Fostering mathematical reasoning in inquiry-based teaching – the role of cognitive conflicts

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Students' independent mathematical inquiry is often endorsed as a valuable teaching method. In this article, we scrutinise in what ways these independent situations entail the students' development of mathematical reasoning. We study the cognitive conflict in one fifth-grade class participating in an inquiry-based intervention study. The findings indicate that cognitive conflicts can support the students' reasoning processes and that the environment has an important role in retaining the conflicting positioning by making the cognitive conflicts available for discussion and scrutiny. The students' processes of resolving cognitive conflicts are stretched over time and involve different routes and exploring approaches and understandings.

A student's reasoning competence is very important in mathematics and is often seen as fundamental for learning mathematics. Here, Ball and Bass (2003) argued that "the notion of mathematical understanding is meaningless without a serious emphasis on reasoning" (p. 28). Indeed, reasoning can be seen as a basis for both mathematical understanding and communication and as a critical part of developing a mathematical approach that, for instance, appreciates convincing arguments (Ball & Bass, 2003; Carpenter et al., 2003; Hanna & Jahnke, 1996; Stylianides & Stylianides, 2008). In the literature, there is a clear distinction between reasoning in mathematics and reasoning in everyday life; an example of this is Harel and Sowder's (1998, 2007) distinction between empirical proof schemes and formal proof schemes. Most students in primary school have empirical proof schemes, and research has shown that changing students' empirical schemes to more formal proof schemes is highly nontrivial (EMS, 2011), and how to make the transition from empirical to more formal deductive reasoning is still open to further research (EMS,

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2011). This transition is the specific interest of this paper. The idea in deductive reasoning is that students use a general rule they already know is true to argue for a new conjecture.

We wish to investigate students' reasoning processes in this transition to gain insight into whether and in which way inquiry-based teaching in this regard proposes to support the transition to formal reasoning.

Many important documents and initiatives have pointed to the relevance of inquiry-based science (IBSE), mathematics (IBME), and engineering education (Artigue & Blomhøj, 2013) in primary school classrooms, where the students' independent formulation and exploration of mathematical problems are the focal point of the teaching processes. Indeed, such an approach has been explored and developed in a number of project and initiatives (Cordis, 2021; PRIMAS, 2013). The question is, however, how the specific relations are constructed between situations in inquiry-based teaching and the reasoning processes in mathematical situations. Our interest in this relation led us to study the students' independent reasoning situations in an inquiry-based teaching approach. When doing this, we realised that cognitive conflicts can play an important – and sometimes positive – role in bringing the students from open exploration towards more directed exploration and reasoning – we refer to such conflicts as productive cognitive conflicts. We refer to students' *independent* reasoning situations as cases where students are working independently; that is, alone or in small groups with little or no involvement from the teacher. In these situations, the teacher typically initiates the activity. The teacher may typically move around between the groups of students to follow their process and is available to support them if they experience problems in their process.

Furthermore, we focus on situations where the students *reason* in the sense that they make conclusions or consider the validity of different approaches to bring them towards a conclusion, a solution or simply a better understanding of the situation.

In the current paper, we more closely examine how a cognitive conflict can entail the students' development of mathematical reasoning when students are engaging in a mathematical inquiry. We follow how a group of students experience a cognitive conflict when they move from making meaning of a mathematical situation using different representations towards a more formal mathematical solution. We characterise the productive cognitive conflict and focus particularly on which aspects of the environment are important to promote the productive cognitive conflicts and the subsequent the development of mathematical reasoning that emerge from this. We ask the following research question:

> What characterises productive cognitive conflicts when students work in independent situations with mathematics in inquiry-based

teaching, and in what way do these productive cognitive conflicts relate to students' mathematical reasoning process?

In this paper, we first introduce the state of the art on inquiry-based teaching as well as the theoretical constructs on which we build our understanding of both students' independent reasoning work and the cognitive conflicts that pupils can experience in such situations. Then we present the development project *Quality in Danish and mathematics*, which is the context of the investigation on which we report. After the theoretical and practical setup, we present the central empirical case on which we build our analysis. The case consists of one double lesson where students are working with triangles using physical manipulatives and digital tools. We analyse this case by focusing on cognitive conflicts. The result shows how cognitive conflicts, given the right material and mathematical environment, can spur the learning of deductive reasoning in inquiry situations.

