Tasks, tools, and mediated actions – promoting collective theoretical work on algebraic expressions

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The aim of this article is to exemplify and discuss what teachers using learning activity need to consider when planning and supporting students' collective theoretical work on algebraic expressions. Data are from two iteratively developed research lessons in two grade 7 classes. The analysis focuses on students' tool-mediated actions, the mathematical content processed, how the content is dealt with, and on identifying the crucial aspects that enable collective theoretical work. The result provides examples of how the content of the task, its design, and its tools, as well as the teacher's and students' tool-mediated actions are crucial factors in the promotion of collective theoretical work.

Today's mathematics educational research emphasises that students must be given the opportunity to develop mathematical thinking. This makes students' theoretical thinking or conceptual understanding a central goal in today's mathematics teaching. In relation to this, whole-class discussions focusing on students' reasoning, argumentations, and problem solving, are considered a means for such development. However, whole-class teaching is challenging, since the teacher must not only invite the students to explore mathematics content collectively but must also create conditions that focus on and maintain creative and collective reasoning (Emanuelsson & Sahlström, 2008; Ryve et al., 2013). In the research literature, there are also noteworthy examples of how to orchestrate whole-class discussions that are student-centred and targeted to advance important mathematical ideas (e.g. Larsson, 2015; Larsson & Ryve, 2011; Stein et al., 2008; Taflin, 2007). In a research project, Larsson and her colleagues

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found that the use of mediating tools, like tables and graphs presented on the blackboard, helped students to focus on the content in whole-class discussions (Larsson, 2015; Larsson & Ryve, 2011; Ryve et al., 2013; see also Sfard, 2008). Principles illuminated in these studies relate to whole-class discussions as a consequence of students' work, often done in pairs or small groups. Viewing whole-class discussion as a phase that follows students' work on a problem is apparently common (Stein et al., 2008). Using whole-class, tool-mediated discussions as a means of helping students to review their own and others' arguments and solutions collectively, especially when using mediating tools, can thus be regarded as a promising teaching strategy (Larsson, 2015; Larsson & Ryve, 2011; Ryve et al., 2013; Liljedahl, 2016). Vygotsky (1978) argues that students develop theoretical thinking or conceptual understanding in a zone of proximal development (ZPD) and that higher order thinking occurs first on a social level, in a social activity (as an interpsychological activity) and later on an individual level, in the form of independent thinking (as an intrapsychological activity). This provides a strong motive for organising the teaching of theoretical concepts as a collective problem-solving activity (Arievich, 2017; Vygotsky, 1978). Viewing whole-class teaching as a collective – or joint – activity, it is possible to take advantage of the idea that theoretical thinking first emerges socially in a collective zone of development where everyone's competences are of value and where it is possible to borrow experiences from others.

Following Vygotsky's legacy, Davydov and his colleagues have developed a theoretical framework for teaching, in the West known as learning activity, in which students collectively use each other's experiences and specially designed mediating tools – learning models – to solve subject-specific problems (Davydov, 2008; Davydov & Rubtsov, 2018). Thus, the whole-class activity can transform into a student-driven, tool-mediated learning activity exploring, for example, theoretical structures and relationships in algebraic expressions.

Therefore, we claim that there is a need for strategies and actions, especially communicative actions, to be used by teachers when realising a learning activity that can support students' agency in collective (whole-class) problem-solving discussions with a specific content or learning outcome in mind. Within the research field of early algebra (see below), Davydov's mathematics programme, framed by the theory of learning activity, has been referred to as a promising alternative route for young students' theoretical learning, that is, development of young students' algebraic thinking and algebraic problem-solving abilities (Cai & Knuth, 2011; Kaput et al., 2008; Kieran et al., 2016). The fundamental idea of learning activity is a teaching that helps students to develop their analytical and theoretical thinking through their reflections (Davydov & Rubtsov, 2018). This suggests a teaching that enables theoretical thinking, a teaching that we refer to as theoretical work. Theoretical work can be described as a process in which the principles and internal relations of specific concepts are explored, a process which can lead to an understanding of a theoretical concept (Davydov, 2008). Given the findings indicating that whole-class discussions benefit from the use of visual mediating tools, and the promising results from the use of Davydov's learning activity (see below), we still need to know more about what to consider when planning and realising tasks, tools, and communicative actions in order to establish a problem-solving theoretical work in a collective form. This knowledge needs to be both fine grained and particularly specified through empirical studies (see Dewey, 1929/2013).

Aim and research questions

Taking the perspective of learning activity, in which students' tool-mediated collective reflections are central, the aim of this article is to narratively exemplify and discuss what teachers need to consider when planning and supporting students' collective theoretical work on algebraic expressions. The aim is specified by the following two research questions.

