Initiating teacher-researcher collaboration to support students' mathematical problem-solving

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Implementing teaching through mathematical problem-solving entails substantial challenges and calls for sustained teacher-researcher collaboration. The joint research and development project "Teaching that supports students' creative mathematical problem-solving" has a fundamental ambition to be symmetric in that both teachers' and researchers' needs and conditions are attended to and complementary in that their different areas of expertise are utilised and valued. In this paper we show how the interplay and development of symmetry and complementarity can function as a means for studying teacher-researcher collaborations.

Educational research has long been criticised for its weak relation to classroom practice (Burkhardt & Schoenfeld, 2003; Coburn & Penuel, 2016), regarding lack of attention to practical key teaching issues, lack of development of empirically-tested and easily-implemented tools and processes, and lack of collaboration between teachers and researchers (Burkhardt & Schoenfeld, 2003; Stylianides & Stylianides, 2013). In mathematics education research, this critique is often linked to *constructive teaching design* that focuses on students' own construction of solutions to mathematical problems, in contrast to an *imitative teaching design* based on presenting procedures and having students imitate them (Gravemeijer et al., 2016; Maass et al., 2019; Munter & Correnti, 2017). One underlying reason for the gap between research and practice is the difficulty in establishing and sustaining teacher-researcher collaborations (TRCs) due to insufficient funding, obstacles for organising professional

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Säfström, A. I., Palmberg, B., Granberg, C., Sidenvall, J. & Lithner, J. (2021). Initiating teacher-researcher collaboration to support students' mathematical problem-solving. *Nordic Studies in Mathematics Education*, 26 (3-4), 131–151.

development for teachers, and systems that do not reward developmental work (Burkhardt & Schoenfeld, 2003). For Swedish mathematics teachers, time for development is scarce, and teachers report spending only about 7 % of their time on reflection and professional development (Swedish National Agency for Education, 2015). Another reason for difficulties in arranging successful TRCs is the lack of insight into how such collaborations actually work. Descriptions of the dynamics of TRCs are rare, and they tend to describe shortcomings rather than good practices and are often only reported when projects are finished (Coburn & Penuel, 2016).

This study analyses the dynamics of the TRC in the initial stages of an ongoing mathematics education design research project focusing on constructive design by exemplifying both challenges and good practices.

The challenges of constructive design

One well-substantiated insight from mathematics education research is the superiority of constructive design: the key to broadening and deepening mathematical competence is to work on problems, i.e. tasks where the solution method is not known in advance but has to be constructed through one's own mathematical reasoning (Hiebert & Grouws 2007; Jonsson et al., 2014; Norqvist, 2018; Norqvist et al., 2019; Olsson & Granberg, 2019; Schoenfeld 1985; 2007; 2015; 2020; Wirebring et al., 2015). Constructive design also entails changed teacher-student interaction, as it requires teachers to abstain from giving a procedure when students ask for help (Brousseau, 1997). Constructive design has impacted policy documents in many countries, for example, by emphasising reasoning and problem-solving, but research results recurrently show that imitative design – practicing a given procedure on routine tasks – still dominates in schools (Boesen et al., 2014; Gravemeijer et al., 2016; Maass et al., 2019).

Implementing constructive design involves specific difficulties. First, teachers might perceive imitative design to be less demanding because it requires less adaptation to students' specific needs. To explain a procedure is relatively simple compared to identifying specific obstacles and guiding students' own reasoning (Ball, 1993, 2001; Leinhardt & Steele, 2005; Schoenfeld, 1998; Sherin, 2002; Tall, 1996). Second, constructing one's own solutions may be perceived as more challenging for students (Hiebert & Grouws, 2007). When students struggle, the teacher may feel a social obligation to provide procedures, falling back on imitative design and thus reducing the learning opportunity (Brousseau, 1997). Third, problems often become a mere addition to the set of tasks to be checked off, even when the original intention was to create opportunities for conceptual learning (Gravemeijer et al., 2016). Fourth, if the standard for evaluation of successful teaching is that students solve many tasks, imitative design is indeed more effective. In the short run, constructive design can seem ineffective and disruptive because the positive effects on deeper conceptual understanding are only evident in a longer perspective (Lee et al., 2014; Lithner, 2008; Jonsson et al., 2014; Ridlon, 2009).

