Fostering an intimate interplay between research and practice: Danish "maths counsellors" for upper secondary school

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The gap between research and practice is a well-known problem (and topic) in educational research, and not least in mathematics education research. This empirically based article discusses the effects of a particular attempt to foster a closer connection between research and practice by involving mathematics education research findings in the activities of selected Danish upper secondary school mathematics teachers, who (have) take(n) part in a research-based so-called "maths counsellors" in-service teacher programme. A key aspect of the programme is to make the activation of research findings a mere necessity for significant aspects of the maths counsellors' practice. To illustrate the teachers' research-based work, the article presents six characteristic examples from the implementation of the programme, i.e. authentic examples of how prospective maths counsellors have identified students with mathematics specific learning difficulties, have diagnosed the nature of these difficulties, and how they have designed interventions to help the students overcome them. A discussion of how these activities draw upon and are grounded in mathematics education research findings serve as a basis for evaluating the "model" behind this further education programme.

> *In theory there is no difference between theory and practice, but in practice there is*
>
> (attributed to the American baseball player Yogi Berra)

Within education in general and in mathematics education in particular, it is a classical problem to establish meaningful and effective connections between research and practice (e.g. NCTM Research Committee, 2006; Cai et al., 2017). What is well known, however, is that the usual "filling-up model" where teachers are merely told what research has to say

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about this or that, does not work; the connection neither becomes clear enough nor applicable for the teachers. Knowing this, several attempts have been made within mathematics education to involve mathematics teachers in research activities in order to create links between research and practice. One such example was the French IREM network (Douady & Henry, 1995). Another example is the Italian "nuclei of research" (Bartolini Bussi, 1998). In both cases the relationship between research and practice consisted of teachers collaborating with researchers on specific research projects for a shorter or longer period of time, implying that the participating teachers would thus become a kind of researchers themselves, at least for a while. Other pertinent examples can be found in Scandinavian countries. Suffice it mention the professional development work done for a lengthy period of time at Kristianstad University (Grevholm, 2001) and the "Learning communities in mathematics" project at University of Agder (Jaworski et al., 2007). However, one would think that it should be possible to involve and exploit research outcomes in teachers' activities without necessarily implicating the teachers in the research activities proper.

The present article presents and discusses a model, designed by the authors, for doing exactly this. Or to be more precise, the model itself is introduced and described in an article in *Educational studies in mathematics* to which the reader is referred (Jankvist & Niss, 2015). The model – "a research-based *maths counsellor* teacher programme" – was implemented at Roskilde University, Denmark, in 2012 and has been in function since then.[1] This means that lots of experiences have been harvested so that we can now consider the following question:

> To what extent does the "maths counsellor model" actually bridge the gap between research and practice in mathematics education?

In the present article, we attempt to answer this question by identifying and analysing strengths, difficulties and challenges which we have encountered with this model of involving mathematics education research findings in selected Danish upper secondary school mathematics teachers' activities. A key point in the model is to make the activation and use of research findings a mere necessity for significant aspects of the teachers' practice.

Below, we first provide a brief outline of the "model" itself and the outcomes it has given rise to so far. Next, we offer six authentic examples of teachers' use of mathematics education research findings as part of their practice. These teachers all completed the programme. The selected examples stem from the teachers' written reports, which they have handed in as part of the programme. Finally, we discuss the affordances and challenges encountered with the model at issue. This implies that this article is definitely empirical in nature, as it relies heavily on empirical data stemming from a development project, but it is not meant to be an "ideal-typical paper" in the sense of Niss (2019), because it does not present an experimental study attempting to answer a traditional research question.

The "maths counsellor" in-service education model

The maths counsellor in-service teacher programme for upper secondary school mathematics teachers at Roskilde University runs part time over three semesters (in total 30 ECTS – European Credit Transfer and Accumulation System), during which the teachers ideally – albeit not always in practice – have a reduced teaching load in their schools. Each semester has an overarching theme:

1 concepts and concept formation in mathematics;

2 reasoning, proofs and proving in mathematics;

3 models and modelling in mathematics.