Inquiry-based teaching

Inquiry-based teaching is an instructional theoretical model that has been developed in several disciplines. The key idea is that the students must solve real or authentic problems about their educational activities in relation to science and mathematics (Artigue & Blomhøj, 2013). In relation to the learning of mathematics, the teaching of inquiry-based mathematics has been the focus of a number projects and empirical studies in which different topics have been studied – e.g., the benefits (Bruder & Prescott, 2013) or the difficulties of the implementation (Dorier & García, 2013; Engeln et al., 2013; Krainer & Zehetmeier, 2013; Larsen et al., 2019; Maaß & Artigue, 2013; Maaß & Doorman, 2013; Schoenfeld & Kilpatrick, 2013). Also, in a Nordic context, several inquiry-based studies have been carried out, including the *SUM project* (Haavold & Blomhøj, 2019) and *Learning communities in mathematics* (Jaworski, 2006).

However, research on the relations between inquiry-based teaching and students' transition from making an argumentation based on rationales and intuition to more deductive reasoning in mathematics has yet to be explored.

Within inquiry-based mathematics education, the teacher must guide the students to select appropriate experiences, to reflect on these experiences so that their educational potential actually emerges, to make the students develop productive inquiries (Artigue & Blomhøj, 2013) and also so they can experience the limitations of their knowledge, hence creating the conditions for achieving the required cognitive evolution (Artigue, 2012).

In this article, the focus is on the student's independent work in inquiry-based teaching, but the role of different manipulatives and artefacts will also be included. This focus is in line with Artigue and Blomhøj (2013) that concluded that inquiry-based teaching includes giving autonomy and responsibility to students, inclusive of a large focus on the experimental dimension of mathematics. The idea behind inquiry-based teaching is that students learn more and better when they can take control of their own learning by defining their goals and monitoring their own progress when making the inquiry. This must, of course, take place with a teacher's guidance. Experiments in inquiry-based teaching can have many approaches, but often, when conducting an investigation, different kinds of manipulatives, representations, or other resources are included in the experimentation, where the context of the investigated problem is also essential (Baptist, 2012).

Proponents of inquiry-based mathematics often take a constructivist approach, suggesting that students construct knowledge following the lines of work of professional mathematicians. Indeed, mathematicians often face nonroutine problems and make investigations where they must search for information and develop conjectures that they then must justify and finally communicate the results (Artigue & Blomhøj, 2013). In this paper, we take a cognitive and constructivist perspective to articulate cognitive conflicts as defined by Piaget (1977) and Tall (1977). It would be possible to take a more social perspective focusing on practices and norms (Yackel & Cobb, 1996) or on the commognitive conflict configurations (Sfard, 2008); however, to focus our analysis on the cognitive conflicts, we chose not to include the more social perspective; rather, we focus on the interplay between the physical surroundings and representations on one side and the students' articulations on the other.

Independent and adidactical situations

In using the word *independent situations*, we refer to those situations where students either individually or often in groups perform an investigation and where the teacher is minimally present or present as a guide to help if needed. We use the notion of *independent situations* from Blomhøj (2016). He uses the term in three steps of inquiry-based teaching: 1) staging the problem, 2) students' independent situations and inquiry, and 3) whole-class discussion and reflection. Using Blomhøj's notion in the terminology from the theory of didactic situations (Brousseau, 2006), these situations will be defined as adidactical situations, which are situations in which the student takes a problem as his or her own and solves it based on its internal logic and not in the light of the teacher's control. By using the term independent situations, we want to make clear that these situations sometimes are without the requirements that Brousseau

(2006) set for a situation to be defined as an adidactical situation. For example, the student must be engaged in a game, and "this game being such that a given piece of knowledge will appear as the means of producing winning strategies" (Brousseau, 2006, p. 57). This is not always the case in independent situations observed in inquiry-based teaching. However, in adidactical situations, the student should not be aware of the teacher's intentions about the knowledge underlying the situation. This might also be the case in the independent situations in inquiry-based teaching; however, the teacher's role in both situations is to organize a milieu for the students to engage with and then withdraw from the scene. In Brousseau's terminology, this will be in "the situation of action" (Måsøval 2011).

By using the notion of independent situations, we want to clarify that we have a broader focus than the specific adidactical situations that Brousseau describes. In independent situations, the teacher can also be in dialogue with the students and groups in small delimited periods of time but without taking control of the processes.