> RQ1: What indicates that collective theoretical work is established in whole-class discussions focusing on algebraic expressions?

> RQ2: What in the task – its content, its design, its staging, and its tools – creates opportunities for students to engage in collective theoretical work on algebraic expressions?

Algebraic thinking

Algebra has a special position in mathematics and is relevant for most other mathematical areas (Hemmi et al., 2021). If the students have a good understanding of algebra, it enables them to succeed in their mathematics studies (Kieran et al., 2016). For a long time, algebra was equated with solving equations, but nowadays it has a much broader meaning. Algebra can be described as generalised arithmetic – arithmetic concerns operations on number, whereas algebra concerns systems of objects and operations on these (e.g. Krutetskii, 1976). Traditionally, algebra has been regarded as difficult for students, and is therefore introduced later in secondary school or in upper secondary school (Bråting et al., 2018; Hemmi et al., 2021; Stacey & Chick, 2004). In recent decades, researchers have focused on students' insufficient algebraic knowledge or failures –

from lower secondary school and onwards (Kieran et al., 2016). This can be seen as a background for the growing interest in the research field of early algebra, in which students' algebraic thinking is addressed (Kieran et al., 2016).

In early algebra research, questions about conditions for students' development of algebraic thinking as a basis for formal algebra have been discussed for some decades (Blanton et al., 2015; Kaput et al., 2008; Kieran, 2018; Warren et al., 2016). Kieran (2004) describes algebraic thinking in early grades as involving:

> the development of ways of thinking within activities for which letter-symbolic algebra can be used as a tool but which are not exclusive to algebra and which could be engaged in without using any letter-symbolic algebra at all, such as, analysing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting.
>
> (Kieran, 2004, p. 149)

From a socio-cultural perspective, with the legacy of Vygotsky (1978), the cultural historical tradition where theoretical thinking is crucial for development of higher order thinking, algebraic thinking needs to be understood in relation to the youngest students as well (Cai & Knuth, 2011; Kieran, 2004; Radford, 2021). Today, there is a growing interest in the El'konin-Davydov curriculum (ED curriculum) in mathematics (e.g. Cai & Knuth, 2011; Kieran et al., 2016) as it introduces algebraic thinking as a foundation for arithmetical operations and as a basis of number sense (Schmittau, 2004). The ED curriculum is built on a specific theory of learning that in English is called *learning activity* (Davydov, 2008).

Learning activity as a theoretical perspective

The theoretical perspective for the research project behind this article was learning activity (Davydov, 2008). As mentioned above, this theory addresses students' development of analytical and theoretical thinking, which in relation to mathematics is understood as algebraic thinking (Schmittau, 2004). Grasping the general in a mathematical concept is thus seen as a solid point of departure for students' learning. Learning activity draws on activity theory which states that becoming knowledgeable requires opportunities to become familiar with knowledge-specific, tool-mediated actions that define a specific practice (Leontiev, 1978). Theoretical thinking, in our case algebraic thinking, within this perspective is seen as a social practice – a *praxis cogitans* (Wartofsky, 1979). Thus, teaching requires that the students be provided opportunities to work theoretically to become familiar with subject-specific tools and their functions. From an activity theory perspective, the development of theoretical thinking presupposes collective agency. If the students identify a problem, and thereby develop a motive, they may collectively establish a learning activity. Thus, the way the teacher constitutes the support in the classroom is pivotal for the students to successfully engage in a learning activity. The teacher's role is not to tell or instruct in a direct way. Instead, the teacher needs to promote student agency since a learning activity is depen-dent on students' actions. Davydov (2008) characterises students' theoretical work in a learning activity as *ascending* from the abstract to the concrete. In relation to early algebra, the "abstract" could, for example, be an algebraic expression such as $ax + b$. When the students are familiar with this type of structure, they can transform it into concrete operational tasks of diverse types. Central concepts in learning activity are briefly described below.

Problem situation as a theoretical concept is generally a well-planned situation with a built-in, contentful contradiction from which it should be possible for the students to identify a problem to be solved (Repkin, 2003). The situation itself is thus not a stated problem to be solved. The students need to analyse what is given in the situation. Further, the students need to evaluate what previous knowledge can be used. If the problem situation is well designed, the students will experience a need for new knowledge. Thus, the problem situation should create a meaningful context in relation to which their current knowledge is, to a certain extent, insufficient. A problem situation can be rather narrow, in which the teacher, for example, says, "in another class, some students gave this solution. How were they thinking about the problem?". The students are then supposed to analyse the given situation to identify what type of problem needs to be solved.

