Variation theory and teaching experiences as tools to generate knowledge about teaching and learning mathematics – the case of pre-service teachers

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The theory-practice divide in teacher education is commonly viewed as there are two separate entities – theory and practice. However, in practice-based research approaches, theory is commonly integrated with existing practical knowledge with the aim to deepen teachers' knowledge about practice or to create new knowledge. In this study, we examine 30 pre-service teachers taking part in a 5-week course in a teacher education program in Sweden, in which an action-research approach termed Learning study was used to deepen the pre-service teachers' thinking and reasoning about mathematics teaching in order to develop primary student learning. Variation theory was used as a tool to support the pre-service teachers' reflections on how different ways of structuring the mathematical content are related to student learning outcomes. This research aims to illustrate how the integration of theory and teaching experiences from the 5-week mathematics education course supported pre-service teachers' generation of knowledge about teaching and learning mathematics. In this study, we regard mathematical tasks created by the pre-service teachers and used in the lessons as generated knowledge about the practice of teaching. Data were collected during the course and consist of written reports about task refinements in the pre-service teachers' lessons. We identified five different ways of re-designing the tasks: expanding tasks, making tasks more explicit, making tasks less explicit, bringing metaphors and representations to the foreground, and creating new tasks.

The gap between theory and practice in teacher education is well known. The reasons for this gap have been repeatedly examined in a variety of studies over the years (e.g. Dewey, 1904; Goodson & Hargreaves, 1996; McGarr et al., 2017). Korthagen (2010) points out some of the most

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common reasons for the gap in his analysis, "The relationship between theory and practice in teacher education". For example, the divide can be explained from a socialization point of view – if novice teachers do not feel ready and well prepared to teach when they leave teacher education, the first year of teaching can be shocking. Under such circumstances, there is a risk that novice teachers abandon their theoretical insights about teaching and learning and adjust their teaching to norms prevailing in the schools or that they even develop a resistance to using theoretical knowledge in future teaching.

Other reasons for the gap, according to Korthagen's analysis, have to do with "the complexity of teaching" (p. 670) and different types of knowledge offered during teacher preparation. In teacher education, theories play an important role in preparing pre-service teachers (PS-Ts) for a variety of dynamic factors and challenges in the classroom and also play a role in helping PS-Ts to reflect on teaching in terms of what they have done, what they are doing, and what they are going to do. However, Sugrue (1997) claims that lived experiences of teaching and learning from previous school days strongly effect decisions about teaching. If the PS-Ts have not encountered concrete teaching problems in their training, they may regard theories as irrelevant to learn or use, and they therefore tend to teach in a way they were taught years ago. Furthermore, it seems as though formal knowledge (theories and knowledge produced by researchers) is more often presented and more in focus in teacher education than is practical knowledge (situated and action-guiding). Under such circumstances, the education in itself contributes, to a high degree, to the theory-practice gap (Korthagen, 2010; Loughran, 2006).

In attempts to promote the relationship between theory and practice in teacher education, but perhaps above all to support PS-Ts' developing their practical knowledge, a number of practice-based research approaches that foreground "teachers as researchers" instead of "teachers as end users of research" have become more common during recent decades. Such approaches are, for instance, action research (e.g. Ax et al., 2006), lesson study (e.g. Huang et al., 2019), and learning study (Davis & Dunnill, 2008). Even though the different approaches stem from diverse research traditions, they all can be explained as collaborative and iterative teacher-driven inquiry that seeks to develop practice but also to generate knowledge about teaching and learning (Elliott, 2012; Morris & Hiebert, 2011; Runesson & Gustafsson, 2012). A common feature in these approaches is that the inquiry starts with a question that emerged from a problem teachers confront in their practice and that they want to better understand. Based on this question or identified problem, teachers in these approaches jointly plan, act, and revise their teaching actions in a number of cycles (sometimes in collaboration with researchers or facilitators from the university). This process of planning, acting, and revising creates a model in which repeated and critical reflections on actions and, commonly, on student learning, play a central role. In order to provide answers to their questions, the teachers themselves collect and analyze data from different sources (e.g. observations, written tests, filmed lessons, or interviews) and various points in time. The collected data serve as the basis for reviewing previous actions and for making pedagogical decisions for new actions in recurring cycles.