We chose these themes both because they are significant and pertinent to upper secondary mathematics education in Denmark, as is spelled out in the national curriculum documents, and because they epitomise key aspects of the eight mathematical competencies in the Danish KOM-project (Niss & Højgaard, 2011; 2019), which constitutes a major part of the theoretical foundation of the maths counsellor programme.

The teachers' work in each semester is structured in three different phases: to *identify* (i.e. detect and select) students with genuine learning difficulties in mathematics; to *diagnose* the learning difficulties of the student(s) identified; and finally to undertake *intervention* according to the diagnosis arrived at with respect to the individual student. For the first phase, the teachers are equipped with a theme-specific so-called "detection test" consisting of questions and tasks administered to the students in relevant classes at their schools (for examples see Jankvist & Niss, 2017). The tests are developed by us and are informed – but not entirely determined – by research literature regarding the specific theme. The purpose of the test is to assist the teachers in detecting students with possible mathematics specific learning difficulties and to help to identify students who would need and benefit from counselling. Each teacher identifies 1–4 students with mathematics specific difficulties, and these students are offered math counselling with the aim of rectifying or reducing the observed difficulties during the semester. It should be noticed that only students who actively wish to be part of the project, and who are willing to invest the needed efforts, are selected. This is to say, the purpose of counselling is not to motivate unmotivated students, but to assist those who work hard in mathematics on a daily basis but do not succeed in their endeavours. In the diagnostic phase, the participating teachers – strongly assisted by the research literature they read as part of the programme (see Jankvist & Niss, 2015) – employ self-constructed tasks, interviews, *etcetera*, to come to grips with the nature and origin of the students' mathematical learning difficulties. Oftentimes, the first – preliminary – diagnosis has to be revised along the road, e.g. because the difficulties turn out to be more deeply rooted than assumed at first. Finally, taking the diagnosis as the point of departure and – again – with support from the research literature and supervision by us, the teachers design and implement an intervention scheme for the students selected. The intervention scheme also includes steps which enable the counsellors to "measure" in what respects and to what extent the intervention has worked as anticipated for the selected students.

For each semester, teachers in groups of 2–3 that remain stable during the three semesters write up a report. During the semester they receive supervision from us. At the end of each semester the teacher groups present their work and their reports to all the other groups in order to receive feedback from their peers as well as from us. After the completion of the third semester, all three reports are combined into one along with an introductory chapter. This final report forms the basis for a final oral exam at the university. The teachers who pass receive diplomas as certified maths counsellors. With this diploma in hand, the teacher is ready to undertake the function as maths counsellor to students and colleagues back home in his or her secondary school.

Before we continue, it should be mentioned that Danish upper secondary school typically takes three years, corresponding to grades 10–12 (ages 16–18/19) (even though there is, in fact, also a two-year option called "hf"). The three-year upper secondary school is organised in three different streams, a general academically oriented stream, called "stx", a technology- and science-oriented stream, called "htx", and a commerce and business-oriented stream called "hhx". Previously, all students had to study mathematics for at least one year (C level), but due to recent reforms, almost all students must now study mathematics for at least two years (B level). A segment of the student population study mathematics for all three years (A level). In principle, a given level is supposed to be the same across different streams, but, as there is diversity in content and approach in the three streams, they should not be taken to be identical, even though the official level label is the same. After completion of the mathematics programme, students take a national exam in mathematics according to their chosen level.

Outcomes up till now

In addition to the substantive outcome of the programme in terms of identification of students, diagnoses of the nature of their mathematical learning difficulties, and intervention measures to counteract the difficulties encountered, the programme has also had some other important ramifications, which we shall briefly outline here. First, it soon became clear to participants that it would be desirable to establish a formal framework for exchange and discussion of information, experiences, and deliberations amongst the maths counsellors, who all work alone or with one other colleague in their respective schools, which are scattered across the country. Therefore it was decided to establish an *association* of the maths counsellors who have successfully completed the programme – and their supervisors – as a forum for such exchange and discussion. The primary outlet of the work of the association is *an annual meeting* for all members, held in May in conjunction with the final residential course of the second semester of the programme. On that occasion we have had the pleasure and privilege to welcome an internationally renowned mathematics education researcher to meet the counsellors and to give a talk, and sometimes a workshop as well, on a topic pertaining to the themes in the programme. Thus, Helen Doerr gave a talk at the first meeting in 2014, Carolyn Kieran in 2015, Tommy Dreyfus in 2016, Morten Blomhøj in 2017, Raymond Duval in 2018, Lieven Verschaffel in 2019. The meeting in 2020 had to be cancelled because of the corona virus pandemic.