The concept of *contradiction* is crucial when designing a problem situation for a learning activity and draws on the idea that every activity is driven by historically developed contradictions (Engeström & Sannino, 2010). A contradiction often manifests as a conflict or a dilemma but in a teaching situation the idea of contradiction can be manifested in a problem as a hook or as an apparently incompatible condition, for example, when two different statements are presented as correct. To determine if both are correct or if something is wrong, the students must analyse the situation. Thus, a contradiction can be a trigger for the students to examine the situation. At best, the teacher succeeds in creating a contradictory situation, but a contradiction can also be unplanned due to a comment, question, or solution given by the students (or by the teacher).

When students work with theoretical mathematics content such as the structure of algebraic expressions, mediating tools in the form of *learning models* can, through modelling, help them to visualise the abstract content (Davydov, 2008). Learning models can be constructed out of symbols (such as x or $a = b + c$), schemes (represented with line segments ), graphs (such as those depicted in coordinate systems), and physical representations (such as Cuisenaire rods or paper models). With learning models as tools, students can explore abstract phenomena (Davydov, 2008; Davydov et al., 2003; Gorbov & Chudinova, 2000; Zuckerman, 2004). A learning model can be created by both the teacher and the students or collectively in the classroom and should not be compared with formal mathematical models but rather seen as a materialised representation of the abstract that can be elaborated and developed.

In learning activity, *collective reflection* is primarily seen as a collective practice. That is, to become aware of one's own thinking, one needs to see one's own arguments and explanations in the light of others'. By reflecting on the explanatory actions of others, the individual student can also become aware of their own thinking and actions (Zuckerman, 2004).

Teaching activity vs. learning activity

Stressing students' collective agency, it is possible to differentiate between teaching activity and learning activity (Eriksson, 2015). In a teaching activity, the teacher is the one who directs the work, often through a communicative question–answer pattern that is usually described with the abbreviation IRE[1] (Mehan, 1979). Further, the teacher retains the main responsibility for defining and presenting the problem to be solved and the students are expected to follow (passively or actively) the teacher's directive actions. In a learning activity, the students must develop a need for new knowledge and thus a motive to engage in the work. The actions taken in a learning activity are thus dependent on the students' agency. Davydov (2008) talks about *learning actions* – actions concerning: problem identification; reflection and analysis of graphical or letter models and actual knowledge in its "pure form"; testing of and elaboration on a general problem-solving method; and finally, assessment of the problem-solving methods.

The learning actions taken by the students require students to activate their current theoretical thinking and test their previously known methods and conceptual tools to analyse the problem situation, and identify and define what the problem was about and how to model it to find plausible solutions.

Methods

Data for this article come from a three-year research project (2017–2019) financed by the *Swedish institute for educational research*.[2] The overarching aim of the project was to develop knowledge of how teaching can be designed to promote students' ability to reason algebraically. Each of the five research lessons, including lesson plans, tasks, learning models, and communicative actions were adjusted and refined iteratively and collectively. The data are taken from the third and the fourth research lesson in a grade 7 class (students aged 13). The teacher's role was to conduct the jointly planned lessons while the researchers were responsible for documenting the lessons with written notes and video. The lessons were transcribed with the aim of capturing oral communication, complemented with gestures and intonation to provide meaning in culturally familiar situations (Nordin & Boistrup, 2018; Roth & Radford, 2011). Transcription followed the principles of a word-for-word, speech-neutral text, organised in dialogic form (Linell, 1994). Gestures and intonations were captured in square brackets and words emphasised by the speaker were underlined. Students' names were changed. Although the teacher conducting the research lessons was part of the school's ordinary staff, he did not ordinarily teach grade 7 that year and was therefore not acquainted with the students.

The design of the research lessons

In relation to the overarching aim of the project, the content was chosen to enable the students to develop a theoretical understanding of an algebraic expression (e.g. $ax + b$), and specifically, to reason about algebraic expressions without assigning specific values to the variables. When designing the research lessons, we followed the principles of learning activity theory (Davydov, 2008; Repkin, 2003). Inspiration for the design of different problem situations and learning models used in the project and the planning of the staging of the research lessons was taken from the ED curriculum (Davydov, 2008; Schmittau, 2004; Zuckerman, 2004). Previously in the project, three critical aspects that indicated what the students needed to discern in an algebraic expression had been identified (Wettergren et al., 2021). They were:

- the components have different functions,
- same variables must have the same value[3], and
- the value of a variable is determined in relation to other components.