Even though it has become clear that collaborative and teacher-driven methodological approaches are important for teachers' professional development and for student learning, there are problems known to be associated with the theory-practice divide. It has been argued that when teaching is not seen as a theoretically informed activity, it can be reduced to an accumulation of experience in terms of imitation or sharing best practices. (e.g. Elliott, 2012; McMahon et al., 2015; Stenhouse, 1981). For instance, McMahon and colleagues claim that "a strong theoretical base is the foundation of decisions about the learning needs of individual pupils and groups of pupils and the ways of creating contexts in which learners can flourish" (2015, p. 173). In a similar way, Nuthall identifies the need for a theoretical base in practice-based research approaches, as he concludes that teachers only occasionally understand the relationship between the quality of teaching and their students' learning. To bridge this gap, he proposes that teachers "require an explanatory theory of how different ways of managing the classroom and creating activities are related to student learning outcomes" (Nuthall, 2004, p. 276).

Even though it has been shown that learning theories may be hard for PS-Ts to apply in practice (e.g. Korthagen, 2010; Sugrue, 1997), it has been demonstrated to be insufficient to regard theories and research results as just something to apply in practice – as a solution for filling or bridging the "gap". For instance, instead of viewing formal and practical knowledge as different entities, Lampert (2010) concludes that it is more helpful to see theory as a tool that allows teachers to deepen their thinking and reasoning about practice or to integrate theory with existing practical knowledge in order to generate new knowledge. The present study is related to this conclusion. It aims to illustrate how the integration of theory and teaching experiences drawn from a 5-week mathematics education course, building on a practice-based research approach termed Learning study (LS), can support PS-Ts to generate knowledge about teaching and learning mathematics.

Learning study in teacher education

Learning study is a teacher-driven, collaborative and iterative inquiry which has much in common with the other practice-based research approaches previously mentioned. A difference, though, is that the cycles involving joint activities such as lesson planning, lesson analysis, and lesson revisions, are framed within a theory of learning, commonly variation theory (Marton, 2015). The theory can be seen as an explanatory tool (Nuthall, 2004) for understanding and analyzing the relationship between teaching and learning and for making decisions about teaching from a content-specific view. As variation theory focuses on the content, the relationship implies that what is made possible to learn depends on how the content is handled in a lesson (Kullberg et al., 2017).

Since the introduction of the LS approach 20 years ago, it appears that it has come to play a valuable role in bridging the gap between theory and practice for supporting professional development, not just for in-service teachers but also for PS-Ts, as the approach – or modifications of it – is an integral part of teacher education around the world. For instance, in Hong Kong, the Institute of Education has incorporated LS in all of its primary school teacher education programs (Lai & Lo-Fu, 2013). The worldwide spread of the approach can also be seen as a response to the challenges of theory and practice in teacher education, as a specific theory is used, in an iterative process, for systematic reflection on the relationship between classroom activities and student learning. Besides using theory as a tool for reflection in LS, teachers also practice the theory in real situations, as they conduct lessons based on theoretical concepts (this is further explained in the next two sections).

Recently, there have been a number of studies reporting on experiences of implementations of LS in initial teacher education. Most of these studies are based on self-reported reflections and/or interviews focusing on the PS-Ts' experiences and learning from participating in an LS. Based on 18 PS-Ts' experiences, Durden (2018) identified five qualitatively different conceptions about the significance of LS; for example, it improves lessons by following a process, and, through conceptual change, it transforms school students' understanding. Even though there are different views of what LS is, there are reasons for expecting LS to offer rich opportunities for PS-Ts to understand the complexity of the relationship between teaching and learning (Davies & Dunnill, 2008; Turu, 2017; Wood, 2013). For example, self-response questionnaires from 475 PS-Ts in Cheng's (2014) study indicate that the respondents tended to agree that the LS course had helped them to develop deeper understanding of how to prepare and improve lessons, and furthermore, to identify students' different ways of understanding. The results from these studies and those of Ko (2012), and Royea and Nicol (2019) suggest that it is possible to learn the most crucial principles of variation theory even in courses that lasted only a few weeks. Turu (2017) points out that variation theory helps the PS-Ts to shift their focus from the "act of teaching" to the "agent of teaching", which means that they start to pay more attention to how students in school learn rather than how to fill lessons with relevant activities. Royea and Nicol found that the PS-Ts in their study tended to emphasize that variation theory stands out in comparison with other theories, as they perceived it to be directly applicable to teaching. Other formal or influential learning theories, which they were more familiar with, were regarded as "philosophy disconnected from teaching practice" (Royea & Nicol, 2019, p. 13).