Already whilst working with the first cohort of prospective maths counsellors it became clear to us that their developmental experiences and findings deserved to be made available to their colleagues and other mathematics educators in the Danish education system. We therefore decided to edit a book, the chapters of which were to be written by the maths counsellors based on some of the cases they had dealt with in their work during the three semesters, with an introduction written by us (Niss & Jankvist, 2016). The publishing house soon agreed to publish a second and a third book as well (Niss & Jankvist, 2017; in press), consisting of chapters by the second, third and fourth cohorts of maths counsellors. [2] The two first books also appeared in Swedish translations (Niss & Jankvist; 2019a; 2019b).

Finally, as part of the three phases of the maths counsellor programme participants collect huge amounts of data from the students involved. At the time of writing we have data from far more than a thousand students. Of course, these data have already been utilised by the prospective maths counsellors as part of the reporting of their work. However, the data are so rich that they easily lend themselves to a broad spectrum of fundamental *empirical research* on mathematics specific learning difficulties. We have published articles based on these data (Jankvist & Niss, 2017; 2018; 2019; in review) and are in the process of conducting a number of further research projects on that basis. In that way, the participating teachers' work and data feed directly into the research process, which constitutes yet another way of creating an intimate interplay between research and practice.

On the selection of the authentic examples

As mentioned, in order to illustrate how the maths counsellors are informed and inspired by and draw upon mathematics education research concepts, results and findings, we present a selection of authentic examples from their written reports and/or the published book chapters. More precisely, we present six different examples – two from each of the published books – chosen in such a manner that each of the programme's three themes (cf. earlier) is represented with two examples. To some extent, the presentations here make use of our chapter summaries provided in the introductory chapters of the books. After the presentation of the examples, we discuss the role served by the mathematics education research results and theoretical constructs in each of the examples of the work of the maths counsellors.

Example 1. "The Amalie interpretation"

Two prospective maths counsellors, Klaus B. Pedersen and Anders Torp (2016), used the constructs of concept definition and concept image (Tall & Vinner, 1981; Vinner & Dreyfus, 1989) to investigate how distorted concept images of the concept of function may cause such a rickety understanding of functions that it almost endangers students' overall mathematical learning. Through a longer series of so-called "lunch sessions" involving diagnostic aspects as well as interventions, with selected third year students (grade 12), the teachers discovered a peculiar phenomenon, which they had not seen described in the literature. One student, given the pseudonym "Amalie", turned out to possess the clear and outspoken misconception that in the linear (or more correctly: affine) relation $y = ax + b$, a denotes the value of x, while b denotes the value of y, and that the graph for the relation hence is the single point with coordinates (a, b) – i.e. not a straight line.

Having observed this phenomenon a number of times, the teachers decided to screen 417 newly accepted first year students in their school. It turned out that 9 % of this population had the very same perception, while another 11 % possessed variants thereof. Neither Amalie nor her classmate Julie, who also took part in the sessions, viewed x and y as being mutually interrelated by the given equation. Furthermore, Julie did not view mathematical conclusions as following from some premises, but as a set of arbitrary rules made up by "some philosophers, or what they're called?". But all in all, it was the distinction between concept definition and concept image that was the primary driver for the two teachers to discover and intervene against this fairly widespread misconception – now oftentimes called "the Amalie interpretation" among Danish mathematics educators – of the constituting elements of the graph of a linear function. And it was the very same distinction that enabled them to assist the students in the "lunch sessions" to eventually pass their mathematics exam.