In summary, implementation of constructive design requires not only viable solutions for teaching, but also a common understanding of the differences between constructive and imitative design and common standards of evaluation of teaching. This, in turn, can only be obtained through sustained TRC.

Symmetry and complementarity in TRC

The long-known difficulties of implementing research results has called for engaging teachers in both implementation and research processes (McLaughlin, 1987; Wagner, 1997), and the complexity of TRC has been acknowledged. Frameworks for TRC have been developed (e.g. Jaworski, 2003; Koichu & Pinto, 2018), and different perspectives on TRC have emerged in multiple areas such as action research (Rönnerman, 2008), implementation research (Century & Cassata, 2016, Maass et al., 2019), and design research (McKenney & Reeves, 2018). In design research, TRC is formed to generate theoretical insights and practical solutions simultaneously in consecutive cycles of three core processes: *exploration* of a problem, *design* of solutions, and *evaluation* of outcomes (Gravemeijer & Cobb, 2006; McKenney & Reeves, 2018).

Two aspects are central in TRC in general and design research in particular: equal commitment and attention to both parties' needs and conditions, here called *symmetry*, and joint contribution as well as utilisation and valuation of the different areas of expertise of the parties, here called *complementarity*.[1] However, the lack of good examples of symmetric and complementary collaborations in the research literature (Wagner, 1997) remains, and most studies focus only on challenges or lack descriptions of the concrete activities conducted (Coburn & Penuel, 2016; Pareja Roblin et al., 2014).

Symmetry

Although expressed in different terms, the idea of symmetry is recurring in research on TRC. In Wagner's (1997) categorisation of collaborations, the degree of "symmetry of understanding and purpose" is an important dimension, ranging from researchers aiming to understand teaching, while making no effort to enable teachers' understanding of the research, to equal commitment to a common aim to understand each other's practices. Jaworski (2003) raises questions regarding symmetry with respect to whose knowledge and learning is explicated in a project and who is inquiring into whose practice. Coburn and Penuel (2016) use the term "mutualistic" to describe the presence of joint negotiation of the focus of the work and shared authority within the project.

In design research, there is a need for joint involvement in all processes to some degree (Kali, 2016; McKenney & Reeves, 2018). During *exploration*, when the instructional problem is defined and analysed, symmetric attention to needs and suggestions is key (McKenney & Reeves, 2018; Reeves, 2006). For solutions to be viable, *design* needs to be aligned with the needs and wishes of different categories of participants (McKenney & Reeves, 2018; Reeves, 2006). During *evaluation*, a shared understanding of successful outcomes facilitates revision and refinement of the solutions that are satisfactory to all parties (McCandliss et al., 2003; Reeves, 2006). A further reason for the importance of symmetry in design research has to do with its iterative nature; if collaboration is to be sustained over multiple iterations, it must be mutually beneficial (McKenney & Reeves, 2018; Reeves, 2006).

Complementarity

The synergy of TRC springs from the teachers' and researchers' contributions coming from different perspectives, experiences, and knowledge bases and thus complementing each other (Farley-Ripple et al., 2018; Pareja Roblin et al., 2014). In cultural-historical activity theory, the idea of complementarity is described as resources and expertise being spread, negotiated, and utilised across systems to solve common problems (Edwards & Kinti, 2010). Allowing different competences to complement each other can also be seen as a way to realise ethical standards, since utilising and valuing all participants' competences is both respectful and effective (Hoffecker et al., 2015).

To develop theoretical insights and practical solutions, design research goes beyond teachers enacting teaching designs and researchers reporting research outcomes, and calls for combining participants' expertise across research activities (Cobb et al., 2003; Kali, 2016). In all three core processes – exploration, design, and evaluation – researchers' theoretical understanding is best complemented by teachers' practical knowledge and ground-level instincts (Kali, 2016; McKenney & Reeves, 2018). Complementary perspectives generate better knowledge of the problem, the target context, and the requirements for solutions, and this makes it possible to set goals and to produce outcomes that are valued by both teachers and researchers.

Aim and research question

We aim to analyse the dynamics of the TRC in the initial stages of an ongoing mathematics education project, exemplifying both challenges and good practices, by answering the following question.

> How did symmetry and complementarity interplay and develop within the core processes during the first year of a TRC focusing on constructive design?