Cognitive conflicts

In addition to inquiry-based teaching, another learning perspective indicates that cognitive conflict is an instructional strategy to promote students' conceptual change. The notion of "cognitive conflict" is in line with Piaget's theory on cognitive development (Piaget, 1977). According to Piaget, a learner constructs new knowledge when he or she encounters input from the environment, and when the new assimilated information conflicts with previously formed mental structures, the result is a cognitive conflict that motivates the learner to seek change and develop new mental structures. It represents the learner's adaptation to the environment's input. Inconsistency between newly realised information and the students' conceptualization of previous experience will therefore present a potential cognitive conflict. However, such an experience will only become a cognitive conflict when explicitly invoked by the students, because the students may simply dismiss or treat this conflict as an exception (Zazkis & Chernoff, 2008). A cognitive conflict is only invoked when a learner accepts that there is a contradiction that needs to be dealt with. The alternative is that these two contradictory theories continue in the students' understanding as two parallel theories as the student fails to see that there is a conflict.

Cognitive conflicts are described in mathematics education, where Tall (1977) stated that understanding in mathematics often occurs in significant jumps that are accompanied by a clear sense of comprehension rather than a smooth and steady process; Tall (1977) used cognitive conflict to explain that after meeting these conflicts, the learner must restructure his or her mathematical schema and restructure them in an appropriate manner. The starting point of the process of reshaping and restructuring this new knowledge is the students' naïve knowledge (Dreyfus et al., 1990). Indeed, implementing a cognitive conflict as a way of teaching has been reported in a variety of topics in mathematics education (Zazkis & Chernoff, 2008).

Zazkis and Chernoff (2008) described an example where cognitive conflicts are used to help students face their misconceptions. In one of their examples, a student is asked to simplify $\frac{13 \cdot 17}{19 \cdot 23}$. The student starts by multiplying the numbers in the numerator and the denominator and get $\frac{221}{437}$. The student now starts to search for common factors in the numerator and the denominator, but after a while concludes that ”two prime numbers multiplied by each other are primes” (Zazkis & Chernoff, 2008, p. 200). The teacher then uses a pivotal example by asking if 15 is a prime. This makes the student acknowledge her misconception and disproves her suggestion.

The *Quality in Danish and mathematics project*

The observed class participated in a design-based, developmental and random controlled intervention experiment called *Quality in Danish and mathematics* (KiDM) (Hansen et al., 2019). The aim of KiDM was to make mathematics teaching in Danish compulsory schools more inquiry based. The overall KiDM programme had several stages that started with surveying the literature on inquiry-based teaching in mathematics and conducting some qualitative interviews with teachers and supervisors (Dreyøe et al., 2017; Michelsen et al., 2017) and then developing an inquiry-based teaching programme for a four-month mathematics teaching approach for both primary school year four and year five, which afterwards was implemented in 145 classes. All the activities in the intervention were all built around a simple three-phase model, where 1) the teacher introduced a problem/investigation, 2) the students themselves made an inquiry with minimal guidance from the teacher – the independent situations – and, 3) the activity ended with a whole-class discussion. The KiDM project was created as a random controlled trial (RCT) where approximately the same number of control schools and intervention schools were randomly selected with respect to their geography, size and ethnicity (Hansen et al., 2019). At the control schools, the intervention was not carried out and the intention was that they should continue their normal teaching; however, the control schools received the same before and after tests as the intervention schools. The programme also encompasses collecting qualitative data that included classroom-video observations from the intervention schools; it is these observations that form the basis of the present study.

Observations from the KiDM experiment

The observations used in the current study were collected in one year five intervention class. The class was randomly selected to be part of the KiDM project and chosen to be observed because their teacher volunteered at a kick-off meeting for the project. It was important that the teacher volunteered as cooperation was needed because the intention was to observe the class once a week over four months (16 lessons). It is important to emphasise that this teacher, therefore, was not selected because of his specific abilities or special interest in inquiry-based mathematics. However, because of different practical issues in the class, the final number of lessons recorded in the class was seven double lessons. All the observed lessons were transcribed in full by two preservice teachers and the first author of this article.

In the observed lessons, the camera followed one group of students, and the group was chosen in collaboration with the teacher with the intention of the group being robust and not immediately disturbed by the camera. The current paper presents one double lesson. This specific lesson is presented as it clearly shows the students' independent work with different representations and manipulatives and allows us to see an example of how these representations relate to a cognitive conflict.