As an interesting addendum, it can be mentioned that two upper secondary mathematics teachers from the fourth cohort of maths counsellors worked with a student who turned out to possess what may be referred to as "the reverse Amalie interpretation." When this student was asked to provide an expression for the straight line passing through the points (0, 0) and (1, 3) in a coordinate system, he simply wrote down: $y = x + 3$. The two prospective maths counsellors interpreted this as the student looking for the values of a and b in $y = ax + b$, believing himself able to read this directly from the set of coordinates, completely disregarding the point (0, 0) and its influence on the equation (Henriksen & Christiansen, 2016).

Example 2. Understanding fractions

Christian Christiansen and John Sorth-Olsen (2017)[3] applied constructs from the American *The rational-number project* (RNP) (e.g. Cramer et al., 1997)[4] to assist a student in developing a much better understanding of fractions than the pretty limited one she actually possessed. Based on the questions related to number sense in the 1st semester detection test, the teachers had singled out a student with particular difficulties related to fractions. Amongst other things, the student believed 13 / 4 to be larger than 13 / 3, and she was not able to transform the number $13\,{}^2\!/_3$ into a decimal number representation. As part of the diagnostic interview, the student was asked to list four given rational numbers in descending order, with the following result: 5/8; 0.99; 0.48; 14/13.

RNP introduces and highlights seven different aspects of the concept of rational number: *fractional measure/part-whole; ratio; rate; operator; quotient; linear coordinate;* and *decimal*. Through a number of specially designed diagnostic tasks and follow-up interviews, the teachers were able to uncover that the student's difficulties could be related especially to a lack of understanding rational numbers as "part-whole", "ratio", and "operator" (in certain contexts at least), and in particular also to the "linear coordinate" and "decimal" aspects of the concept. So, the student experienced some amount of difficulty in five of the seven aspects of rational number sense as laid out by the RNP.

Taking Sfard's (1991) hierarchical model of mathematical concept formation (interiorisation, condensation, and reification) as their basis, the teachers designed a targeted intervention focussing on the relationship between the process and object aspects of the concept of rational number (and fraction). In particular, the intervention focused on the role and meaning of the numerator and the denominator in a fraction as well as on the translation between a part-whole representation, e.g. a circular diagramme, and a linear coordinate representation. The intervention turned out to be successful on the students' behalf. In the following semester, the same student was the subject in an intervention on mathematical reasoning in problem solving. The teachers reported that the two interventions seemed to provide the student with more "mathematical self-confidence" (not only related to fractions). As a matter of fact, the entire enterprise turned out to be a bit of a success story, since the student ended up receiving the highest grade (12 on the Danish scale which translates to an A) at the final oral and the second highest (10 which translates to a B) in the final written exam at the mathematics level B in the upper secondary school mathematics programme.

Example 3. Reasoning and students' beliefs

Jesper Nymann Madsen and Tina Vejsig Nielsen (in press)[5] took as their point of departure Flavell's (1979) notion of meta-cognition, taken from psychology. The authors linked this notion to a selection of theoretical constructs from mathematics education research, e.g. semiotic registers (Duval, 2006), socio-mathematical norms (Yackel & Cobb, 1996), and not least mathematics-related beliefs (e.g. Op't Eynde, de Corte & Verschaffel, 2002), doing so in the context of students' difficulties related to mathematical reasoning, proofs and proving (the theme for the second semester of the programme). Also the KOM-framework's (Niss & Højgaard, 2011; 2019) reasoning competency is relevant for the identification and analysis of the students' difficulties as well as for considerations on how to intervene.

We meet the second year student Mette, who – due to her beliefs that mathematical proofs do not necessarily generate truth – is willing to accept and defend (!) – that "all numbers are equal to 0" since she has been provided with a mathematical "proof" of this claim (in a detection test 2 item, where the illegal operation of dividing by 0 is performed in disguise). Apart from this, Mette is largely driven by empirical proof schemes (Harel & Sowder, 2007). Drawing on Bell's (1976) use of so-called proof-trees, the prospective maths counsellors designed an intervention that assisted Mette not only in relation to developing her mathematical reasoning competency, but also to "clean up" some of the mathematics related beliefs that impeded her learning of the subject.