Even though it can be concluded that LS has the potential to support PS-Ts' use of theory to make decisions about practice, there are also known challenges with the approach. For instance, it can be difficult to distinguish between variation theory and the LS approach (Royea & Nicol, 2019), and it seems easier to focus on activities and methods than on how to use concepts from the theory to structure the content (Davies & Dunnill, 2008; Turu, 2017). Furthermore, all the studies about LS in teacher education reported on here show that initially in the courses, the PS-Ts regarded variation theory to be confusing and hard to learn and use as a tool for planning and analyzing lessons and student learning. However, it seems that, to some extent, these difficulties are transient, as some of the studies describe that the PS-Ts found the theory to be more valuable at the end of the courses, and that, during the courses, they developed theoretical insights. Other challenges have to do with the implementation of LS in teacher education. As an intervention cycle is time- consuming, the courses are commonly built on modifications of a regular LS arrangement with fewer cycles (Royea & Nicol, 2019), fewer students to teach (Turu, 2017), or fewer opportunities to teach real classes (Wood, 2013). It has also been shown that it is important for teacher educators to consider the lecture design (Brante et al., 2015) and how the collaboration between the PS-Ts works (Ko, 2012; Turu, 2017), as these seem to be crucial for the PS-Ts' learning the theoretical concepts, and hence, for their developing practical knowledge in such areas as preparing and improving lessons.

Although many benefits of and promising results from using LS in teacher education have been reported, a more explicit focus on how PS-Ts employ variation theory as a design and analytical tool for developing their practical knowledge is called for (Royea & Nicol, 2019; Larssen et al., 2018; Wood, 2013). To examine this, our study aims to illustrate how the integration of theory and teaching experiences, drawn from a 5-week mathematics education course building on LS cycles, can support PS-Ts' generation of knowledge about teaching and learning mathematics. One question guided this study: In what different ways do the PS-Ts redesign mathematical tasks when using variation theory as a tool for reflecting on their planned and taught lessons? The next two sections clarify the theoretical concepts used by the PS-Ts and the context in which the study was conducted, that is, in the LS-designed mathematics education course.

Variation theory – a tool for lesson design and re-design

Variation theory states that learning is always directed towards something to be learned. "The key point is that one cannot simply experience without experience something. Similarly, one cannot think without thinking about something, nor can one learn without learning something" (Lo et al., 2005, p. 24). In the context of teaching, this "something" is commonly a particular learning goal, termed the object of learning.

From a variation theory perspective, learning implies a person's becoming able to see something (the object of learning) in a new, more qualitative way by experiencing aspects that the person had not yet discerned but needs to discern – the critical aspects (Marton & Booth, 1997). Consequently, the object of learning can be defined as an answer to the question "What is to be learned?" For any given object of learning, what the students must learn does – of course – follow from disciplinary features of the concepts dealt with, that is, their properties and characteristics. But it should also be noted that it follows from the students' different understandings of the concepts (Lo et al, 2005; Pang & Ki, 2016). One way of using variation theory as a design tool is, thus, to identify the critical aspects relative the one's own students and, based on that knowledge, to plan and create learning situations. Analyses of observed lessons and pre- and posttests can give teachers clues about what is critical. But as the critical aspects are dynamic and relative to the students, teachers' conceptualizations of critical aspects may emerge, through the inquiry activities, from a "list" of features of mathematics known in advance to be a more detailed and pedagogically powerful answer to the question "What is to be learned?" (Mårtensson, 2019).

The variation theory of learning states that rather than telling the students the critical aspects, the aspects must be structured in terms of patterns of variation and invariance to be discerned (Pang & Marton, 2003; Marton, 2015). Watson and Mason suggest that the theory can serve as a tool to support teachers in selecting and designing tasks or learning situations that invite the students to discern the critical aspects. They claim that "tasks that carefully display constrained variation are generally likely to result in progress in ways that unstructured sets of tasks do not" (Watson & Mason, 2006, p. 91). Imagine a task designed to improve students' conceptions of angles. If the task highlights the difference between two straight lines meeting in a common end point and two similar (invariant) lines not having a common end point, it is most likely that the common end point of the angle will be discerned. On the other hand, if the end point and the rays are invariant, but the degrees of the angles are varied, it is possible to learn that there are different types of angles (acute, right, obtuse, straight, reflex, or full rotation). Over and above considerations about which aspect or aspects of a concept must be varied against a background of invariance, micro-level considerations about what structure and regularities are most likely to be exploited and identified may also be required when designing mathematics tasks (Watson & Mason, 2006).