Presenting one double lesson

The observed lesson from the KiDM project follows the three-phased progression envisaged. The activity used can be found on the KiDM website (KiDM, 2017). The lesson will first be described with specific quotes from the teachers and the students and afterwards analysed.

The case of "rope-triangles"

This activity focused on conducting a systematic examination of different triangles. The activity was also used in other classes and analysed in the Danish part of PRIMAS (Artigue & Blomhøj, 2013). The intention was for the students to find all the possible triangles (whole number on the sides) with a rope of a circumference of 12 metres (and a knot tied at each metre); in this process, the students should learn about the triangle inequality. The triangle inequality states that in any triangle, the sum of the length of the two shorter sides is always larger than the length of the longest side.

In this case, the teacher introduced the double lesson by talking about different kinds of triangles (right-angled triangles, obtuse triangles, acute triangles, equilateral triangles and isosceles triangles). The key problem of the lesson was then introduced to the students by first asking them how many integer triangles they think they can make with a 12-metre-long rope. The students were then invited to make guesses that the teacher wrote on the blackboard, after which he or she briefly introduced the students to how they could subsequently draw the triangles in GeoGebra. The situation where the teacher facilitates a guessing process is not an independent situation, but it is the first step towards the students subsequently having to examine and for themselves whether their guess is correct.

The students then were handed the ropes in groups of four or five and started making different kinds of triangles outside their classroom. An example of students doing this activity is shown in figure 1.

Figure 1. *Students investigating rope triangles* (*picture from* www.kidm.dk)

These introductions are in line with the teacher guide. The assumption in the teacher guide is that the students, by using the rope, will discover that it is not possible to construct a triangle with a side length of seven or more metres. Furthermore, the guide assumes that it is not until the students must draw the triangles into GeoGebra that they will discover that it is not possible to draw triangles like 6-3-3 and 6-4-2, because in most cases the rope can stretch a little on one of the sides, while it may be less stretched – curved downwards – on another side. So, it looks like a triangle though it is very flat. In GeoGebra, on the other hand, it is not possible to form these triangles; they will constitute a straight line.

The students Ella, Alma and Nikolaj quickly found that by standing in different formations, six different ”triangles” can be made: (4-4-4; 2-6-4; 2-5-5; 6-1-5; 6-3-3 & 5-4-3). In the process, the group tried to see if it is possible to make a triangle where one side is seven metres long, which they quickly reject because the rope cannot be stretched this way. They then went back to the classroom, and without any intervention from the teacher, they started to use GeoGebra to check if it was possible to draw these triangles. Ella drew some triangles in GeoGebra, but when drawing the ”triangle” 1-5-6, she experienced some problems. However, she quickly announced that: ”they all work”. Ella's figure can be seen in figure 2.

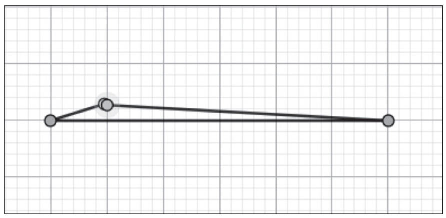

Figure 2. *Ella's drawing of the ”triangle” 1-5-6*

The teacher afterwards pointed at the 1-5-6 triangle and asked her, ”Are you sure that this is a triangle?” Ella answered, ”Yes because it has three sides”. When Nikolaj tried to make the 1-5-6 triangle in GeoGebra, he said, ”That is not a triangle”. Ella answered him by saying, ”But it has three sides”. The third student (Alma) announced, ”It is very ugly – I am not able to make this triangle”. The conversation continued as follows.

Nikolaj: Try to see mine!

Alma: That one is also very ugly.

Ella: [looking at her own triangle] Okay, maybe now it fits better.

Nikolaj: [asking loudly to their teacher] We are finished, what should we do now?

Ella: I would say that it is a triangle because it has three sides but wait a second ... Wait a second ... It does not fit. No, Nikolaj wait, it is not a triangle because these two (points at two sides) are not long enough to come together. That is why I don't think it is a triangle. Those two together have the sum of six, and that is why it is not a triangle, so Nikolaj, it does not fit. There are only four triangles.

Nikolaj: But what about this drawing?