We also hear about the first year student Sonja, who experienced upper secondary school mathematics as something crucially different in nature to what she had previously been exposed to in primary and secondary school. The teachers designed an intervention, relying, once again, on Bell's proof-trees, but also on Pimm's (1987) work with the language aspect of the acquisition of mathematics. This assisted Sonja in understanding, among other things, elements of representational shifts between different semiotic registers (Duval, 2006) as part of a mathematical reasoning process.

In both interventions, directed towards Mette and Sonja, respectively, the teachers applied metacognition as a tool to make the students explicitly reflect upon their own activities and learning in relation to their work with mathematical reasoning and proving.

Example 4. Integration and reasoning

Ama El-Nazzal, Lone Stilling Karlsen and Mette Juhl Christensen (in press) took their point of departure in the manifest and well-known transition problems between the different mathematics programmes and levels in the Danish education system. Yet, whilst Madsen and Nielsen (in press) in the case of Sonja considered transitions between lower and upper secondary school, these authors addressed the transition problem between level C and level B, i.e. a problem within upper secondary school.

We follow Student S and Student M, both of whom experienced the transition from level C to level B as challenging – not due to the involved change of teacher but rather to the manifest increase in mathematical demands and also to the fact that they found the textbook more difficult to follow. Both students, however, persistently showed up to class and put a fair amount of effort into their work. Yet, they both experienced severe difficulties in relation to mathematical reasoning and proof. Whilst Student M actually perceived mathematical proofs as being an integral part of the discipline of mathematics, Student S rather believed mathematics to be a "do-task-and-find-answer" kind of discipline. The prospective maths counsellors addressed these observations through the notion of belief (e.g. Jankvist, 2015; Schoenfeld, 1992) and by attempting to change the standard didactical contract (Brousseau, 1997).

Upon performing a classroom based diagnosis of the entire second year class' mathematics related difficulties concerning reasoning and proof, the class was exposed to a classroom based intervention. This intervention addressed reasoning, proofs and proving within the area of infinitesimal calculus, and was designed with inspiration from Freudenthal's *Realistic mathematics education* (RME) (e.g. Treffers, 1993). The positive effects of the intervention were measured in terms of "didactical contract", "sociomathematical norms" (Yackel & Cobb, 1996) as well as of Duval's (2006) discussion of conversion between different semiotic registers.

Example 5. The modelling game

The two teachers Jørgen C. Ebbesen og Signe K. Mengel (2017) [6] based their intervention design for all three semesters on one and the same idea inspired by theoretical constructs by Brousseau's (1997) and Duval (2006). More precisely, building on Brousseau, the teachers designed mathematical "games" in the form of puzzles. In the first semester, the teachers designed a "letter game", which consisted of 18 pieces with different algebraic expressions. The game consisted in having the students carry out 9 simplification tasks by matching the pieces in the puzzle, e.g. $a^2 + b^2 + 2ab$ with $(a + b)^2$. In the second semester, the teachers used Duval's theoretical construct of semiotic registers, which among other things offers a potential explanation of students' difficulties in shifting between different representations of mathematical concepts and objects. The purpose of this game was to make the students do exactly this, i.e. engaging in shifts between different representations of, say, linear functions.

We shall, however, focus more on explaining the "culmination" of the teachers' idea of designing games, which occurred in the third semester with the so-called modelling game. In this game the students were to work with 9 "families" of mathematical representations related to mathematical models and modelling. One such family was on the story of continuingly halving piece of A4 paper – which made up one piece of the "puzzle". Another piece provided the description: "Every time the x value increases by 1, the y value decreases by 50 %." A third piece showed a graph of a decreasing exponential function, and a fourth and final piece stated a differential equation, supposed to model the situation.

