timperley, 2007). It often entails a monologue through which the teacher informs the students. In this study, feedback on task level means that students who ask for help will receive concrete advice on how to solve a task. If they have started solving a task with a fruitless strategy, they will receive information on effective alternative ways to reach a solution. When they have reached a solution, students will be informed whether or not they are correct.

Feedback on process level is aimed at students' thinking, and often entails a dialogue between the provider and the receiver of the feedback (Hattie & Timperley, 2007). Furthermore, feedback strategies avoiding giving students concrete solution methods and/or confirming correct answers often entail supporting their thinking process and developing their solution methods; and, if a method is not working, they themselves determine why this is the case (Hattie & Timperley, 2007). In this study, feedback on process level means that questions will be posed to help students themselves determine whether their solution methods are working. For example, they will not receive confirmation of whether a solution is correct but will instead be asked "Can you verify your solution in some way?". The feedback will be given in dialogues, and it is the students who will determine whether a solution is correct.

In reality, feedback on task level and process level may be parts of the same unit of feedback. It may start on task level and develop into process level (Hattie & Timperley, 2007). This means that the distinction between the two is not always clear. This study will investigate how feedback on either task or process level can support students' reasoning in an experimental environment. Therefore, feedback on task level must be clearly distinguished from feedback on process level, and vice versa. Compared to Hattie and Timperley's (2007) definitions, the feedback in this study is less general, but is associated with the difficulties students can be expected to face in the task-solving process. The reason for this is to ensure that the feedback given is on either task or process level rather than being a mixture of the two. These interpretations of the feedback concept will support the preparation of feedback to students.

Algorithmic and creative mathematical reasoning

In this study, reasoning is defined as "the line of thought adopted to produce assertions and reach conclusions in task solving" (Lithner, 2008, p. 257). The analysis will distinguish between AR and CMR. Typical signs of AR are trying to recall remembered procedures and facts. A variant of AR is guided AR; that is, the teacher guides the students stepwise to a solution. Students who try to recall memorized procedures and/or facts will be regarded as being engaged in AR. Likewise, students who follow the researcher's instructions for solving a task will be regarded as being engaged in guided AR. Students who express original solution methods they themselves have constructed, and formulate arguments for these methods and their solutions, will be regarded as being engaged in CMR. Students who fulfil some of these criteria will be regarded as being engaged in parts of CMR.

Method

The method is based on observing students solving tasks aimed at engaging them in CMR, and when they ask for help, giving them feedback on either task or process level. When they reach a solution, they will be encouraged to explain their thinking. Attention will be paid to whether they engage in CMR before receiving feedback and whether they refer to their CMR when explaining their thinking. It is assumed that students' explanations of their thinking will reveal the characters of their reasoning. A possible outcome of feedback on task level is that students' explanations, when they have received instructions in how to find a solution to a task, connect to their CMR from earlier attempts to find the solution. A possible outcome of feedback on process level is that, in their explanations, students elaborate on their CMR from earlier attempts to solve the task. Both approaches may give a teacher the possibility to support CMR.

Sample

All students in a Year 5 class (11–12 years old) in a Swedish school were asked to participate, and 16 of them consented. Their regular teacher considered that they were used to solving problems in pairs. In consultation with the teacher, the students were divided into two groups assessed to be equal with respect to their mathematical abilities. Both groups included high- and low-achieving students, as well as students who had Swedish as their native or second language. They were paired based on the regular teacher's experiences of those who worked well together.

Task design

Two tasks were chosen, both having been used in earlier studies examining reasoning. The tasks were designed in line with Lithner's (2017) guidelines for CMR tasks. That is: (1) no complete solution method is available to a particular student from the start; (2) it is reasonable for the current student to construct a solution stepwise; and (3) it is reasonable for students to justify the construction and implementation of a solution. As the tasks had been used for older students in earlier studies, it was determined that the students in the present study would need assistance in constructing and implementing a solution, which was desirable for investigating feedback. This was piloted and found to work well. The two tasks had different levels of difficulty, with the intention that all students would face a task containing difficulties for which they would need help.