Methods

This study was conducted within an ongoing project aiming to develop principles and tools for constructive design. Due to the complexity of devising and implementing constructive design, the project entails iterations of exploration, design, and evaluation on different levels, from microlevel – regarding the focus and issues of particular lessons – to macrolevel – regarding the focus and issues of the project as a whole (McKenney & Reeves, 2018). The project engages a *research team* of eight researchers and 51 mathematics teachers from seven primary and secondary schools spread over three municipalities. The teachers were already organised in *collegial teams* before collaboration was initiated and, in most cases, the whole team participates in the project. Each team participates regularly (once a week or every second week) in *joint meetings* together with the researchers and is led by one or two teachers who co-plan the joint meetings with a designated researcher. All teachers have given informed consent to participate in the research project, and particularly to notes and audio recordings from joint meetings being used for research.

A key endeavour is the joint development of a *teacher guide* aiming to aid teachers in supporting students' own mathematical reasoning when they encounter difficulties during problem-solving, rather than giving a procedure for solving the task (Sidenvall et al., 2019). The teacher guide is based on the principles of formative assessment and includes the following elements: *diagnostic questions* to invite students to share their line of reasoning, a framework for *diagnosis* of the students' difficulties and suggested *feedback* for each difficulty, consisting of suggestions for heuristic and metacognitive strategies – such as reading the text again, making a drawing, or solving an easier task – and questions aimed at spurring reflection on the problem-solving process.

The data used for the present study includes documentation from 38 joint and 17 research group meetings in the form of audio recordings, presentations, and notes collected from the initial discussions and first term of the project. At least two researchers attended most meetings, and in those cases one researcher was designated to take extensive notes. Data also includes teacher-researcher and researcher-researcher communication and project documents from this period.

Method of analysis

To begin with, we overviewed the central activities conducted by each collegial team and selected activities to achieve breadth regarding the three core processes. Symmetry and complementarity are characteristics of collaborative processes and cannot be identified using short excerpts from data without placing them in the context in which they are expressed. Therefore, the analysis required two main steps. In the first step, one or two researchers reviewed the available data and wrote a concise account of each activity, including the main events. These accounts were reviewed by the other authors to ensure descriptive validity (Maxwell, 1992/2002).

In the second step, symmetry and complementarity, as defined above, were used as analytic concepts (figure 1). For each account, presence of the three parts of symmetry (equal commitment, attention to own and other's needs, and attention to own and other's conditions) and of complementarity (joint contribution, utilisation of own and other's expertise,

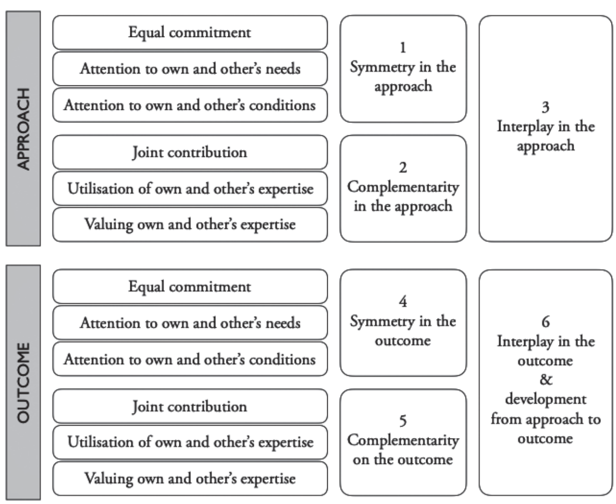

Figure 1. O*verview of second step of analysis and the six elements of the results*

and valuing own and other's expertise) were identified in the approach (1 and 2, figure 1) and the outcome (4 and 5, figure 1) of the activity. Whether different parts were intended to or did facilitate or hinder one another was also noted, resulting in a description of the interplay between symmetry and complementarity in the approach (3, figure 1) and the outcome (6, figure 1). For the outcome, a comparison with the approach was also made to reveal whether symmetry and complementarity increased or decreased throughout the activity (6, figure 1).

The analysis of each activity was done by one or two authors. To check for theoretical validity (Maxwell, 1992/2002), each analysis was critically reviewed by the other authors.

Results

The analyses of four activities are presented below, including one activity within each of the three core processes and a fourth activity describing two microlevel iterations of all three processes. For each case, we first present the brief account of the activity and then the results structured by the elements 1–6 (figure 1). That is, the analysis concerning symmetry, complementarity and the interplay between them are presented under the headlines "Approach" (elements 1 to 3) and under "Outcome" (elements 4 to 6). In these sections, the links to the definitions of symmetry and complementarity are in italics.