The prospective maths counsellors illustrated how these games – besides functioning as classroom based or small group interventions – also can provide opportunities for uncovering the students' difficulties. An example was that several students believed the following two statements "... the *y* value decreases by 50 %" and "... the *y* value becomes 0.5 smaller" to be identical. Presumably this is to do with these students' perception of 50 % as a number, namely 0.5, rather than as an operator. In this respect, the modelling game can serve both the purposes of modelling as a goal and modelling as a vehicle for concept formation (Niss, 2008).

Example 6. Modelling and students' beliefs

Another prospective maths counsellor, Solveig Witting (2016), considered how students' mathematical beliefs may have either a constraining or a furthering impact on their work in mathematical modelling. The point of departure for this teacher's investigation was the experience that mathematical modelling, applications of mathematics, and real life contexts do not – contrary to what one might expect – automatically appeal to all students, not even to those who usually do well in mathematics. Modelling does not necessarily make them better in doing usual mathematics, or happier. The teacher put forward the hypothesis that this might stem from the students' beliefs about mathematics as a discipline. She employed Grigutsch' (1998) typology for students' mathematics-related beliefs. Grigutsch distinguishes between five types of belief: formalism-oriented; process-oriented; application-oriented; schema-oriented; and rigid schema-oriented. By studying two students as part of the diagnostic phase, the teacher was able to establish qualitative correlations between the students' mathematics-related beliefs and their modelling capabilities, as understood by carrying out the different phases of the modelling cycle (e.g. Ferri, 2006).

The first student – a first year girl, usually with a just below average performance, whose mathematics-related beliefs were application-oriented – was able to navigate in open situations concerning real life contexts and also to perform pre-mathematisation in a reasonable manner (cf. Niss, 2010), but difficulties emerged when the situation required specific mathematisation, which she found almost unpleasant. Aspects of the student's own mathematics-related beliefs simply appeared to collide with those implicitly underlying the teaching situation. The second student was a third year girl who usually performed just above average. Her mathematics-related beliefs were schema-oriented, with a strong focus on the involvement of standard methods (often used erroneously), on choosing formulas, on rote learning, etc. When such a focus was sufficient, she was happy with mathematics – but she could not do modelling! She did not take the phrasing of modelling assignments seriously but immediately began hunting for random, but typically irrelevant, formulae that somehow reminded her of something in the assignment. She was not at all able to handle the pre-mathematisation phase of the modelling cycle, for this cannot be schematised. Since both of these students actually performed around average, also in their written exams, the teacher concluded that the students' difficulties with mathematics cannot be explained solely by the lack of mathematical competencies. Hence, their mathematics-related beliefs seemed to play a significant role.

Maths counsellors' use of theoretical constructs

One observation to be made in relation to the authentic examples presented above concerns *when* and *how* the use of mathematics education theoretical constructs enters into the maths counsellors' practice.

The Danish KOM-framework (Niss & Højgaard, 2011; 2019) on mathematical competencies play a permeating role in almost all of the maths counsellors' projects, and often forms an underlying basis for their work with diagnosing students' mathematics specific difficulties across all three semesters of the programme.

In terms of mathematical concepts and concept formation (1st semester), the constructs of concept definition and concept image due to Vinner and colleagues (e.g. Tall & Vinner, 1981; Vinner & Dreyfus, 1989) as well as Sfard's (1991) tripartite hierarchy of mathematical concept formation (internalisation, condensation, reification) – as seen e.g. in examples 1 and 2 – often play a crucial role in the teachers' work to obtain the first narrowing down of the difficulties of the students. Although not evident from the examples presented above, Skemp's (1976) distinction (inspired by Stieg Mellin Olsen) between instrumental understanding and relational understanding also often play a role in the maths counsellors' initial diagnoses of the students' difficulties. Duval's (1996) description of transformations within and conversions between semiotic registers is another construct often applied by the maths counsellors. Due to its schematic nature, Duval's model of representational shifts not only functions well in diagnosing students' difficulties, but also in targeting interventions (e.g. example 5) as well as in measuring their effects (e.g. examples 3 and 4).