The first task assigned to the students, the matchstick task (figure 1), includes mathematic content like variables and constant terms, and ideas like relationship and change. Investigating 16 students aged 11–13 from four countries who solved the matchstick task, Reinhardtsen and Givvin (2019) found that most of them used draw and count as a strategy and employed arithmetic experience when proceeding. When solving the task there are possibilities to use different representations such as figures, arithmetic, and algebraic expressions, and strategies like drawing, counting, and looking for patterns can be applied. Based on experiences from pilot tests, it was considered that students would find themselves in processes like: (a) understanding the instructions and conditions for solving the task; (b) approaching how many matchsticks are needed for seven squares, through either drawing and counting them or starting with four squares and realizing there must be three more sticks for every new square and based on this calculating for seven squares; (c) for 50 squares, either figuring out how many sticks are needed for ten squares and then multiplying by 5 (which will result in an incorrect answer), multiplying 50 by 3 and adding 1, multiplying 49 by 3 and adding 4, or performing any calculation built on the insight that every square except the first includes three sticks; or (d) for the general calculation, students who failed on 50 squares having to explore and analyse the pattern and figure out that every extra square requires three sticks, and students who have already come to this insight building their solution on this. The key to calculating how many sticks are needed for 50 squares, and the general solution, was considered to be the realization that one square requires four sticks and all the others three.

The second task, the tower task (figure 2), includes combinatorics and was assigned only to students who had managed to solve the matchstick task without support from the researcher. Possible processes through the solution could be (a) understanding the instructions and conditions for the task; (b) creating a first example in line with the instructions (e.g. drawing, using concrete material, a thought experiment); (c) exploring all possibilities with, for instance, one red and two blue blocks; (d) either exploring one blue and two red blocks or drawing the conclusion that there must be equally many possibilities as with one red and two blue blocks; (e) realizing that ”two at the most” means it is also allowed to use only one colour, which adds two possibilities; or (f) verifying the solution

Matchsticks may be formed as squares in a row like below:

- How many matchsticks would be needed for 7 squares in a row?
- How many matchsticks would be needed for 50 squares in a row?
- Explain with a rule or a formula how many matchsticks are needed for an unknown number of squares in a row

Figure 1. *The matchstick task*

You have a large number of red and blue blocks. How many different towers can you build where exactly three blocks are standing on each other using two colours at the most?

Figure 2. *The tower task*

by reasoning (e.g. "If there are one red and two blue, the red can be in three positions"). These processes do not necessarily appear one after the other. The key to approaching the task was considered to be the understanding that "two colours at the most" includes the possibility to build towers with only one colour.

Feedback design

Based on all expected processes of the task solutions, a protocol was established with specific feedback, on both task and process level (figure 3). During the intervention, the protocol supported the way the students received feedback. The definitions presented in the framework were used to secure feedback on task level that would not contain elements that could be referred to feedback on process level, and vice versa. In summary, the guidelines for feedback on task level were to: explain the conditions for the task, make students aware of mistakes and incorrect solutions,

Process	Task level	Process level
Approach how many sticks are needed for 50 squares	If no action is undertaken - Point out that the first square needs 4 sticks and every additional square needs 3 sticks. If this does not engage the student in solution – explain that it is possible to multiply 50 by 3 and add 1, because the first one needs 3+1. If the solution is based on how many for 10 multiplied by 5 – make the student aware that for every 10 there will be one stick too many except for the first one.	If no action is undertaken - Ask the students to explain the way they are thinking. If the students have a solution – ask them how they are thinking, ask them if they can check the answer in some way, ask them if they can explain why they think the solution works.

Figure 3. *Excerpt from the feedback protocol*

supply students with appropriate methods, confirm correct answers, and ask questions addressing whether instructions are correctly perceived. For feedback on process level, the guidelines were to: ask students to explain how they are thinking about the conditions for solving the task, encourage students to explain their methods, ask students if they can verify their answers, give students alternative tasks to allow them to test their methods, and when interpreting students' thinking as correct, ask them to clarify how they are thinking.