Exploration: formulating common goals

As the researchers applied for funding for developing and studying constructive design in collaborations with teachers, they used their existing networks to initiate discussions with schools and municipalities with either ongoing development projects or an interest in this line of work. The researchers offered funding for freeing up time for teachers' participation, and the intention was that the school and the acquired funding would each finance half of the teachers' time spent on the project. Before the funding was granted, the teachers and researchers had several meetings to explore research and development issues and to formulate project goals. One of the themes in these discussions was student motivation and beliefs in relation to problem-solving, as seen in notes from the joint meetings: "See effects on self-confidence in mathematics among students [...] Resolve the mathematics knot – how do we get away from a belief that mathematics is something only for the smart ones" (School A). "Students have problems with problems. Teachers refer to students not even trying, even though they could figure things out themselves. 'They don't try.' A lot of discussions among the teachers about many students focusing only on remembering algorithms" (School C). "Lack of confidence, stuck in not being able to, hard to trust that one can learn something one doesn't already know, confidence in one's own ability, not to give up" (School E, F, G).

Based on these discussions, the researchers drafted project goals aiming to find formulations common for several or all schools that were in line with the funding applications, as that would benefit the researchers by simplifying the coordination between schools. For the motivation and belief theme, the researchers formulated two goals for students' learning: "To, in relation to problem-solving, increase motivation and confidence in one's own ability" and "To develop both independence and collaborative skills in problem-solving". In addition, they included two specific goals for Schools A and D because they were unsure whether the common goals captured the same meaning: "That students feel more confident working outside the textbook and with problems" (School D) and "Strengthen students' motivation, positive beliefs about mathematics, self-confidence, and ability to reflect" (School A). The goals were then discussed with the seven collegial teams during half-day meetings organised by the research team. All schools agreed on the two common goals, but School D chose to keep their specific goal as well. Of 15 goals in total, all schools agreed on nine common goals, while six goals were agreed upon by 1–3 schools. In addition, the initial discussions resulted in established collaboration with all the contacted schools.

Approach

1. That both the teachers and researchers were willing to attend meetings and share experiences and suggestions, even before funding was granted, is seen as a sign of *equal commitmen*t. The purpose of the meetings was to explore the content and focus of the project together, *attending to each other's needs*. The researchers tried to take responsibility for guiding and documenting the exploration, *attending to differences in conditions* regarding time in the project.

2. Teachers and researchers were both invited to *contribute*, but in different ways: researchers intended to *utilise* teachers' experiences and suggestions shared verbally during meetings, and their own greater experience in writing for documentation.

3. Altogether, in approaching the activity, the intended interplay was that symmetry regarding attention to needs and conditions would be *facilitated* by complementary utilisation of participants' expertise.

Outcome

4. In the beginning of the project, teachers had very limited time for work and researchers *attended to these conditions* by taking a larger responsibility and a larger workload. All contacted schools initiated collaboration agreeing on common goals, thus signalling sustained *commitment* and *attention to researchers' needs*. In a few cases, teachers also kept specific goals, indicating that *teachers' needs were attended* to.

5. As intended, teachers and researchers jointly *contributed* to goal formulation: teachers by *utilising* their experience, and researchers by *utilising* their writing skills.

6. However, there are factors that call for caution in the evaluation of the symmetry and complementarity of the outcome, and consequently of how they *developed*, and whether they *facilitated* one another. First, the research team contacted several schools, but each school had contacts with only one research team. If there was a discrepancy in goals between the researchers and one of the schools, the researchers would be able to continue the project with the other schools, while it might be harder for the school to find another research team. Second, the researchers initiated contact with an offer of freeing time for teachers' developmental work. Participation in the project would allow teachers to spend more time reflecting on and discussing their teaching, regardless of whether the project focus was aligned with teachers' specific interests. These factors may have hidden lack of symmetry and complementarity, as teachers might have been unwilling to voice that their needs were not sufficiently attended to, that they were not given sufficient opportunity to contribute, or that their expertise was not sufficiently utilised or valued during goal formulation, believing that it might risk the collaboration, even though there were no explicit indications that this was the case.