Learning study as design of the mathematics education course

Since 2013, LS has been incorporated as an integral part of a mathematics education course in two primary teacher education programs (Grades 1–3 and Grades 4–6) at Jönköping University in Sweden. In this course, the variation theory of learning (Marton & Both, 1997; Marton, 2015) was used as a tool to frame the inquiry activities (plan-teach-analyze-revise) and to thereby empower the PS-Ts' knowledge about teaching and student learning. This mathematics education course at Jönköping University is the last of four 5-week courses (one single course is 7.5 credits) in mathematics education. In their third course, the PS-Ts come across variation theory for the first time when they undertake a lesson-analysis task. Therefore, they have limited experience of variation theory when they start their last course, and commonly none of them have prior experience of LS. The course design consists of two intervention cycles closely following the LS steps of planning, teaching, analyzing, and revising lessons set out by Lo ey al. (2005), and Pang and Marton (2003). In helping the groups through the two cycles of LS, we provide them with a schedule guide (explaining the different steps of LS). Additionally, seminars, lectures, and tutorial meetings are included in the design to give close guidance and feedback throughout the whole 5-week process.

During the first week, the LS procedure and variation theory principles are introduced in two lectures. To support the PS-Ts' understanding of variation theory, the lecture designs follow the suggestions made by Brante et al. (2015) that it is important to use several examples to illustrate the difference between critical aspects and general aspects of the content to be taught. In a "conceptual analysis" (Thompson, 2008) seminar and on the basis of variation theory principles, literature, and articles from journals about teaching and learning elementary mathematics and the standards in the Swedish curriculum for compulsory school (Skolverket, 2019), groups of PS-Ts (3–4 in each group) explore a given mathematics topic to gain knowledge about common misconceptions and important ideas related to that topic. The given topics are selected by teachers from our partner schools, which, besides selecting topics, provide opportunities for the PS-Ts to teach primary students. Each group formulates objects of learning for their lessons, for instance, *compare and order units of length*. At the end of the first week, each group designs a pretest for their students, based on the ideas from the conceptual analysis. For the purpose of making instructional decisions, the primary reason for collecting data on student understandings is to capture what may be critical for learning. Another reason for collecting the data, for use in a later part of the course, is to systematically evaluate the relationship between the lessons and student learning outcomes by juxtaposing pre- and posttests.

During the second week, the pretests are administrated by the PS-Ts to their classes (each group teaches two classes). Furthermore, the first lesson is planned based on a pretest analysis and variation theory principles. For each sequence of the lesson, each group is asked to design tasks that could address the critical aspects and to clearly explain the patterns of variation they will use. To emphasize the importance of variation theory as a design tool when planning lessons, the concepts of variation theory are examined in a seminar. During the third week, the groups give their first lessons (lesson 1). Normally, two of the PS-Ts in each group teach the lesson. The other group members observe the lesson and collect information on instruction and student learning by busing an observation protocol. After the lesson, each group evaluates their lesson and student learning by examining the protocols and worksheets. We encourage the PS-Ts to play close attention to whether the patterns of variation and invariance used in the tasks are appropriate and if not, how the tasks could be adjusted to make students experience variation corresponding to the critical aspects and whether there are other critical aspects. Using the results of this evaluation, they refine their lesson plan. Also, to support the PS-Ts in the evaluation and revision process, we discuss, in a seminar, how communication and choosing types of tasks according to the level of cognitive demands (Skott et al., 2010) are essential parts of mathematics education.

In Cycle 2, which is conducted during the fourth week, the groups teach the revised lesson (lesson 2) to their second class at their partner school. This normally means that the PS-Ts in each group switch roles so that those who previously taught now observe, and vice versa. Each group also administers a posttest in their two classes in order to carry out the systematic evaluation previously mentioned. As in Cycle 1, the evaluation is also based on observation protocols and reflections about the critical aspects and the patterns of variation. However, the second lesson plan is not revised, but the PS-Ts are required to submit an individual report about reflections on their experiences of teaching, student learning outcomes, and further improvements of the lesson. Furthermore, each group compiles the results from the two cycles, which they present and disseminate in an examination seminar during the last week.

Data collection and data analysis

In 2018, there were 40 PS-Ts enrolled in the mathematics education course, and of these, 30 gave their written permission for us to use their final reports for research purposes. All the participants were in their fourth year of the initial teacher education program for primary school (Grades 4–6). In their final reports they were supposed to individually describe how to improve the lessons by reflecting on the teaching in lessons 1 and 2, and furthermore, on the students' performance during the lessons and on the written posttests. According to the instruction, they were also asked to argue for their revisions of the lesson plans based on variation theory principles and to present concrete examples of how to re-design previously used tasks and/or to add new tasks, if needed. Whereas the PS-Ts had worked in groups, in some case the original designed task reflected on could be the same, but the re-design of that task differed in the individual reports. Each report consisted of three to four written pages. Data were collected from the written final reports on eight different objects of learning. In total there were 64 mathematics tasks associated with arguments for the revisions made. Over and above the fact that the PS-Ts were asked to refine the mathematics tasks in the instruction of the report, the rationale for selecting mathematics tasks is twofold. Firstly, tasks play a crucial role in enabling us to identify how variation theory is employed by the PS-Ts, as the theory is supposed to serve as a tool for design and re-design. Secondly, "tasks do not in themselves generate learning" (Watson & Mason, 2007, p. 207), hence there may be pedagogical reasons beyond patterns of variation that may impact task re-design.