The construction of the problem situations included algebraic expressions and illustrations. In the first part of the lesson (not analysed in this article), the problem situation contained an expression and some geometrical illustrations that the students were to collectively reflect on. In the second, an expression was given and the students in pairs were asked to draw matching representations. However, when planning for *staging* the problem situation we decided to use premade illustrations as fictive students' solutions instead of drawings that different pairs of students had created. Experience from earlier research lessons was that students' discussions around "neutral" examples flowed more freely than with examples produced by the students themselves. Another experience was that when using premade illustrations, the teacher could more easily make statements or ask questions that directed the students' awareness towards theoretical aspects of algebraic expressions. Constantly presenting the premade illustrations using pairs of different representations created an implicit contradiction in the form of conflicting information between two versions. The conflicting information could be used by the teacher as a tool to urge the students to further their analysis. In other words, the premade illustrations together with the prepared teacher questions/ statements was supposed to trigger the students to further analyse theoretical aspects of the algebraic expressions.

Analysis

The analysis was done in two steps. First, a narrative of the students' and the teacher's actions in the classroom was constructed. The construction of the narrative was based on an analysis of what learning actions the students, supported by the teacher's actions, realised in relation to the given problem situation. These learning actions in a classroom situation are expressed as communicative actions (verbal and non-verbal). In the video recordings and the transcriptions, we paid special attention to situations in which the students found something to be problematic; for example, where they got stuck or opposed a contradiction in the given situation. The focus was the communicative actions where the students contributed to the discussion, for example, by providing arguments and challenging these, especially the actions on the board aimed at the other students. Of specific interest were the parts where the learning models were used. In the next analytical step, we used the constructed narrative to highlight some core findings in relation to the two research questions. In the analysis, the main analytical focus was on students' elaborations in the form of communicative actions relating to the problem situation. More precisely, we focused on how, with the help of tools and each other's suggestions, they jointly built up an argument and an explanation to solve the problem in the problem situation – thus, focusing on the interrelation between the design of the task and the potential of the task to help students discern the mathematical content. In relation to the research questions, the main analytical focus was directed towards those aspects of the teacher's and students' tool-mediated actions that facilitated the theoretical work. In the analysis, questions like: ”who is doing what?”, ”with what tools?”, and ”what are actors trying to accomplish?” were posed.

Result

The result is presented in two parts. In part 1 the narrative is divided into two sequences. In part 2 some aspects of the narrative are highlighted. The examples chosen come from two different groups participating in the lesson iterations three and four, respectively. The two sequences (see below) followed the same design idea and built on an algebraic expression that the teacher wrote on the whiteboard.

Part 1. Establishing collective theoretical work

Before the narration starts, the students have been presented with two different schemes to be used as learning models. One was a line segment model (figure 1) and the other was an area model (figure 2).

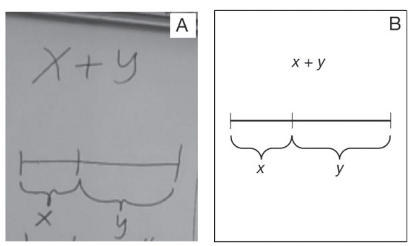

Figure 1. *Line segments as a model of the addition* $x+y$
Note. Panel A is from the lesson video. Panel B is a reproduction of the same illustration.

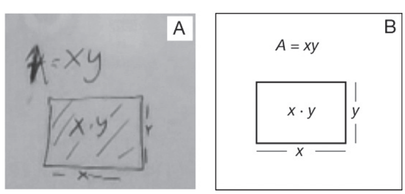

Figure 2. *Area as a model of the multiplikation xy*
Note. Panel A is from the lesson video. Panel B is a reproduction of the same illustration.

The learning models were introduced in a task where the students had to argue for how the algebraic expression $3x+9y$ could be understood in relation to two geometric representations. The learning models were supposed to work as mediating tools in the students' exploration of the algebraic expressions by visualising the theoretical aspects of addition (perimeter) and multiplication (area).

In the next task, the students in pairs were asked to produce representations of the expression $2(k+2m)$ that the teacher had written on the whiteboard.

Sequence 1. A planned contradiction inherent in the tasks

When the paired-up students have worked for about three minutes, the teacher walks around looking at their different suggestions. The teacher pretends to make copies of their work on a flip chart out of sight. After about six minutes the teacher interrupts the working phase and puts one flip chart, representing two different suggestions made by two pairs of students, in front of the class (figure 3), saying that he has copied two different student suggestions, in the form of rectangle representations of the expression $2(k+2m)$.