Procedure

Both groups solved the same task pairwise, one pair at a time, in a separate room. The task was presented in writing. Students were informed that they could ask the researcher for help whenever they needed to. Students who solved the matchstick task without needing help from the researcher were given the tower task. When asking for help, students in the first group were given feedback on task level and those in the other group on process level. When students reached a solution, the intention was to give them the chance to explain their thinking. It was noted that the results of this were different between the tower task and the matchstick task. The two pairs solving the tower task were asked the same question, while those solving the matchstick task had differently formulated feedback. However, even though this most likely affected the results, we consider that the different approaches contribute to answering how feedback can support CMR. The sessions were captured through audio recording.

Method of analysis

The research question focused on students' reasoning before and in connection to receiving feedback. Thereafter, the sequences in which students and researcher interacted were identified. The following steps were then taken.

1 It was ensured that all feedback identified in each session was defined as intended (on task or process level).

2 Reasoning before and in connection to feedback was characterized as CMR or AR.

3 Possible connections between the character of the feedback and students' reasoning were interpreted. Particular attention was paid to the way students explained, nuances in the way the feedback was expressed (e.g. choice of words), and whether the students managed to translate the feedback into CMR.

The analysis was performed in collaboration between the two authors. In order to test the method, two transcripts from pilot tests were analysed together with a research group that examines learning through reasoning. After this, the processes the students were expected to be involved in when solving the tasks were clarified in the feedback protocol.

Results

All eight student pairs managed to reach a solution for the matchstick task. Two pairs (students I & J, and K & L) reached a solution without needing help and were therefore given the tower task. All observed task-solving processes were within the expected ones for each task. Thus, the feedback protocol could be followed and both groups received the intended feedback on either task or process level. However, in some instances the feedback could not be expressed in the exact words used in the protocol, and it was observed that for feedback on task level the choice of words may be crucial with respect to how the students' reasoning was guided.

Without exception, all eight pairs initiated their solutions through CMR. The only ones expressing AR were those who received feedback on task level; students who received feedback on process level tended to elaborate on their initial CMR. However, there were some instances in which students solving the tower task engaged in CMR after receiving task-level feedback.

The matchstick task

All six student pairs who had feedback asked for help after solving part 1 correctly and part 2 incorrectly, in both instances engaging in CMR. Common to all six pairs is that in part 1 they observed that the first square needed one more stick than the following one, and that they did not take this into account when approaching part 2. Students A and B were scheduled to receive feedback on process level.

1 B If four squares need 13 sticks ... we can calculate how many sticks there are for one square ... but 13 isn't in the multiplication table for 4 ...

2 A But these need one less [pointing at the second square].

They use the strategy of calculating how many sticks are needed for one square and then multiplying this number by the number of squares. On line 2, however, student A realizes that the number of sticks differs in the first square compared to the others. They then decide to build seven squares and count the number of sticks. When they proceed to figuring out how many sticks are needed for 50 squares, they return to the numerical approach.

3 A Okay ... for seven squares there are 22 sticks ... how many are there for 50?
4 B Uhm ... we can use the multiplication table for 7 ...
5 A Yes ... there were 22 for seven.
6 B But the multiplication table for 7 includes 49.
7 A Yes ... listen ... we can calculate 22 x 7 ... because 7 x 7 is 49.
8 B Yes ... I'll do it with the calculator ... 154.
9 A Then we need one more ...
10 B Yes ... but on that one we have to remove one ... that makes 154 + 3 ... 157.

In this sequence, students A and B are constructing a solution. They formulate arguments for the solution (lines 6 and 7), which is CMR. On line 10 student B expresses that the last square only needs three matchsticks, but they do not recognize that their solution means that every seventh square has one stick too many. After coming to the solution that 157 sticks are needed for 50 squares, they turn to the researcher (R).

11 B Is that correct?
12 R Can you explain how you were thinking and figure out a way to check whether your answer is correct?
13 A We were thinking that in the multiplication table for 7, 49 is the closest to 50.
14 B And for the last one we only need three sticks ... but wait ... 7 x 7 means we have too many ... there are only 21 for all the next seven except the first one ...
15 A We can count 21 x 7 ... that makes 147 for 49 squares ... and then we add four ... 151 ... the first one has four ... the other 49 have three.