Design: the teacher guide

The teachers at School A started early in using the teacher guide during their problem-solving lessons and they invited one of the researchers to observe their lessons. All quotations in the following are from the joint meetings.

To some extent, using the guide seemed to address the teachers' needs, and they observed a positive change in students' behaviour, as one of the teachers expressed: ”The students' questions have, to some extent, changed from 'I do not understand anything' to questions concerning more specific mathematical difficulties”. One of the teachers suggested the following explanation: ”They know what (diagnostic) questions to expect so they prepare an answer before asking, and by doing that they sometimes come up with an answer without my support". Using the guide furthermore resulted in teachers having more time to help the students who were in greater need of support because "There are significantly fewer hands in the air". Teachers also expressed notions like "Now I ask more exploratory questions and give less direct help".

However, using the guide was also challenging, and the teachers saw this as their own shortcomings rather than due to the complexity of diagnosing students' difficulties and finding appropriate feedback. As one teacher said: "I often fail to choose the right feedback to help my students solve math problems". Some of the teachers were anxious to live up to perceived expectations from the researchers and did not view their own experiences and insights as important for the project. In addition, both teachers and researchers were interested in discussing the lessons and the teacher-student interactions, but it was difficult to find the time to have such discussions. The teachers' schedules did not allow any room for dialogue between lessons, thus making it impossible to communicate valuable information for teachers to use in preparation for the next lesson. The researchers addressed this at a joint meeting by pointing out the value of sharing the insights on how the guide functioned in teachers' regular practice, and they reminded the teachers that the researchers were dependent on input from the teachers to develop the guide. Considerable joint meeting time was then set aside for the teachers and researchers to share their experiences from the lessons. During one of these meeting the teachers chose to show how they had adjusted the layout and content of the guide to better fit the classroom conditions, and they explained how and why they did this. Their suggestions for clarifications and additional feedback were also added to the teacher guide by the researchers, e.g. the question "What do you already know about [e.g. calculating areas]?" as feedback when the difficulty is in initiating exploration of a problem.

Approach

1. Initially, teachers and researchers were *both committed* to engage in the design and testing of the guide, but neither teachers nor researchers paid enough *attention to their common need* for dialogue after lessons or to the *conditions* regarding lack of time.

2. There was also a lack of complementarity in that the teachers initially kept their adaptations of the guide to themselves and did not *value their own expertise* in relation to the project. Therefore, neither the researchers' nor teachers' *expertise were utilised*, and the teachers were not able to use

researchers' experiences from the lessons and the researchers were not able to use the teachers' adaptations in the development of the guide.

3. When it comes to interplay, insufficient attention to the different conditions for teachers and researchers regarding time led to insufficient opportunities to utilise teachers' expertise in practical use of the guide. In approaching the development of the teacher guide, lack of symmetry thus *hindered* complementarity.

Outcome

4. In the outcome, researchers and teachers paid more *attention to the needs and conditions of both parties*, by allocating joint meeting time to issues that were important to teachers.

5. A positive outcome with respect to increased complementarity was observed when the participants *utilised one another's expertise*. In particular, insights from *teachers' expertise* regarding practical use of the teacher's guide, were utilised to develop the guide further.

6. The *increase* of symmetry, in terms of paying more attention to one another's conditions and needs *facilitated increased* complementarity, as the teachers were able to utilise the researchers' expertise to develop their skills on how to use the guide and the teachers' experience was utilised to develop the guide.

Evaluation: construction of a survey

In parallel to joint meetings and development work with the teachers, the researchers worked with methods of data collection within the research group. At the outset, the researchers did not involve the teachers in this work. However, the question of documentation of changes in practice was raised by the teachers at School D at a joint meeting after using the diagnostic questions from the guide for some time. At this meeting, one of the teachers said: "I was thinking about this part with the diagnostic questions. How do we document this? [...] we haven't written anything. Could it be as simple as at these meetings we take some notes: How does it feel now? Where are we now?" Another teacher then emphasised the value of consistency over time: "If you write things at random, then you might focus on different things. So, isn't it a good idea to have some questions to keep to?" At this point, methods for documentation were identified as a common need for both teachers and researchers, and the next joint meeting was devoted to devising a survey with understandable questions and reasonable response alternatives that was feasible to fill out every week. For example, one question was: "To what extent were the posed diagnosis questions helpful for me as a teacher?" with four response alternatives: "Very little", "Fairly little", "Fairly much", and "Very much". Analyses of the teacher responses were thereafter used as a means of reflection at the following joint meetings, and the researchers presented the survey to four other schools that also wanted to use it, thus giving researchers rough estimates of how well the diagnostic questions worked for different age groups.