In order to understand in what different ways the PS-Ts re-designed the mathematical tasks, we employed a qualitative data analysis following several steps (Denscombe, 2017). In the first step of analysis, the authors of this paper repeatedly read the PS-Ts' reports individually, marking what we interpreted as suggestions of changes to tasks. The question

”How does the task change, and on what basis?” was used to maintain focus on the research question.

In the second step, each example of changes was scrutinized collectively. The suggestions of task re-design in the reports which were not argued for in relation to the lesson plan 1 or 2 were weeded out. However, some tasks in the reports were not modified from lesson plan 1 or 2 but were still included in the analysis as they were designed based on identifications of *new* critical aspects. In some reports, a single task was selected, whereas in other reports several tasks were selected for analysis.

In the third step, excerpts with examples of tasks from the reports were selected. These examples were compared to each other, and similar examples were grouped together. This enabled us to identify different categories, that is, different ways of re-designing a previously used task or of creating new tasks. In the last step of the analysis, we worked together to describe the characteristics of each category.

Results

The analysis indicates that the PS-Ts re-designed mathematical tasks in five different ways – *expanding tasks*, *making tasks more explicit*, *making tasks less explicit*, *bringing metaphors and representations to the foreground* and *creating new tasks* – when employing variation theory as a design and re-design tool during the mathematics education course. These different ways are presented as five categories below. In all categories except making tasks less explicit, the tasks were adjusted in relation to student learning outcomes and task implementation.

Expanding tasks

Expanding tasks was a result of the PS-Ts' new and more distinct insights into what the students must learn, gained through the process of lesson analysis and their reflections on student understanding. Depending on the nature of how the new insights were incorporated into the tasks, the tasks were expanded in two different ways: 1) by adding new examples to the original sets of examples, and 2) by broadening the set of numbers or units used in the task. To illustrate the first way of expanding tasks, the following case is given: One group regarded unit squares as critical for enabling students to understand the concept of area and to find the area for different two-dimensional shapes. Therefore, in the first two lessons, the group designed and used a task consisting of three examples highlighting how different shapes have area, even though the unit square does not cover the entire shape (figure 1).

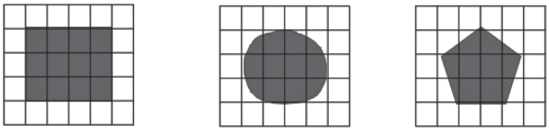


Figure 1. *The pattern of variation in the task consists of different shapes and an invariant unit square*

When planning the third lesson, one PS-T reconsidered the critical aspect as it seemed as though the students had discerned that a unit square can be used to estimate the area of plane shapes, but the level of demand was too low. Therefore, she argued that discernment of different sizes of unit squares is required to more fully understand the concept of area. To create opportunities for students to compare different units, a new set of examples (figure 2) was added to the first version of the task. Moreover, prompts that would call on students to make generalizations about finding the area were also included. For instance, "Why is the area of the same rectangle 12 and 6?"

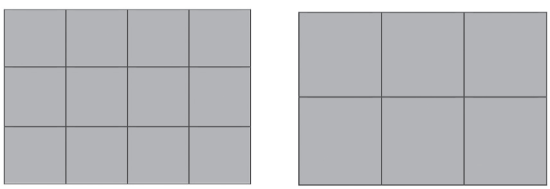


Figure 2. *The sequence added in the re-designed version consists of two shaded rectangles with identical size (invariant), but the size of the unit squares within each rectangle varied*

The second way of expanding tasks means that new insights about what students must learn were used to modify the original task, as in the example above. The most significant difference, though, is that only the range of numbers, algebra symbols, or units (such as length units) was increased within the original example – a new sequence of examples was not added. For instance, one group wanted their students to learn how to order negative numbers on a number line and to tell which numbers are greater or lesser. Discerning the symmetry of positive and negative numbers was considered to be a critical aspect related to that goal. In lessons 1 and 2, the PS-Ts used a number line (the number line at the top in figure 3) and asked the students to explain where the integers between 5 and -5 are located and to describe pairs of opposite numbers (e.g. 2 and -2, and 5 and -5), focusing on the distance from zero. Based on the analysis of students' low performances on the exit-note in lesson 2, one PS-T reconsidered the critical aspect and argued for the necessity of broadening the range of numbers to open up the possibility for students to pay attention not just to the symmetry of positive and negative numbers between 5 and -5 in the specific example, but also to the general structure of opposite numbers. Pairs of opposite numbers beyond 5 in each direction were therefore selected and added to the new task (figure 3), for instance 10,000 and –10,000.