Instead of pointing out the mistake, the researcher encourages the students to explain their thinking, which is feedback on process level. On line 13, student A refers in a reasonable way to the reasoning from lines 4–7. On line 14, it seems as if expressing the reasoning makes student B realize that every additional seven squares only needs 21 matchsticks. On line 15, student A elaborates into a correct solution with plausible arguments. So far, the only feedback from the researcher has involved encouraging the students to explain their way of thinking, and the students have come to all the insights necessary for solving the complete task. When they proceed, students A and B ask what a mathematical formula is. Instead of explaining what a formula is, the researcher asks.

16 R Can you explain a way to calculate the number of sticks for any number of squares?

17 A We always use a lot of 3s ...

18 B Then we add 4 ... or ... instead we can multiply 3 by all squares and add 1 ... the first one has 3 + 1 ...

19 R So ... if I ask you how many sticks there are in 80 squares?

20 B 80 x 3 ... plus 1 ... that makes 241.

21 R What's your rule?

22 B Three times the number of squares ... plus 1 ... we didn't count the whole first square.

The researcher's question on line 16 can be posed because the students have already expressed the basics for a general solution on line 15: that all squares need three sticks except for the first one, which needs four sticks. On lines 17–18 they further elaborate to a general solution, that the number of matchsticks is the number of squares x 3 + 1, backed up with an argument. The researcher's question on line 19 gives them the opportunity to try and confirm their solution and express a correct formula (line 22). Notable in the example is that the students elaborate on their own reasoning, that all explanations are formulated by the students, and that the researcher's feedback on lines 12, 16, and 21 makes them refer to how they have reasoned.

Students G and H were scheduled to receive feedback on task level. They read the instructions and start with the first part.

1 G If you do one square and then another ... they don't fit together ... you have to remove one stick.

2 H Exactly ... then we can put them together three and three until we have seven squares [draws seven squares].

3 G Now we can count them [counts the sticks] ... 22.

Students G and H construct this part of the solution themselves, and formulate arguments (line 1). Therefore, their reasoning is regarded as CMR. They proceed.

4 H Okay ... next question ... how many sticks it takes to build 50 squares ... that's too many to draw.

5 G We can figure out how many it takes for ten ... and then multiply by five.

6 H We can do as we did with seven but with ten [draws and counts] ... 31.

7 [G and H calculate 31 x 5]

8 G 155 [looks at the researcher].

9 R This isn't correct ... if you look at this [points at the drawing with ten squares] ... you count the first stick five times ... it should only be counted the first time ... so you have four too many then.

10 G Okay ... so we can calculate 155 – 4 then ...

11 H 151.

12 R That's correct.

In the first part (lines 4–7), the students continue engaging in CMR (constructing the solution). They combine their previous strategy of drawing and counting with a numerical approach, multiplying 10 (the number of squares) by 5. The feedback on line 9 addresses an incorrect solution (why it does not work) and how to resolve the mistake, which is on task level. On line 10 the students use the last part of the feedback to correct their solution. It is unclear whether they have connected the researcher's feedback to their reasoning on line 1, "If you do one square and then another ... they don't fit together". They may have done so, but might as well simply be following the researcher's suggestion to reach the correct answer. This indicates some limitations of feedback on task level. The researcher's explanation on line 9 does not explicitly connect the students' reasoning on line 5 with that on lines 1 and 2. Furthermore, the feedback also provides a shortcut to the correct answer: they need to remove four sticks. This may remove the students' incitement to connect the explanation to their CMR.

When proceeding, students G and H ask what a mathematical formula is. The researcher explains.

13 R A formula is like a rule where you can take any number and do the right calculation. Let's say you have 88 squares and using that in a formula will calculate how many sticks you need to build them ... look here [pointing at the drawing of ten squares] ... if you remove the first stick every square has three sticks ... so you just add 3 + 3 + 3 88 times ... can you do that in another way?

14 H You can calculate 88 x 3.

15 R Yes ... and what's that?

16 G 264.

17 R Yes ... but remember you set the first stick aside, so what do you need to do?

18 H We have to add 1 ... 265.

19 R So, you take the number of squares times 3 ... and add 1 ... that's the rule ... if I asked you to explain how it works ... what would you say?