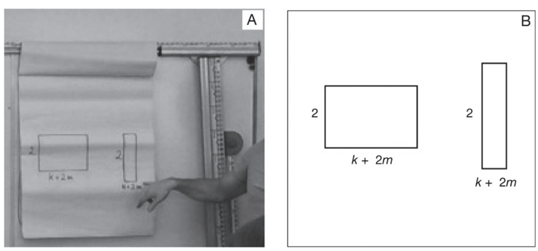

Figure 3. *The teacher shows the first illustrations of the expression* $2(k+2m)$
Note. Panel A is from the lesson video. Panel B is a reproduction of the same illustration.

With an astonished voice the teacher asks the students if both examples can be correct even though they do not look the same, and if both are correct, how that can be. The teacher also encourages the students to walk up to the flip chart and visualise their arguments by pointing or making marks on the flip chart or the whiteboard. Leya points at the flip chart, writes $k=?$, and explains that both representations can be correct since k is unknown and therefore can be anything. Another student, Hampus walks up to the flip chart and continues Leya's argument, saying that both examples are correct by talking about and visualising the meaning of the invisible multiplication sign.

Excerpt 1

Hampus: We had this [points at the expression given, $2(k + 2m)$]. It's the same as that [points at the expression and at the two rectangles]. Two times [points between the numeral 2 and the parentheses], there's an invisible multiplication sign [writes the multiplication sign and points to the class as if marking the importance of this (figure 4)] so then it makes two times that [points at the k] and two times that [points at the $2m$]. And over there [points at the bigger rectangle] it is the same. So, there it will be $2k + 4m$ [writes under the expression given] and over there [points at the side of the bigger rectangle where it says $k + 2m$] it's also like that.

Teacher: Wait a second, that one to the left ... [slowly stops talking, his voice signalling that he cannot follow Hampus's argument].

Hampus: It's the parentheses [points to $k + 2m$] that is marked on one side. Or both of these are correct [points at both rectangles]. Or, sort of right [hesitantly].

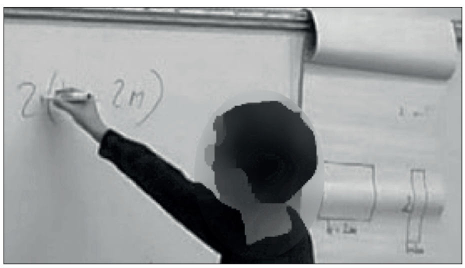

Figure 4. *Hampus draws a multiplication sign in the expression* $2(k+2m)$
Note. The content shown on the flipchart is reproduced in figure 3, Panel B.

As Hampus states his arguments, he pauses and looks at the teacher for confirmation. The teacher, however, is hesitant, saying, ”wait a moment. But ...?” as if he cannot follow Hampus's argument. Hampus then says that both rectangles are ”kind of” correct.

A third student, Elin, then questions how both rectangles can be representations of the same expression since they do not have the same area. At the same time, she says that the variables k and m can symbolise any number.

Excerpt 2

Elin: [...] But if the two are separated [draws a line between the two rectangles] then it can be correct because then that k is equal to that k and that m is equal to that m [points at k and m in the illustration].

Another student enters the discussion and argues that the two rectangles can be representations of the same expression.

Excerpt 3

Dante: It doesn't look like the same area, but it is, because you can't know what it is. It can be anything [...] because that's two different examples of what it can be [pause] you don't get to know enough to calculate what it is [looks at the teacher, questioningly].

Teacher: Mm ... [in a thoughtful tone, signalling that he isn't fully satisfied].

During Dante's argumentation, he tries to get confirmation from the teacher, who is again hesitant, saying "mm", thoughtfully. Hampus then enters the discussion again and tests his argument further by using determined numbers.

Excerpt 4

Hampus: They are both two times k plus m, and regardless of order you get the same answer. But if you, like, do it over again and k is maybe two and m is four, you get, like, two times six [Hampus says six not four] on both and then it's twelve.

The teacher encourages Hampus to go back to the flip chart where he starts to illustrate his argument by first pointing at the variables on the sides of the rectangles and then starting to write on the whiteboard $k = 2$ and $m = 4$ and concludes that the two rectangles have the same area. The teacher sums up, based on Dante's and Hampus's arguments, by saying that both rectangles can be a representation of the expression, although both represent an area. In this summary, the teacher repeatedly checks with Hampus that he has correctly understood what Hampus said.

When the students seem satisfied with having solved the problem and having established that the two rectangles were, in fact, equally adequate representations of the expression, the teacher moves on to discuss two other illustrations.