Ella: But it does not fit because this small space here shows that because 1 + 5 this is 6, and 6 + 6 is 12 or 5 + 1 this is 6, so it does not fit because then they are equal.

Nikolaj: Arhh okay!

Alma: But then, we need to find new triangles.

Ella: Yes, and 7 does not work still. The number cannot be 6 or bigger than 6 because then, it does not fit at all, but I do not understand why that one fits [points at the 6-4-2 triangle].

Nikolaj: What?

Ella: I do not understand why that one fits – this one with 6-4-2 – when the other one does not fit [Ella draws the triangle in GeoGebra again]. But Nikolaj, try to see here – this one does not fit either.

Afterwards, the group tried to find new triangles by using GeoGebra, but they did not have any success.

Ella: If you started with 2 as the first line, then there could be one with 4, so no, it can only be 5 and 5 – we have that one. Okay, then we go on and take 3 as the first line and then 4 and 5, and we also have that one. 3-2 and then there must be 7, but it cannot be done. 3 and 5 and then the last is 4 – we have made that one. Then, we move on to 4 for the first line and then 6 and 2, but it cannot be done either. Then, there are 4, 5, and 3, which we have. And 4 and 2 do not work. And 4 and 4 and 4, we have that one. Then, 5 as the first line and 2 and 5 – yes and then 5-4-3 we have. We cannot find any more. And if we get to 6, then it doesn't fit anymore.

The lesson ended with a very brief summary where the teacher comments on the 6-3-3 and 6-4-2 triangles and starts to present the triangle inequality in an informal way.

Different cognitive conflicts in the students' reasoning processes

After a thorough introduction, the students believed that they could construct many different triangles with the rope, but after the students experimented with the rope, they realised that this was not the case. This was done in a setting where the students were outside the classroom in the schoolyard and worked independently in their own in a group, with no involvement from the teacher.

The activity with the rope showed the students that it is not possible to make a triangle with a circumference of 12 that has a side of seven or larger because then, the rope cannot be stretched out any further. In this situation, the rope visually helped the students identify this cognitive conflict. Here, there is a conflict between what the students thought (all triplets x, y, z where $x + y + z = 12$ are possible; see step 1 in figure 3) and what the rope actually showed them (a side of seven or larger is not possible; see step 2 in figure 3). It was, however, still possible for the students

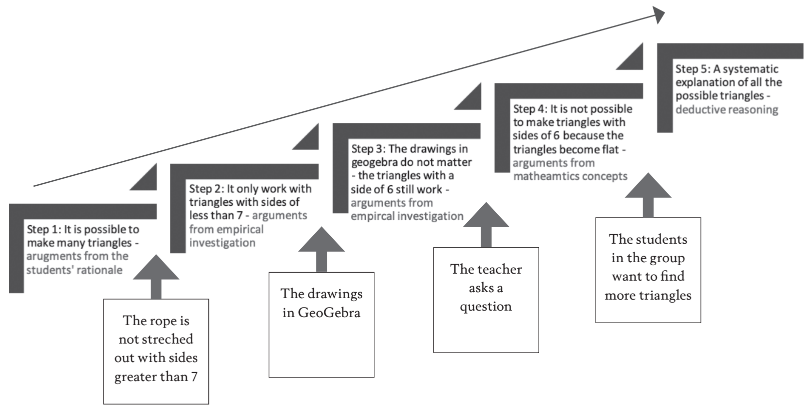

Figure 3. *The reasoning process in the rope-triangle activity*

to make a 1-5-6 ”triangle”, a 6-4-2 ”triangle” and a 6-3-3 ”triangle” with the rope, even though it looked a little flat. The students' work with the rope does not make it clear that this is impossible, and the students were still not aware of the triangle's inequality. However, when the students then worked in the GeoGebra programme, some of them became more suspicious and sceptical; the 1-5-6 ”triangle” did not fit very well, and ”it looks ugly”, but they still – after some discussion – concluded that the triangles with a side of six still work (see step 3 in figure 3). In this situation, the cognitive conflict that was intended to arise when trying to construct the ”triangles” in GeoGebra and they did not appear, at least in the first case. The teacher, however, observed this and intervened by asking Ella if she was sure that the 1-5-6 triangle was a triangle. Initially, this does not trigger any response from Ella, but after a while, by studying the drawings in GeoGebra again, she suddenly realised that the triangles with a side of six do not work either. Her arguments this time were not based on the rope or GeoGebra drawings but rather on the fact that the two sides ”1” and ”5” summed up to the same length as ”6” – the other side of the triangle – and that means the sides coincided. This argument involved more formal mathematics in the understanding of the triangle inequality (step 4 in figure 3) when she argued that the sum of the two sides became the same as the last side (1 + 5 is 6 and then they were equal). Afterwards, the rest of the group now wanted to find more triangles, and these initiatives led Ella to finally come up with a systematic argument for how many triangles they could find with this rope, explicitly articulating the triangle inequality (step 5 in figure 3).