20 G We removed one stick and multiplied the number of squares by 3 and then removed 1 ... no ... we added one.

In the sequence above, the researcher's feedback guides the students to make the correct calculations (lines 15 & 17). Thereafter, she formulates arguments for the general calculation (line 19), and then the students repeat what she has explained (line 20). Notable in the example is that the students initially attempt to implement the solution, whereafter the researcher points out their mistakes and explains how to resolve the task. This means that the students, compared to students A and B, have extended information, and in order to continue CMR must synthesize the teacher's explanations with their own reasoning. It also means that they have a choice when explaining the solution, to either use the researcher's explanations to elaborate on their initial CMR or repeat the researcher's explanations. However, if the researcher's question on line 19 had asked *why* instead of *how* the formula works, this might have guided the students to use their initial CMR when explaining. This indicates that word choice is crucial when delivering feedback on task level.

The tower task

Students K and L were scheduled to have feedback on process level. When approaching the tower task, they initially do not understand the instructions. Instead of explaining, the researcher encourages them to read the instructions again.

1	K	Okay ... if they can be built with two colours at the most ... exactly three blocks high ... there can be two ... one colour of two and another one [looks at the researcher].
2	R	Can you explain how you're thinking?
3	L	Uhm ... at the most two colours ... uhm ... then as you said you can have two blocks with the same colour ... two blue and one red.
4	K	Yes ... you have to have the same colours on each one [looks at the researcher].
5	R	What were the instructions like?
6	K	At the most two colours...
7	L	Then we can build one with only blue blocks...
8	K	Yes ... it says two colours at the most.
9	L	And we can build another one with only red blocks.

On line 1, student K expresses his understanding of the instructions. Looking at the researcher may be perceived as an expectation that the researcher should verify whether this understanding is correct. Instead of verifying student K's expression, though, the researcher encourages the students to explain how they are thinking (feedback on process level). On line 5, again, the researcher does not verify the students' implicit question. Instead, the encouragement to explain their thinking and read the instructions entails that they must articulate the part – important for the solution – that two colours at the most means that it is also allowed to build towers with only one colour. This part of the solution is an example of creating an original solution method, which is part of CMR, following the researcher's feedback on process level. Students K and L proceed, exploring how many ways a tower can be arranged with one red and two blue blocks . When they conclude that one red and two blue blocks can be arranged in three ways, they turn to the researcher.

10 L But what's the correct answer?

11 R That's what you're supposed to figure out.

12 K Okay … with this combination you can have three … two blue at the bottom … two blue at the top … and the red in the middle …

13 L There are seven combinations … with two red and one blue you can build these ones [pointing at towers with two reds on top, and two blue with a red in the middle].

14 K Eight … you can turn this one [pointing at the one with two reds on top of the blue] upside down … that makes eight …

After exploring how many ways one red and two blue blocks can be arranged, student L asks the teacher for the correct number (line 10). The researcher's answer on line 11 encourages student K to formulate arguments for their solution (line 12). What is not explicit, but reasonable, is that the conclusion that one red and two blue blocks can be arranged in three ways leads student K to search for the missing combination of one blue and two red blocks (line 14). In terms of feedback, the researcher's question on line 11 is an example of process level, and on the following lines students K and L fulfil the CMR they initiated before receiving feedback. After reaching a solution, students L and K are asked to explain how they reasoned.

15 R Can you tell me how you were thinking when you solved the task?

16 L I thought you could have different designs … only one colour … blue or red … and two blue and one red, and two red and one blue …

17 K Then you can have the blue on top, in the middle and at the bottom … and the same if there's one red and two blue … that makes 2 + 3 + 3, which is 8.

The students' explanation builds on their reasoning when solving the task, and their arguments are based on content relevant to the solution. Notably, the feedback encourages the students to express, repeat, and elaborate on their CMR. The feedback does not explicitly contribute to the solution, which arguably entails the students using their CMR to explain how they reasoned.

Students I and J were scheduled to receive feedback on task level. Initially they receive some support from the researcher, clarifying the instructions for the task.