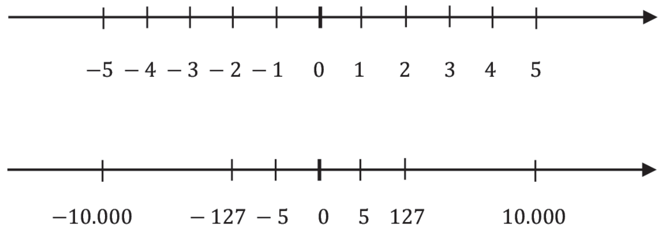


Figure 3. *In the original task (the number line at the top), the range of numbers was between -5 to 5. In the re-designed task beneath, the number range were broadened to -10,000 to 10,000*[1] *to make it possible for students to see the general structure*

Making tasks more explicit

Making tasks more explicit is fundamentally an effect of taking poor learning outcomes as a point of departure for reflections about task improvement and how the task can provide better opportunities for learning. There were PS-Ts who made the tasks more explicit by eliminating distractions to help the students pay more attention to the core of the task. This means that irrelevant features such as pictures, colors, and tables were removed from the original task or that the extent of the specific mathematical content, such as the range of numbers, was limited. In some cases, the only adjustment in this category was that specific prompts were added to the original task. This is not to say that the task was expanded by adding new features of mathematics, as in the category expanding tasks. On the contrary, the prompts were added exclusively to make the task clearer in a way that may help students focus on the most important parts. One such example is shown in figure 4, below.

The task shown in figure 4 was designed to highlight similarities (the growing structure) and differences (each figure in the example increases by 2 and 3 squares, respectively) between the examples. In addition to discuss the growing structure of the patterns, the students were supposed to express each figure numerically (e.g. figure 2 = 2 x 2) and to predict the number of squares that would comprise the hypothetical figures 5, 20, 45, and 150. In one report, the PS-T did not deem it sufficient to let the

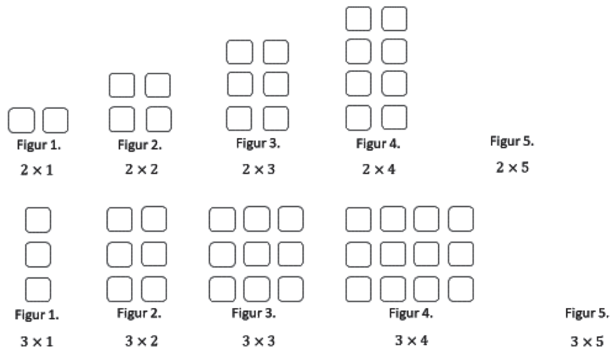


Figure 4. *The same examples were used in the original and in the new task. The task was made more explicit by the inclusion of specific prompts*

students just answer how many squares there are in each figure; he wanted to encourage the students to discern the relationship between the changing figures to predict the next figure. He did not change the pattern of variation used in the original version of the task, both patterns are growing (invariant) and the ways in which they increase varies. However, he did not exclusively relay to the pattern of variation and invariance, as he argued that specific prompts are required to make it easier for students to discern the relationship. Therefore, the task was adjusted to include the following prompts: "Can we use the differences between the figures to predict the figures 20, 45, and 150?" and "Why are there 40 squares in figure 20 in the first pattern? Please explain!"

Making tasks less explicit

In some cases, even though the rationale proposed by the PS-Ts for their task adjustments was to make it easier for students to learn, tasks were not re-designed based on careful consideration of how much scaffolding might be needed or how implementation of the task can be improved. Instead, variation theory was, to a great extent, mechanically used for making decisions about modifications. Given this, the new version of the task was made less explicit or more diffuse than the original one, primarily due to that the possibilities for directing students' attention to a specific critical aspect were limited. One such example is taken from a report about teaching the concept of angles. In the two previous lessons, the task in figure 5 was used to highlight that the angle mark (different sizes) is not crucial for determining the size of an angle.

In the modified task, the pattern of variation, in which an individual aspect varied (different sizes of the angle mark), was abandoned on the

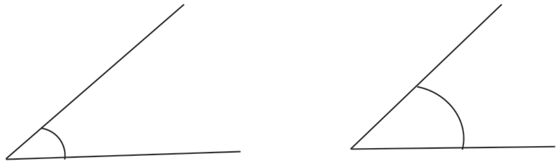


Figure 5. O*ne single aspect – the angle mark – varied in the original task*

basis of the argument that ”fusion is a pattern of variation that provides the opportunity for students to discern more”. The contrast between the sizes of the angle marks was therefore replaced with a pattern of variation in which several aspects varied – the angle mark, the lengths of the rays, the figures, and the sizes of the angles (figure 6).