Figure 3 illustrates the steps in Ella's reasoning process, with the different elements affecting the process exemplified by blue arrows. Ella's initial understanding was that it is possible to make many different triangles and that there were no geometrical constraints apart from the fact that the circumference should be 12. However, based on the interactions with the environment, Ella realised that the triangles' sides should be less than seven. Later – because of the interactions with GeoGebra – the conflict became overt, and the students discussed if 1-5-6 was actually a triangle. Helped by the teacher, Ella realised the triangle's inequality, and because she needed to explain her reasoning to the group, she finally constructed an argument for the number of whole number triangles with a circumference of 12.

The example shows that the cognitive conflict that exists in the analysed episode is productive and important in relation to the students' mathematical reasoning processes. In this sense, the cognitive conflict is the driving force for the students' reasoning processes, where the environment has the role of retaining the conflicting positioning, and making them available for discussion and scrutiny. The process of resolving a cognitive conflict is – at least in the examples provided here – a process stretched over time and one that does not necessarily entail large, significant jumps in the students' understandings but rather small steps towards resolving the conflict (steps 1 to 5 in figure 3). We suggest that it makes sense to view these small cognitive conflicts as elements in the students' reasoning processes.

Cognitive conflict can support reasoning

In the analysed situation, Ella experienced a cognitive conflict between the manifestation of the triangle's inequality as ropes and as part of the GeoGebra mathematical environment. This situation spurred Ella's mathematical reasoning because her initial rationale (i.e., it is possible to create all sorts of triangles) was confronted with reality: first in the situation with ropes, where she was able to accommodate her rationale to the empirical situation without creating a cognitive conflict, and later in relation to GeoGebra, where her rationale became difficult to maintain. In this specific situation, Ella was able to experiment with and explore her (partly wrong) rationale (i.e., it is possible to create all sorts of triangles). Hence, the conflict with the mathematical reality expressed in GeoGebra is deeper than if she was confronted with just the triangle's inequality or the GeoGebra environment. This seemed to increase Ella's motivation for engaging in mathematical reasoning. Looking at the literature about reasoning in mathematics education, it has mainly focused on characterising the different types of argumentation and reasoning (Brousseau & Gibel, 2005; Harel & Sowder, 1998, 2007) and less so about what makes students engage in reasoning processes and how to make the transition from making an argumentation based on rationales and intuition to more deductive reasoning (EMS, 2011). We suggest, however, that Ella's inclination towards more deductive reasoning was spurred by a cognitive conflict and that deeper and more articulated conflicting understandings were the larger the internal motivation for addressing this conflict with mathematical reasoning.

A good environment retains a cognitive conflict

One of the things that makes Ella's experience particularly strong is that her wrong understanding/rationale was accommodated in the first exploration and her work with the rope. Therefore, Ella had a justified trust in her own rationale that made it more difficult – we can imagine – to give up this reasoning afterwards. By supporting the fact that Ella must retain and reinforce her rationale while also experiencing that it cannot be true, this pressed Ella to address the conflict. Students can typically and easily dismiss conflicting views as exceptions and disturbances that do not need to be taken into account (Zazkis & Chernoff, 2008); hence, the cognitive conflict must be explicitly invoked by the students. Therefore, the material and representations that support productive cognitive conflicts become more important because the students are more inclined to realise a conflict if the material and representations for conflicting views are present at the same time and if the students independently have explored these conflicting views. In the analysed episode, the conflicts were retained by the material aspects of the environment and activities, such as the rope exercise and the GeoGebra exercise. The material aspects of the environment used in the lessons played an important role in producing cognitive conflicts because the materials – here represented in a visual way – advocated for the conflict, and the feedback from the materials had different weight/power in the reasoning process. The feedback from the rope did not make it possible for the students to realise that the sides in the triangles could not be six or larger and neither did the exercise in GeoGebra, but all together, including the teacher's hint, these items forced Ella to resolve the cognitive conflict and understand the triangle's inequality.