1 I What ... wait ... we could use a lot of red blocks and a lot of blue ones...
2 J Three blocks high and two colours ... uhm [looks at the researcher].
3 R I can show you an example [builds a tower with one red and two blue blocks] ... this is a three-block-high tower with two colours ...
4 I Ah ... and you can use two red and one blue as well ...
5 R Yes ... two colours at the most.

The feedback on line 3 is considered to be on task level because it suggests how to initiate the solution, and on line 5 confirms that the students' reasoning is correct. In this case, feedback on task level engaged the students in systematically exploring the combinations of one red and two blue blocks and of one blue and two red, and the fact that one tower could be only red or only blue, which is CMR. They conclude that the maximum number of towers is seven, and ask the researcher if this is correct.

6 R You've missed one combination.
7 I That must be something tricky ...
8 J It can't be that tricky ... [looks at the researcher]
9 R Well ... it could be something you've already built that you can turn upside down.
10 I Aha ... if we turn this one upside down, we have two instead of one ... then we have eight combinations.
11 J Is that correct? [looks at the researcher]
12 R Yes ... that's correct.

The researcher's feedback on line 6 draws the students' attention to the fact that they are not correct. Instead of formulating arguments for the solution, they wait for further information from the researcher (lines 7 & 8). The researcher's hint on line 9 guides them to search for the particular combination that will add the missing combination. When they find it (line 10), the researcher confirms that they are correct (line 12). Making students aware of the incorrect solution, suggesting how to find the missing alternative, and verifying the correct answer are all examples of feedback on task level. In the sequence above, the students are attentive to the teacher's information and use it to solve the task without formulating arguments for the solution, which is regarded as guided AR. In this part, the students' initial attempt at CMR turns into guided AR in connection to receiving feedback from the researcher. After reaching a solution, students I and J are asked to explain how they reasoned.

13 R Now then ... can you explain the way you were thinking when solving the task?

14 I First we found seven combinations ...

15 J And then we found out that we could turn one upside down ... then we had eight combinations which is the most you can build.

Their explanation is shallow, and does not include their reasoning in the first phase when they systematically explored possible combinations. Notably, even though the feedback is in line with the task level, the students initially engage in CMR. A possible reason for this is that the feedback on line 3 only gives the students part of the solution, based on which they can proceed by CMR. The feedback on line 5 may leave space for them to be creative. However, they are not engaging in formulating arguments, which is a criterion for CMR. Compared to students K and L, the researcher does not encourage them to tell how they are thinking until they have reached a solution (line 13). It is reasonable to believe that although their reasoning in the early stages, albeit not explicitly expressed, included an explanation for why there are a certain number of possible combinations, their explanations use only what the researcher has made them aware of. It is also reasonable to believe that if the researcher had asked the students to explain instead of verifying the correct answer (line 12) they might have maintained CMR.

Synthesis of results

Looking more closely into what distinguishes the two feedback levels reveals that feedback on process level entails the students repeatedly being encouraged to express their reasoning, which means that they have opportunities to repeat, frame, and elaborate on their attempts to solve the task. Meanwhile, feedback on task level entailed the students receiving new information that they were to include in their reasoning. Thus, feedback on process level supports students in overviewing and framing the solution and elaborating on their CMR, while feedback on task level adds more information for them to process, and they can choose to replace their reasoning with the researcher's explanations.

If students have not made any progress towards the solution, feedback can encourage them to take a first step in solving the task, for example asking them to read the instructions again (as with students K and L). However, what contradicts the conclusion in the previous paragraph is that, when solving the tower task, students R and J engaged in CMR after the researcher had explained how to initiate the solution, which is feedback on task level. What differs from task-level feedback in the matchstick task is that the students had not solved any part of the task and the feedback only contained part of the solution and did not explicitly explain how to use it. Thus, if students have not expressed important parts of the solution in their reasoning, feedback on task level can guide them to engage in CMR if the feedback does not include a complete solution or explanation.