Sequence 2. An unplanned contradiction inherent in the lesson flow

In this lesson the teacher by mistake wrote $2(k + 4m)$ instead of $2(k + 2m)$ on the whiteboard. This situation was chosen because since neither of the premade solutions on the flip chart (figure 5) corresponded to the expression, the problem became more complex. The teacher introduced the new problem situation, by saying "since both were correct before, isn't it reasonable that both of these are also correct?" That is, the teacher tried to signal to the students that if two different representations could be used to explain an expression in one task this must always be the case. The dilemma created by the teacher's mistake – the expression did not correspond to either of the two illustrations – seemed to trigger the students to further analyse the components and variables in the algebraic expression. Oscar identifies the problem and starts to explain why it is problematic.

Excerpt 5

Oscar: They don't show the expression, not really.

Teacher: Don't they? [surprised]. You'll have to go up and show it, you'll have to explain why [Oscar walks to the flip chart].

Oscar: If we start with seeing the perimeter as the expression, then we write *k* plus two *m* plus *k* plus two *m* [writing the operation]. That makes two *k* plus four *m*. And that is not the same as that expression [points to the expression given].

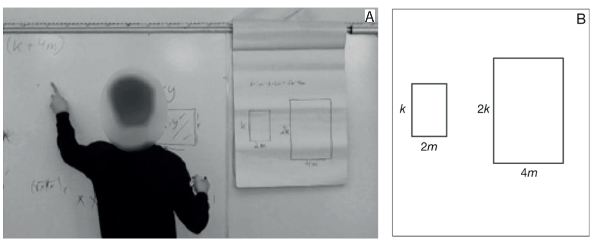

Figure 5. O*scar points to the expression given: 2(k + 4m)*
Note. Panel A is from the lesson video. Panel B is a reproduction of the same illustration.

Oscar starts to explain why both cannot be correct by using the perimeter and area introduced in the beginning of the lesson as mediating tools (see figure 5).

Excerpt 6

Oscar: [continues] Because if we were to reduce this [points to $2(k + 4m)$], it would be two *k* plus eight *m* [writes it after the expression given] which is not the same as two *k* plus four *m* [...] And this [points to the bigger rectangle], would be two *k* plus four *m* plus two *k* plus four *m* which makes four *k* plus eight *m* [writes on the flip chart], which still isn't the same as [points to the expression given] two *k* plus eight *m*. And then the area would be *k* times two *m* [writes on the flip chart] on this [points to the left rectangle] and that's not the same either, as two *k* plus eight *m* [...]. And on this [right rectangle] two *k* times four *m* [writes on the flip chart] it's not [points to the expression given] two *k* plus eight *m* either.

After Oscar's explanation, the teacher asks the other students if they agree with Oscar's statement that the rectangles cannot be representations of the expression, and further, what should be written on the sides of the rectangles to make them correctly represent the written expression. Another student, Albin, walks up to the flip chart and gives an example of what variables to write on the sides of each rectangle.

Excerpt 7

Albin: This one [points to the left rectangle] can be changed to be like that [points to the expression given] two ***k*** plus four ***m***. So, we can change it like this: [writes 4***m*** under 2***m*** on the flip chart] from two ***m*** to four, that gives ***k*** plus [writes and speaks simultaneously (figure 6)] four ***m*** plus ***k*** plus four ***m***. [Pauses and looks at what he has written]. It's really, we see it, two ***k*** and four ***m*** [points to what he has written]. So, we can write it in this way [writes $2(k + 4m)$ adjacent to where he has written $k + 4m + k + 4m$]. And for the other we can [shrugs and points at the flip chart], well, we can really just cross this over [voice signalling that it is a suggestion] the two [crosses the numeral 2 in front of the variable ***k***] then we only have ***k*** and four ***m*** again. And that fits the expression.

While Albin writes his suggestions of what variables could be written on the sides of the rectangles, he explains why they can be changed in this way.

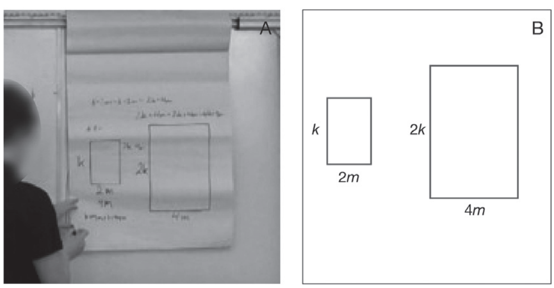

Figure 6. *Albin writes:* $k+4m+k+4m=2(k+4m)$

Note. Panel A is from the lesson video. Panel B is a reproduction of the same illustration.

Albin's work visualises the function of the parenthetical notation by showing how the two representations can be adjusted to fit the expression given.

Thus, the problem presented on the flip chart enabled the students to continuously work together on the same problem. This together with the staging of the lesson where different representations of the same expression one by one were latched to each other, revealed an abundance of algebraic properties.