For mathematical reasoning, proofs and proving (2nd semester), the construct of proof schemes (external proof schemes, empirical proof schemes, and deductive proof schemes) by Harel and Sowder (2007), typically is an important instrument for the counsellors' diagnoses of the students' difficulties as well as for measuring the effect of an intervention (e.g. example 3). The notion of beliefs – those addressing the classroom context (e.g. Op't Eynde, de Corte & Verschaffel, 2002), those addressing aspects of problem solving (Schoenfeld, 1992), and those addressing aspects of mathematics as a discipline (Jankvist, 2015) – usually serve as a basis for trying to understand the students' actions in relation to the activity of proving in mathematics, also when it comes to finding out whether or not they understand the need for and functions of mathematical proof. The construct of sociomathematical norms (Yackel & Cobb, 1996) is often activated in explaining what a student sees as sufficient for putting forward a mathematical argument. On the other hand, as shown in example 4 it also becomes an objective for the maths counsellors to establish new sociomathematical norms as the"taken-as-shared" through an intervention in class. Example 3 illustrates how a construct from the literature, i.e. that of Bell's (1976) proof-trees, serves as a tool for the design of interventions.

As regards mathematical models and modelling, the construct of didactical contract (Brousseau, 1997) usually plays a key role in the work of the maths counsellors – as does also, to some extent, the notion of beliefs. This is due, not least, to the fact that many students believe that a fair mathematical task should have one and only one correct answer, which is certainly not the case in the building of mathematical models. Hence, some students find the very activity of mathematical modelling to violate the didactical contract and to jeopardise their image of what mathematics is all about. The modelling cycle (e.g. Blomhøj & Jensen, 2003; Ferri, 2006; Niss, 2010) typically serves as a means, both for pinpointing students' difficulties related to the different phases of the modelling process – in combination with Niss' (2010) notion of implemented anticipation – and for designing interventions.

In addition to the extensive amount of literature provided to the prospective maths counsellors during the programme (see Jankvist & Niss, 2015), the counsellors also find literature on their own. In example 2 on understanding fractions, the teachers needed more specialised and targeted literature than offered by the course readings, in this case because fractions are not explicitly on the agenda of the upper secondary school curriculum. For that reason, they searched for relevant literature and found the RNP (Cramer et al., 1997). Example 3 illustrates another instance of this, with the use of Pimm's (1987) work on language in mathematics and Fauvell's (1979) notion of metacognition, the latter being inspired by one of the maths counsellors' background in psychology. In example 4, the teachers discovered RME (e.g. Treffers, 1993) on their own. And in example 6, the maths counsellor herself found the article by Grigutsch (1998), linking modelling and beliefs, and made this the focal point of her work. Numerous other examples exist in the comprehensive pool of other prospective mathematics counsellors' written reports. [7]

Potentials and challenges of the "model" and its role in schools

The approach just presented to linking research and practice has been shown to have several qualities and potentials. Firstly, the scheme puts research literature to direct, relevant and effective use in teachers' own work on detecting, diagnosing and intervening against mathematics specific learning difficulties with their students. Thus, research literature is not studied out of context for practice but is, on the contrary, essential for practice. More often than not, in-service teachers with a diploma from the programme ask for hints to literature that deals with specific topics and issues that they have encountered in their endeavour to come to grips with phenomena and problems observed with their students. By way of the literature, teachers become acquainted with theoretical constructs that equip them with a set of lenses through which they can "see" and interpret empirical observations.

Secondly, the projects done by the teachers resemble research projects proper in terms of the questions posed, the theories invoked, the methods adopted, the analyses performed and the reporting produced, except that due to time constraints the amount of effort that teachers can put into their projects does not match that of real research projects; hence findings may not take the shape of publishable research findings. As the projects are focused on guiding teachers' practice as maths counsellors they are best perceived as development projects.

Thirdly, even though the primary objective of the programme is to educate teachers to undertake the function of maths counsellor, there is ample evidence that their experiences from the programme also change their own teaching of all their students, not just those with learning difficulties. Moreover, their experiences give rise to significant discussions and changes of practice amongst their mathematics teacher colleagues. In many places the maths counsellors serve as the first amongst "peers" by conducting seminars at which findings and insights from their project work are presented and discussed for the benefit of all their colleagues. In some cases, maths counsellors have been asked to orchestrate local projects aimed at developing mathematics teaching at large in their schools.