Approach

1. When developing and devising methods for data collection, the researchers made assumptions about *their and the teachers' needs and conditions*: that the teachers had limited need for a survey and did not have time to engage in the development and analysis. Not *attending to the teachers' needs* thus hindered *equal commitment* to developing data collection methods.

2. The approach also did not allow for *joint contribution*; the teachers had no opportunity to *contribute* to the development of the methods, and the researchers had no opportunity to *utilise and value the teachers' expertise*.

3. In the approach to construction of surveys, lack of symmetry thus *hindered* complementarity.

Outcome

4. In the outcome, *attention was paid to the needs of both the teachers and researchers* in that both parties wanted to reveal possible changes in classroom practice. Both parties also believed that the survey data could be a good basis for discussions leading the project forward. When these common *needs were attended to*, the teachers and researchers *equally committed* to the development of the survey.

5. Both parties also *valued and utilized each other's expertise* in the joint work. Instead of simply filling out questionnaires, the teachers *contributed* to the construction of the questionnaire by complementing the researchers' perspective with knowledge of what questions were meaningful for teachers. Such questions are also preferable with respect to rigour because they can be argued to lead to greater validity.

6. Altogether, as a result of the teachers' initiative, commitment turned out to be more equal than was envisioned in the approach, and through attention to both parties' needs, better opportunities for joint contribution were created, and both parties' expertise was more valued and utilised. *Increased* symmetry thus *facilitated increased* complementarity.

Exploration, design and evaluation: problem-solving activities

The researchers' intention, when starting the project, was to limit the time spent on selecting problems in favour of working with the teacher guide. However, teachers at School C voiced a need for finding suitable problems, and therefore the joint meetings, including the activity described below, were split between working with problems and working with the guide. Over three meetings, two microlevel iterations of exploration, design, and evaluation were carried out, and tasks regarding both simultaneous and quadratic equations were developed. In the first meeting, one researcher gave a presentation on the difference between problems and routine tasks and qualities of problems aimed for learning of mathematical content. The teachers used this as input for discussing and developing a lesson where students were to guess the solutions to a number of simultaneous equations and then construct solutions by drawing graphs in the dynamic software GeoGebra. The intention was to make students see patterns and draw conclusions. The initial design was tested in a couple of classes, together with the guide, and experiences were brought back to the second joint meeting. When evaluating the initial design, the teachers described how many students did not look for patterns or try to draw conclusions across subtasks unless there was something unexpected in a task. Both teachers and researchers suggested revisions of the tasks, such as reducing the number of subtasks, including questions of the form "What if...?", and including reversed tasks where students are asked to formulate a pair of equations for a specific number of solutions. The revised tasks were then tried in a new class and discussed again at a third joint meeting. After revision, the tasks worked better and gave opportunities for reflection for students at different levels. Between the second and third meeting, the researchers summarised the joint work and formulated general ideas for developing and documenting problems.

Approach

1. The researchers *attended* to teachers' *conditions and needs* – seeing them as key and acknowledging that teachers had little time for finding suitable problems except for during joint meetings – by allocating time for developing problems, and the teachers *attended* to researchers' *wishes* to keep working with the guide in parallel with developing problems.

2. The work was planned to give room for *joint contribution* during joint meetings, *utilising* teachers' and researchers' *expertise* of practice and research respectively.

3. Both teachers and researchers can thereby be said to have approached this activity in a way where symmetry and complementarity was thus meant to *facilitate* each other.

Outcome

4. Both parties were equally *committed* to the activity, and they raised their own and *attended* to each other's *needs and conditions*. For example, teachers continued to use the teacher guide along with testing designed problems, and researchers provided input and engaged in discussions on the design of problem-solving activities.

5. The teachers *contributed* by developing and testing designs and sharing reflections, while the researchers *contributed* by giving presentations, giving input on design choices, and taking responsibility for documenting and summarising the process. The latter made it possible to draw more general conclusions that might aid future development of problems. Both parties' *expertise* was thus *valued* and *utilised* in the initial and revised design.