Resolving cognitive conflicts nevertheless takes time. Ella experienced what can be described as a eureka moment when she realised the triangle's inequality. This, however, did not happen as a response only to the environment she was engaged in at the time (the GeoGebra activity). We suggest that this moment also can be credited to the work that Ella did with the ropes, where she experimented with and reinforced her ideas about all the triangles they could make. In this sense, the quick realisation of how a conflict can be resolved is due to all the work that goes into exploring, articulating and reasoning about the different understandings that were in conflict. In this sense, the manifestation of conflicting views that can lead to student reasoning becomes a central design parameter in inquiry-based mathematic education.

Discussion and implications

Today, there is a common understanding that there must be an immediate learning benefit from all activities that students work on in the classroom, as in, for example, the goal-oriented approach towards teaching (Hansen, 2015). This may entail that if the students do not arrive immediately at a solution, the teacher will interrupt and give them the answer. However, it also entails that teachers will not allow for complex – perhaps not immediately solvable – tasks in the lessons, which, in addition to suppressing a natural urge to investigate, also denies the students the chance to construct the answer themselves by using productive cognitive conflicts, which can take time to resolve. During the KiDM interventions, we experienced teachers who would not apply the triangle-rope activity because the students were able to make ”not triangles”, like the 3-3-6 triangle. They found this prevented the students from coming to understand the triangle's inequality because the students – at least for a short period – had an incorrect understanding. Therefore, it is necessary for mathematics teachers to be aware that resolving a cognitive conflict can include small steps and will take time. The teachers need to be patient before they interact and interrupt the students; otherwise, it will destroy the students' motivation for making their own inquiries and to be part of the reasoning process, possibly preventing a deep understanding of the target concept.

Additionally, it is important that the teachers plan ahead so that the material aspects of the environment advocate for resolving the conflicts. It is not necessary that the material aspects provide a definitive resolution, but they must advocate for a later resolution.

Conclusion

By analysing video observations from one year five primary school mathematics lesson where the teacher used an inquiry-based approach, we were able to study the students' reasoning process and characterise the students' cognitive conflicts. We found that productive cognitive conflicts are important to mathematical reasoning because they can be seen as the motor that helps the students in their transition from empirical to more formal deductive reasoning in inquiry-based teaching. In these cases, the environment will have an important role in retaining the conflicting positioning, making them available for discussion and scrutiny.

References

Artigue, M. (2012). What is inquiry-based mathematics education (IBME)? Background resources for implementing inquiry in science and mathematics at school. In M. Artigue & P. Baptist (Eds.), *Inquiry in mathematics education* (pp. 3–13). Fibonacci Project.

Artigue, M. & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM*, 45 (6), 797–810. doi: 10.1007/s11858-013-0506-6

Ball, D. L. & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 27–44). NCTM.

Baptist, P. (2012). Towards teaching and learning inquiry-based mathematics. In M. Artigue & P. Baptist (Eds.), *Inquiry in mathematics education* (pp. 13–25). Fibonacci Project.

Blomhøj, M. (2016). *Fagdidaktik i matematik*. Frydenlund.

Brousseau, G. (2006). *Theory of didactical situations in mathematics: didactique des mathématiques, 1970–1990* (19th ed.). Springer.

Brousseau, G. & Gibel, P. (2005). Didactical handling of students' reasoning processes in problem solving situations. In C. Laborde, M.-J. Perrin-Glorian & A. Sierpinska (Eds.), *Beyond the apparent banality of the mathematics classroom* (pp. 13–58). Springer.

Bruder, R. & Prescott, A. (2013). Research evidence on the benefits of IBL. *ZDM*, 45 (6), 811–822. doi: 10.1007/s11858-013-0542-2

Carpenter, T. P., Franke, M. L. & Levi, L. (2003). *Thinking mathematically: integrating arithmetic and algebra in elementary school*. Heinemann.