Naturally, our approach to linking research and practice also meets various challenges. In (Jankvist & Niss, 2015), we pointed to the fact that prospective maths counsellors in the beginning fall back into the role of teaching the students instead of diagnosing their learning difficulties. Here, we shall point to another challenge. Given that the participating teachers oftentimes have a fairly strong mathematics background from their master's degrees and typically a somewhat weaker background in the didactics of mathematics, it usually takes a while for them to come to grips with the special nature and scope of research findings in mathematics education, be they of a theoretical nature (e.g. a proposed theoretical construct or distinction or a full-fledged theoretical edifice) or of an empirical nature. The fact that empirical research findings require interpretation and typically are not set in stone clashes with their notions of empirical research results as being solid and indisputable. Also the fact that our field admits several different, and even competing, schools of thought, theoretical constructs or empirical findings concerning a given *problématique* may seem to suggest insufficient solidity of the research findings. However, when participants have learnt to use research as a means for understanding and interpreting their own observations this problem diminishes.

The professional development model presented here presupposes that schools are able and willing to make use of the competencies that their maths counsellors have gained through the programme. This of course requires material and immaterial resources to be invested in the role and functioning of the counsellors. The fact that, in Denmark, the education sector at large has suffered from massive budget cuts during the last couple of decades, undermines this prerequisite in many schools. So, it has happened that some schools cannot afford to ask their maths counsellors to engage in counselling activities to the extent initially expected and desired. It is unclear to what degree this problem will be remedied in the future.

Finally, this report on the maths counsellor model has been based on experiences and evidence compiled by direct contact with the mathematics counsellors during and after their education programme. It has not – yet? – been possible to undertake a systematic empirical investigation of the actual roles played by the counsellors in their schools, neither of the long term effects of the programme on teachers' perception of mathematics specific learning difficulties and effective ways to counteract them.

Perspectives

Even if the maths counsellor programme described here has upper secondary mathematics education in Denmark as its focus, it is clear that upper secondary mathematics education is only one part of the system. Most of the learning problems encountered at this level have their roots in primary and lower secondary mathematics education. Moreover, unremedied learning problems propagate into tertiary education as well. So, the issues dealt with call for an *extension to all levels* of the education system in Denmark. For a variety of reasons this is very difficult to establish in Denmark. One problem of extending the programme to primary and lower secondary teachers is that not all of them have the mathematical and didactical backgrounds that are needed to really benefit from the programme as it stands. Therefore, substantial investments in pre- and in-service education for such teachers are badly needed.

However, the fact that we have provided an ”existence proof” of the viability of the programme we have established may eventually lead to an extension of the programme to other education levels as well, at least pointwise.

We can safely assume that the learning problems encountered in Denmark are not specific to that country, even though the problems are likely to manifest themselves differently in other countries and contexts. Here, it is interesting to note that this programme has significant similarities in aim and spirit, and in some aspects of implementation, with the Kristianstad and Agder projects mentioned above (Grevholm, 2001; Jaworski et al., 2007). It would be worthwhile exploring whether the ideas and measures described here and in (Jankvist & Niss, 2015) could be subjected to some kind and degree of *internationalisation*. We would be very interested in discussing such possibilities with colleagues in other countries.

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Notes

1 The authors also are in charge of carrying out the programme, since 2018 accompanied by Morten Blomhøj who was also involved in the earliest development of the ideas behind it. Sif Skjoldager and Jesper Schmidt-Hansen were our collaborators in 2013–2014 and 2015–2016, respectively.

2 The first two books were reviewed by Barbro Grevholm (2016; 2017) in NOMAD.

3 The project was done in collaboration with Hans-Henrik Wollert Torp.

4 See also http://www.cehd.umn.edu/ci/rationalnumberproject/default.html

5 The project was done in collaboration with Christian Skjødt Pedersen.

6 The project was written in collaboration with Jens O. C. Malmquist.

7 These are available through Roskilde University Library.

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