To sum up, the students are apparently engaged in theoretical work in the two sequences given in the narrative above. What is indicating this? First, the content of the discussion is dense and focused. Second, the students build on each other's arguments. Third, when they do not receive confirmation from the teacher, they continue to substantiate their suggestions themselves.

Part 2. Collective theoretical work enhanced by task, design and tools

In Sequence 1, the two rectangles were different, but their sides were labelled identically (2 and $k + 2m$, respectively). This created an opportunity to discuss the varying property of variables. One side being labelled $k + 2m$ also highlighted the fact that a singular entity, like the length of the side of a rectangle, could potentially be in a complex form, like the expression $k + 2m$.

To connect the expression to the rectangle representations, the students had to identify the additive structure of length when represented as the perimeter of the rectangle. This way, focus is placed on the equality $2(k + 2m) = 2k + 4m$, and enables the students to process the invisible multiplication signs in the expressions.

In Sequence 2, the two rectangles were also different but here their labelling was not identical: one was labelled $k \times 2m$, the other $2k \times 4m$ (figure 5). In Sequence 2, the teacher made an error and gave the students the expression $2(k + 4m)$ instead of $2(k + 2m)$. Thus, the premade rectangles on the flip chart were both incorrect representations of the expression. This created a problem that triggered the students to elaborate on, and in this case the solution required an algebraic manipulation either on the expression or the labelling of the rectangles. It also opened the task for interpreting the expression's connection to the rectangles both through the additive structure of length and through the multiplicative structure of area.

Promoting students' collective theoretical work

As highlighted above, the task and its transformation into a problem situation provided opportunities for the students to reflect upon the structural aspects of the algebraic expression. But the transformation of the task into a problem situation and the students' theoretical work was dependent on the teacher's actions as well. Thus, the theoretical work can be seen as a joint action in which the teacher strove towards promoting students' agency.

The teacher's actions in these two sequences were using intonation and gesture to signal his hesitancy or signalling that he could not follow some of the arguments. Apparently, this type of action (response) encouraged the students to develop their collective theoretical work. The students' learning actions during these two sequences consisted mainly of identifying problems, testing and modelling structural aspects, transformations of the models used, and testing and proving possible solutions. But the teacher's actions also included constantly balancing the students' theoretical work, using claims, suggestions, and questions, thus pushing the collaborative discussion forwards and in the direction of the intended goal for the lesson. The teacher adopted an "unhelpful" position and carefully "sabotaged" the discussion with incorrect claims and suggestions rather than filling the logical gaps in the students' reasoning. Only when the teacher believed that the students had discerned the most central parts of the content did he confirm by sensitively summarising.

It is difficult for the teacher to plan exactly what to say during such a phase; it demands clearly defined ideas of what actions the students could take and what could be mathematically advantageous for the lesson and then being very attentive to what the students say and do during the discussion. In this delicate balance, the teacher can be guided by students' suggestions or their attempts to explicate, rather than pointing out mistakes or providing correct solutions.

In addition, it became clear that promoting the discussion to be accessible for the whole class, by writing on the board or on the flip chart, avoiding erasing what others had written, and thus enabling the students to clarify their arguments by pointing, played a crucial role in the whole class discussions. In other words, creating a collective memory on the board or on the flip chart was important for maintaining the theoretical work as collective one.

Discussion

In relation to the aim of the article, we want to discuss what teachers working within a learning activity perspective need to consider when planning for and staging students' collective theoretical work on algebraic expressions. In the result, we specifically highlighted what in the content of the task, its design, and its tools creates opportunities for students' theoretical work and what in the teacher's and students' collective tool-mediated actions facilitates the students' theoretical work.

The content of the task, its design, and its tools

The result indicates, in accordance with criteria from Taflin (2007, p. 237), that problems the students work with must introduce important mathematical ideas. From a learning activity perspective, a specific mathematical idea or concept, for instance, algebraic expressions and their components and functions, must be introduced in a way that enables the students to explore the relational and structural features. That is, the general and theoretical aspects must first be the focus of the students' work for them to ascend from the abstract to the concrete (Davydov, 2008). Thus, what from a learning activity perspective can be considered a rich task, must capture the core theoretical principles (Davydov, 2008; Schmittau, 2004). Taflin argues further that the problem should be easy to understand. It may seem like a minor discrepancy between the idea of a problem that is easy to understand (Taflin, 2007) and students being responsible for identifying a problem. However, our findings suggest that the students' problem-identifying actions are essential if a learning activity is to be robust.