Discussion

In the context of this study, all students initially approached the tasks through CMR and needed help in similar instances. Mainly, their responses to feedback on process level involved elaborating on their initial CMR while those who had feedback on task level mainly changed to AR. This is within the expected results, and is in line with previous research (se e.g. Brousseau, 1997; Blomhøj, 2016). However, a more fine-grained analysis reveals some issues that might be of interest when designing teaching for CMR. When solving the matchstick task, students A and B as well as G and H came to the important insight that the first square needed one more matchstick compared to the next one, started solving the task by drawing and counting, and struggled with using numerical approaches when attempting to calculate the number of sticks needed for 50 squares. In this instance, feedback on process level entailed encouraging them to tell how they had reasoned, while feedback on task level pointed out their mistake, explained why their solution was incorrect, and suggested how to solve the task. This means that students A and B had an opportunity to recapture and assess their reasoning, which in turn helped them to spot and correct their mistakes, while students G and H received information that was substantive enough to reach the solution without reflecting on their initial CMR. Possible consequences of this are that students A and B in all instances of feedback framed their solution and added new parts to it through CMR, while students G and H were not encouraged to express their reasoning, and the feedback meant that they had new information to consider. Arguably, it was a challenge to incorporate the researcher's information into their initial CMR, and as the feedback covered the solution, students G and H could use it to explain the solution. Even though there was also a challenge to elaborate on their initial CMR, students A and B had no other options. Research that questions teaching in which students learn from problem-solving claims that constructing solutions places heavy demands on their working memory (Mayer, 2004; Kirschner et al., 2006). This study does not deviate from this, but indicates that encouraging students to express their reasoning may support them in framing and elaborating on the solution while explanations and instructions for how to solve the task may mean that they have further information which can be used in their explanations to replace their initial CMR. A reflection is that if feedback on task level, like that given in the matchstick task in this study, is routinely given , in the long run students will likely pay less attention to their own way of thinking (see e.g. Hiebert, 2003; Airasian, 1997). This means that they will make less effort to formulate arguments for their solutions, which may foster them into AR, which has been shown to entail superficial learning compared to CMR (Jonsson et al., 2014; Olsson & Granberg, 2019). An objection, based on the feedback students G and H received, is that they were asked to explain *how* the solution worked, which may have guided them to repeat the researcher's explanation. Had they instead been asked *why* the solution worked, they might have returned to their initial CMR. However, even if *how* had been replaced by *why*, it is still reasonable to assume that they would have had more information to process, and would have had to do so in the space of a moment. It is also reasonable to believe that, if the researcher had insisted on asking the students to explain further, they might have referred to their initial CMR; but this would have meant feedback on process level, which was not the intention.

A necessary condition for asking students how they have reasoned is that they have made progress when solving the task, which both students A and B as well as G and H had done when they received their first feedback when solving the matchstick task. On the contrary, when solving the tower task, students K and L as well as I and J asked for help before they had made any progress. Feedback on process level was delivered as an encouragement to read and re-consider the instructions of the task, while feedback on task level entailed showing an example of a tower built with two colours at the most. It is interesting in this case that feedback on task level encouraged the students to employ similar CMR as those who received feedback on process level. Compared to the feedback given in the matchstick task, no complete solution or explanation was given. Thus, the students had to think for themselves regarding how to proceed in order to solve the task. Brousseau (1997) suggests that an option for the teacher to support students' problem-solving is to adjust the instructions in a way that allows them to continue constructing the solution. Furthermore, in order to learn, the student must construct at least parts of the solution (Ibid.). An important part of Lithner's (2008) theory of learning by CMR is that constructing a solution method entails a need to formulate arguments for why the solution method works. If these arguments are based on mathematics, there is an opportunity for learning. In the example of students I and J (as well as that of students G and H), this could have been achieved if the researcher had encouraged them to explain how they were reasoning instead of guiding them to the solution. Thus, if students have not recognized important parts of the solution, a possible way to support them is to give them part of the solution. When they have made progress, however, the teacher should challenge them to explain their thinking.

The small sample in this study means that there are limited possibilities to generalize. However, the students come from the same everyday context and can be reasonably assumed to share perceptions of mathematics in school. Even though the students apparently engage in CMR when solving the tasks, the different forms of feedback seem to affect whether or not they refer to their own reasoning when explaining their solutions. This study indicates that a challenge to teachers involves supporting students' engagement in CMR and being sensitive to whether their reasoning is substantial enough to elaborate on, and if not, being careful when providing parts for them to use in solving the task at hand.

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