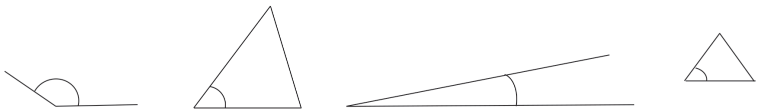


Figure 6. *Several aspects varied in the new task*

The reason for the change, according to the PS-T, was to create a more complex task to deepen the students' understanding of the concept. But as the new example was not added to the first version of the task (lesson 1 and 2), the angle mark consequently became more peripheral, and hence, from a variation theory perspective more difficult for students to discern than it had been before.

Bringing metaphors and representations to the foreground

In this category, the re-design is characterized by changes which are not explicitly linked to variation of the mathematical content but rather to variation of metaphors or representations. The rationale for these adjustments was to deepen the students' knowledge of specific concepts, since the PS-Ts noticed that many students had not learned what was intended in lessons 1 and 2. For instance, in one case, the group used a thermometer to illustrate negative numbers, but in one report it was suggested that the metaphors should vary, such as ”above or below sea level”, ”assets and liabilities” and ”credit card credits and debits”. One PS-T argued that this way of re-designing the original task may therefore offer better possibilities for deepening the students' understanding of negative numbers. In another report, the purpose of the adjustment was to enable the students to relate negative numbers to different contexts and situations. In another case, a seesaw with the same number of children on each side was suggested as a metaphor for equations.

There were several examples in the reports of suggestions to reformulate tasks by bringing representations to the foreground. One example of adjusting the task by adding representations was identified in a report on *comparing and ordering units of length*. The original task was presented in a table format to draw the students' attention to the procedure of converting numbers and units (graphically represented in figure 7), whereas the proposed adjusted task brought different representations to the foreground as students were asked to express length conversion in concrete, symbolic, and verbal forms in the four-fields schema shown in figure 7 (McIntosh, 2008).

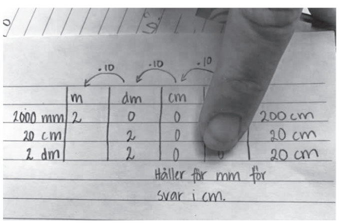
Concrete To measure and cut off pieces of strings according to 2000 *mm*, 20 *cm* and 2 *dm*.	**Symbolic** To write the relationships between 2000 *mm*, 20 *cm*, and 2 *dm* with symbols.
Graphic 	**Verbal** To describe with words the relationships between 2000 *mm*, 200 *cm*, and 2 *dm*. For instance, "I know that… therefore…." and "I also know that..."

Figure 7. *The adjusted task enabled the students to express their knowledge of length using several representations*

Asking the students to compare and convert different length units by using different representations would, according to the PS-T, contribute to students' ability to reason about length. Therefore, the proposed adjustment of the task was to offer variation in terms of the same lengths (invariant) being expressed by various representations (concrete, symbolic, and verbal).

Creating new tasks

In previous categories, the adjustments were made by reducing or expanding the tasks in different ways in relation to one or several critical aspects. In the written reports, we also identified PS-Ts' arguments for including new tasks in lesson 3 because they had identified new critical aspects when analyzing lesson 2 and/or student learning outcomes. These tasks were carefully planned, using principles from variation theory. For instance, in one report on making distinctions between area and circumference, it was proposed that the students must discern that length and area have different dimensions. In the report, the PS-T wanted to emphasize the differences between units in one dimension (circumference) and two dimensions (area) and how the units – centimeters and square centimeters – are expressed in symbolic form using the invisible number 1 (in m1) and the visible number 2 (in m2). In the new task, the PS-T suggested a sequence of examples in which the students were asked to juxtapose and discuss the following pairs of expressions:

a cm^1 and cm^2

b m^1 and m^2

c dm^1 and cm^1

The pairs of expressions were offered simultaneously, thereby providing the learners the opportunity to discern the differences and similarities between the expressions. In the first two pairs, the dimensions of the circumference and area varied, whereas the unit, were invariant in each pair (*cm* in a and *m* in b). In the last example, the dimension was the same while the units of length varied (dm and cm). The PS-T argued that the task thereby made visible for the learners the differences between the written form of area and length as well as the different dimensions.