The examples chosen from the research lessons illustrate that the students initially scrutinised the teacher's claims about the correctness of the representations. Thus, defining the problem as acknowledging that an algebraic expression can be interpreted in different ways. Their work subsequently involved dismantling the expression to describe its functional components and in so doing discerned the critical aspect identified for the groups. Thus, if students identify a problem they regard as worth solving, they may perceive it as a challenge and hence will develop a motive to solve it.

Furthermore, a task should be framed and staged as a problem situation into which the teacher must build contradictions or hooks. If the problem is too easy for the students or if the teacher explains it, this may hinder students' agency. The teacher should not explain what the students are supposed to find out, which tends to happen with the teaching tradition that Schwartz et al. (2011) call "tell and practice". Actively promoting students' theoretical work without verbal or non-verbal evaluation or correction creates a different communicative pattern to the dominant IRE-pattern (Mehan, 1979).

In learning activity, the concept of learning model (Gorbov & Chudinova, 2000) is crucial. However, in the data from the selected lesson it is not obvious that the students transformed and tested the models given in the beginning of the lesson (see figures 1 and 2). Instead, the results indicate that the students used the geometrical representations and the symbolised expressions as learning models (e.g. excerpts 2 and 6).

Teachers' and students' tool-mediated actions

From the narrative and the analysis, we draw the conclusion that when planning and staging a learning activity in which the students are given opportunities to work theoretically on algebraic expressions, teachers need didactical tools and strategies. This is of great importance in teaching situations where the students (and the teacher) are unfami-liar with learning activity. The didactical tools identified in this study are categorised as ”contradictions”, ”playful format” and ”collective reflections”. These can be regarded as specified didactical tools that have emerged from empirical and experimental research lessons of a type that are argued by Dewey (1929/2013).

Contradiction – with the idea that contradiction is a core driving force behind development in societal activities, this concept can also play a role in the design of teaching. Especially if the overarching goal is student agency in theoretical work. When planning for a learning activity to emerge, the task needs to be designed in such a way that the students are motivated to explore it. With built-in contradictions, it seems that students engage in problem-solving discussions – not only seeking a solution but also seeking to understand what is problematic. In the design of the task exemplified in this article, the idea was to provide students with illustrations depicting alternative representations of an algebraic expression. To ensure that the illustrations would create a situation of contradiction, we used premade illustrations (pretending that they were examples from the students). This made it possible for the teacher, for example, as in sequence 1, to ”choose” two representations that looked different but in fact could be considered to be equally good representations of the expression. Thus, a productive contradiction can be planned and built into the task. But a contradiction can also arise during the lesson as a consequence of the teacher's or the students' action (as in sequence 2). This is something that the teacher must be attentive to and if such a situation emerges, the teacher must be sufficiently alert to make productive use of it to enhance the lesson's theoretical investigations.

Playful format – in planning for a certain learning object, the problem situation can be ”dressed” in a fictive frame, or as van Oers (2009) says, in a playful format. In the result section, we described how this was exemplified by the teacher's signalling that he did not understand, could not follow the argument, wanted someone else to explain, etc. For a playful format to be a powerful didactical tool, the teacher must be serious while staging it and the students must believe and play along. In the research lessons, we noted that this happened.

Collective reflections – the result indicates how powerful it is to create a common working space on the board, encouraging students to work in this space, not only to vaguely point to or say something but to explicitly write, adjust, and add while explaining. During the first iteration of the research lesson, the teacher allowed previous suggestions to be erased. However, it became clear to the research group that the students needed access to all the suggestions, and therefore, our conclusion that previous suggestions on the board should not be erased. The suggestions on the board functioned as a collective memory and thus enhanced the students' collective reflections on the problem (Davydov & Rubtsov, 2018; Zuckerman, 2004, see also Eriksson et.al., 2019). Further, the use of gestures such as pointing to a concrete aspect of the task or the model on the board helps to visualise one student's thinking for the others.

To summarise, a task, its tools, and its staging as a learning activity in which students' (and teachers') learning actions have the potential to realise what Wartofsky (1979) sees as a social practice or a *praxis cogitans*, as in our case concerns algebraic thinking.

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Notes

1 IRE: the teacher *initiates* a question, the student *responds*, and the teacher *evaluates* the answer.

2 The research project underlying this article consisted of four different series of research lessons following the principles of learning study (Carlgren et al., 2017). Approximately 310 students in four different schools (grades 1, 5, and 7 and year 1 in an upper secondary school) participated in the project. The teachers collaborated with the researchers related to their own school, respectively. Over 90% of the participating students, the teachers, and the research group, were Swedish born.

3 Two different variables, for example, *c* and *b*, may have the same value; however, it is not appropriate to represent a general relation with a specific relation in which the variables have the same value.

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