Cordis (2021). *The Fibonacci project*. European comission. https://cordis.europa.eu/project/id/244684

Dorier, J.-L. & García, F. J. (2013). Challenges and opportunities for the implementation of inquiry-based learning in day-to-day teaching. *ZDM*, 45 (6), 837–849. doi: 10.1007/s11858-013-0512-8

Dreyøe, J., Larsen, D. M., Hjelmborg, M. D., Michelsen, C. & Misfeldt, M. (2017). Inquiry-based learning in mathematics education: important themes in the literature. In E. Norén, H. Palmér & A. Cooke (Eds.), *Nordic research in mathematics education: papers of NORMA 17* (pp. 329–337). SMDF

EMS (2011). Do theorems admit exceptions? Solid findings in mathematics education on empirical proof schemes. *Newsletter of the European Mathematical Society*, 82, 50–53.

Engeln, K., Euler, M. & Maass, K. (2013). Inquiry-based learning in mathematics and science: a comparative baseline study of teachers' beliefs and practices across 12 European countries. *ZDM*, 45 (6), 823–836. doi: org/10.1007/s11858-013-0507-5

Hanna, G. & Jahnke, H. N. (1996). Proof and proving. In K. C. A. Bishop, C. Keitel, J. Kilpatrick & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 877–908). Kluwer Academic.

Hansen, R. (2015). At styre efter målet i matematik – hvad ved vi egentlig om elevers og læreres målorientering? *MONA*, (1), 7–23.

Hansen, T. I., Elf, N., Misfeldt, M., Gissel, S. T. & Lindhardt, B. (2019). *Kvalitet i dansk og matematik – et lodtræningsforsøg med fokus på undersøgelsorienteret dansk- og matematikundervisning. Slutrapport*. UCL. https://laeremiddel.dk/wp-content/uploads/2020/01/Slutrapport-Kvalitet-i-dansk-og-matematik.pdf

Harel, G. & Sowder, L. (1998). Students' proof schemes: results from exploratory studies. In A. H. Schoenfeld, J. Kaput & E. Dubinsky (Eds.), *Research in collegiate mathematics education III* (pp. 234–283). AMS.

Harel, G. & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805–842). Information Age.

KiDM (2017). *Kvalitet i dansk og matematik* [Quality in Danish and mathematics]. UCL. http://kidm.dk/matematik/laerer/

Krainer, K. & Zehetmeier, S. (2013). Inquiry-based learning for students, teachers, researchers, and representatives of educational administration and policy: reflections on a nation-wide initiative fostering educational innovations. *ZDM*, 45 (6), 875–886. doi: org/10.1007/s11858-013-0537-z

Larsen, D. M., Hjelmborg, M. D., Lindhardt, B., Dreyøer, J., Michelsen, C. & Misfeldt, M. (2019). Designing inquiry-based teaching at scale: central factors for implementation. In U. T. Jankvist, M. Van den Heuvel-Panhuizen & M. Veldhuis (Eds.), *Proceedings of* CERME11. Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Michelsen, C., Dreyøe, J., Hjelmborg, M. D., Larsen, D. M., Lindhardt, B. K. & Misfeldt, M. (2017). *Forskningsbaseret viden om undersøgende matematikundervisning*. Undervisningsministeriet

Maaß, K. & Artigue, M. (2013). Implementation of inquiry-based learning in day-to-day teaching: a synthesis. *ZDM*, 45 (6), 779–795. doi: 10.1007/s11858-013-0528-0

Maaß, K. & Doorman, M. (2013). A model for a widespread implementation of inquiry-based learning. *ZDM*, 45 (6), 887–899. doi: 10.1007/s11858-013-0505-7

PRIMAS (2013). *Primas Project*. Utrecht university. https://primas-project.eu/

Schoenfeld, A. H. & Kilpatrick, J. (2013). A US perspective on the implementation of inquiry-based learning in mathematics. *ZDM*, 45 (6), 901–909. doi: 10.1007/s11858-013-0531-5

Stylianides, G. J. & Stylianides, A. J. (2008). Proof in school mathematics: insights from psychological research into students' ability for deductive reasoning. *Mathematical Thinking and Learning*, 10 (2), 103–133. doi: 10.1080/10986060701854425

Tall, D. (1977). *Cognitive conflict and the learning of mathematics*. Paper presented at PME 1, Utrecht. http://www.warwick.ac.uk/staff/David.Tall/pdfs/dot1977a-cog-confl-pme.pdf

Zazkis, R. & Chernoff, E. (2008). What makes a counterexample exemplary? *Educational Studies in Mathematics*, 68 (3), 195–208. doi: 10.1007/s10649-007-9110-4

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