6. The symmetry and complementarity of the approach thus, in this case, seem to have continued to facilitate the symmetry and complementarity in the outcome. Teachers', as well as researchers', needs were attended to in the activity, and both parties showed commitment and contributed their expertise.

Discussion

In this article, we focus on symmetry and complementarity in the approach to and outcomes of four examples of activities in the initial phase of a design research project. The results demonstrate the complexity of TRC by describing both challenges and good examples, thus providing perspectives that are less common in previous research (Coburn & Penuel, 2016; Pareja Roblin et al., 2014; Wagner, 1997). We show how commitment, attention to needs and conditions, contribution, and utilisation and valuation of different expertise can interplay and develop in TRCs. The results highlight the potential of being mindful of these aspects in each core process of design research. By comparing the approach and the outcome of each activity, we find that initial asymmetric and non-complementary approaches can sometimes develop into symmetric and complementary outcomes. This was apparent when the researchers initially assumed that the teachers did not want to take part in the construction of the survey, but it turned out that the teachers regarded the surveys as an opportunity to document their progress. This process, from approach to outcome, is an example of how it is possible to move from one end to the other of Wagner's (1997) spectrum of understanding and purpose. In the final outcome, the teachers and researchers ended up with a better understanding of each other's practices. This opened up opportunities for improving the work the researchers had started and enabled greater rigour and validity through the documentation of progress in the project. When focusing on the core process of design, the lack of attention to differing conditions, especially regarding time, emerged as a constraint for other aspects of symmetry and complementarity. When the teachers and researchers did not devote sufficient time to discussing the design and content of the teacher guide or to meeting between lessons to exchange experiences, opportunities for mutual contribution to research and development were lost. But, by realigning the joint meeting time to better attend to teachers' needs, and by more clearly communicating how the teachers' experiences were valuable, both teachers' and researchers' needs could better be satisfied through more effective utilisation of both parties' contributions. These examples also reveal that even when symmetric and complementary intentions are explicitly voiced, it can be challenging for both teachers and researchers to act accordingly.

Furthermore, our examples show that a complementary approach risks hiding asymmetry, for example, regarding whose knowledge and learning is explicated in the project (Jaworski, 2003). During goal formulation, researchers had more time and therefore took more responsibility for planning and documentation, which was appreciated by teachers, but this resulted in researchers having greater influence on the definition of key components of the project, such as the project goals and the teacher guide. The project ran the risk of, in Coburn and Penuel's (2016) terms, becoming less "mutualistic", and naturally this also meant that the teachers' opportunities to complement the researchers' expertise and engage in joint contribution were restricted. Our results, however, show that it is possible to overcome such threats by emphasising the value of teachers' input, wishes, and needs and by devoting time to discussing teachers' experiences.

Perceived inequalities in terms of time available to engage in the project can be further problematised and described as different modes of work. Evidently, researchers spent more time theorising, while teachers spent more time practicing and in doing so gaining different and, if utilised, complementary insights. If efforts are made to bridge this divide between different modes of work, the collaboration can achieve symmetry and acknowledge the participants' complementary roles in that teachers and researchers will bring complementary resources to the project. The activity of developing problems through micro-level iterations constitutes one example thereof.

Our study shows promising results regarding sustained collaboration through honouring symmetry and complementarity, as a means for implementation of constructive design. Through cycles of exploration, design, and evaluation, the teachers and researchers collaboratively found functional designs for activities where students were engaged in problem-solving. Dedicating time and effort to aligning the activities with the current learning goals enhanced a constructive mathematics teaching design. By iterating micro-levels of the design, the students' difficulties became more manageable. With more manageable student difficulties during the problem-solving process, the risk of teachers feeling pressured to give the students a solution method, rather than supporting the students' own reasoning, was lowered.

In summary, we agree with Coburn and Penuel (2016) that there is a need for analysing the dynamics of TRCs of ongoing projects, and we find the notions of symmetry and complementarity to be a useful tool for this. Attending to symmetry and complementarity can also serve as a means for developing and sustaining collaboration – which is known to be difficult (Burkhardt & Schoenfeld, 2003) – since collaboration promotes mutual contribution and benefit.

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Note

1 The terms *symmetry* and *complementarity* are also used by the Umeå node of ULF, a national program commissioned by the Swedish government, aiming to develop sustainable models for collaboration between schools and universities concerning practice-based research.

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