Another PS-T, dealing with methods for solving equations, had identified in his taught lessons and in the posttest that it is not enough for students to solve an equation formally, they must also understand when a formal method for solving equations can be more useful than an informal method. The PS-T proposed the task in figure 8.

$$x + 3 = 5$$
$$25 + x = 37$$
$$x + 57 = 121$$
$$x - 57 = 121$$

Figure 8. *The design of the new task included two equations that students could solve with an informal method, followed by two equations that provided a formal method*

The PS-T argued that when solving for *x* in the first two equations (figure 8), informal methods (e.g. we are asked for a number such that when we add 3 to it, we get 5) and formal methods (e.g. we subtract 3 from each side) must vary. Moreover, by presenting the next two examples using a higher number range, the task should cause students to detect that solving for x is not as quickly done as in the first two examples, and therefore they need to learn a formal method. The only variation between the two equations $x + 57 = 121$ and $x - 57 = 121$ is the operator signs. The rationale for this was to draw the students' attention to the procedure of doing the same operation on each side, starting with subtraction in the first equation and addition in the second equation.

Discussion

The course design reported on in this study consisted of repeated activities in which theory was used with the aim of supporting the PS-Ts' decisions about practice. In alignment with other studies on LS in teacher education (e.g. Cheng, 2014; Royea & Nicol, 2019; Wood, 2013), this study also shows that it is quite possible, in a limited time, not just to learn the most crucial principles of variation theory but also to develop a deeper understanding of these principles. However, these other studies have been mainly limited to PS-Ts' lived experiences of the joint process of planning, evaluating, and revising lessons using variation theory. Therefore, the main contribution of this paper is not *that* theory may assist in preparing and improving lessons from the PS-Ts' point of view, it is the examples it provides of *how* variation theory and teaching experiences supported the PS-Ts in generating knowledge about teaching and learning mathematics.

The analysis showed five different ways in which this was done – expanding tasks, making tasks more explicit, making tasks less explicit, bringing metaphors and representations to the foreground, and creating new tasks. Our data provide evidence that variation theory was not being employed in a strictly technical manner in terms of an exclusive focus on how to create and modify patterns of variation and invariance. On the contrary, adjustments were, in most cases, based on a sensitivity to students' understanding and the realization of instructional purposes in the classroom. Many examples show that the PS-Ts increased their demands on students as tasks were redesigned to encourage the students to explain and reason about a relationship or a concept, to find patterns, or to compare and use different methods, even though, from a variation theory perspective, the original task offered students the opportunity to learn what was initially intended. On this basis, our interpretation is that the theory was not just used for developing practice or filling the "gap". Instead, theory guided practice and practice guided theory – the theoretical tool was used for planning lessons and analyzing student learning outcomes and, as these activities were situated in real teaching situations and real problems were confronted, practice seemed to change the PS-Ts' intentions about what the students must know, understand, and be able to do. The fact that both theory (variation theory but also mathematics education research used in the LS-process) and teaching experiences frequently influenced the new tasks can be seen as an example of Lampert's (2010) conclusion that the integration of theory and practice plays an important role in generating new knowledge.

Of course, we are not arguing that all tasks were adjusted with a delicate touch informed by both theory and practice. In the category of making the tasks less explicit, it is almost impossible to talk about refinements, as the changes operated in a way that made the critical aspect peripheral, due to a simultaneous variation of several aspects. This is not to say that the experience of simultaneous variation is not relevant. On the contrary, from a variation theory perspective, simultaneously taking into account the relationships between critical aspects is particularly relevant for learning. However, a necessary condition for learning is that individual critical aspects should also be varied separately (Marton, 2015). Our suggestion, though, is that the examples in this category should preferably be used in courses building on LS in teacher education to make discernable the lack of taking practice (in terms of student learning) into account. The different categories found in this study might also become a tool for reflecting about task design and redesign in the planning and evaluating of lessons in school-based studies. Instead of rejecting a task and trying a new one in the next lesson, PS-Ts might reflect on why the task did not work and if it would be appropriate, for instance, to expand the task or to make it more explicit.

A limitation of the study that should be addressed, though, concerns the unit of analysis – single mathematical tasks separated from other instructions or activities in the lesson. Due to the nature of such a sample choice, this study does not make any claims about whether the lesson plans and the quality of teaching were improved. It is of utmost importance to emphasize here that teaching mathematics is not about creating single tasks based on patterns of variation without carefully considering how connections between different tasks and activities can support student learning in the classroom.

The findings presented here could contribute to reflections and discussions about course design in teacher education. We strongly believe that what is required is changes in course design that are more fundamental than simply learning theories with the aim to apply them in practice. What is needed to bridge the gap between theory and practice is a design in which theory is not disconnected from the practice of teaching.

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Note

1 When the number range were broadened, disproportionate intervals between numbers were used.

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