Guidelines for utilizing affordances of dynamic geometry environments to support development of reasoning competency

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This article reports on guidelines developed based on an extensive research literature review investigating the potentials of dynamic geometry environments (DGEs) when the educational aim is to support students' development of mathematical reasoning competency. Four types of potentials were identified – feedback, dragging, measuring, and tracing – and used in three dimensions of guidelines: students' cognition, task design, and the role of the teacher. Using constructs from the *Instrumental approach*, the *Theory of semiotic mediation*, and the van Hiele model of levels, affordances and guidelines are elaborated upon and their potentials for reasoning competency are analyzed.

Research on dynamic geometry environments (DGE) affordances has revealed potentials regarding the development of students' mathematical reasoning (e.g. Leung, 2015; Edwards et al., 2014). This is promising, because there is research in Denmark and internationally indicating that students' reasoning abilities are inadequate (e.g. Jessen et al., 2015; Hoyles & Healy, 2007). ICT is accessible at all levels of the Danish educational system, so, in principle, the potentials are available in the mathematics classrooms. However, students' access to DGEs does not guarantee greater learning outcome. The manner in which DGEs are used is essential (Jones, 2005). Therefore, it is an important research objective to develop guidelines for fruitful teaching with DGEs.

Since DGEs can be used for different purposes, it is necessary to clarify the mathematical aim of the teaching guidelines. The notion of mathematical competencies, which has gained substantial traction in mathematics education, can be used for this purpose. Niss et al. (2016) call for research into teaching that can support students' development of mathematical competencies.

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Højsted, I. H. (2020). Guidelines for utilizing affordances of dynamic geometry environments to support development of reasoning competency. *Nordic Studies in Mathematics Education*, 25 (2), 71–98.

> Fostering, developing and furthering mathematical competencies with students by way of teaching is a crucial [...] priority for the teaching and learning of mathematics [...] We now need to understand the specific nature of the contexts and other factors that help create such progress. (Niss et al., 2016, p. 630).

Focusing on reasoning competency in relation to using DGEs, this study aims to contribute to this end by asking: *Which research-based guidelines may be formulated for teaching with DGEs in order to support students' development of reasoning competency*?

The main question gives rise to two auxiliary questions: (i) *Which affordances of DGEs can be considered as potentials when the educational aim is to support students' development of reasoning competency*? In addition, (ii) *Which dimensions should such research-based guidelines entail*?

To address these questions, a review is undertaken of existing DGE research, using reasoning competency as the searching and sorting lens. The review findings serve a dual purpose – to establish which dimensions the guidelines should entail, and to identify DGE affordances that can be considered potentials for reasoning competency. Theoretical constructs from the literature that are found to be useful in the conceptual development of the guidelines are also included. On the basis of this work, an analysis of the possible development of reasoning competency in relation to DGEs is conducted, and finally, to answer the main research question, guidelines are suggested. Since reasoning competency plays a crucial role in the article, an elaboration of the notion is in order.

The KOM framework and its reasoning competency

The KOM framework[1] introduces a competency-based approach comprising eight mathematical competencies, illustrated in the so-called KOM flower (figure 1). The framework is integrated in the Danish mathematics education curriculum, and has also had an impact on mathematics education around the globe (e.g. OECD, 2017; for a detailed account, see Niss et al., 2016).

In the reasoning competency (hereinafter referred to as RC), reasoning is defined as "*a chain of argument […] in writing or orally, in support of a claim*" (Niss & Højgaard, 2011, p. 60). RC consists of the ability to create and present formal and informal arguments, as well as the ability to follow and evaluate arguments made by others. It involves understanding what a mathematical proof is, the role of counterexamples, and the difference between a proof and other forms of mathematical reasoning, such as explanations based on examples. In addition, it includes the ability to develop an argument based on heuristics into a formal proof. RC

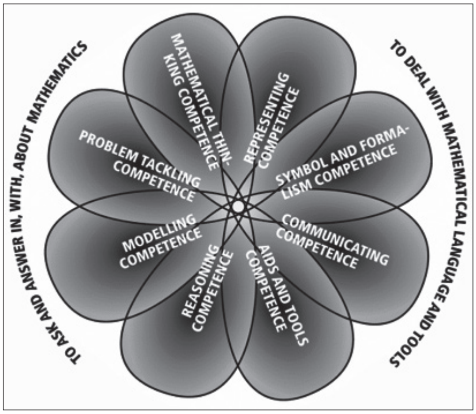

Figure 1. *KOM flower (Niss & Højgaard, 2011, p.1)*

is not only about justification of mathematical theorems, but also about creating and justifying mathematical claims in general, such as answers to questions and solutions to problems (Niss & Højgaard, 2011). In this article, the notion of proof is understood as the product of a proving process (Mariotti, 2012), which includes exploration and conjecturing as well as proving conjectures.

A person's attainment of a competency is qualified in three dimensions: *Degree of coverage* indicates to what extent the characteristics of the competency can be activated. *Radius of action* refers to the situations and contexts in which the competency can be mobilized, while *technical level* describes how advanced the mobilization is (Niss & Højgaard, 2011). To exemplify with regard to RC, a person might be able to follow reasoning put forward by others, but unable to put forward reasoning herself, thereby lacking in *degree of coverage*. She might be able to follow mathematical reasoning in the area of statistics but not in geometry and therefore has a limited *radius of action*. She might be able to follow complicated and technically advanced reasoning and therefore has high a *technical level*. The dimensions have a subjective character, since, for example, a high technical level depends on a person's age and peers.

Review method

The review was anchored in the hermeneutic framework for literature reviewing (Boell & Cecez-Kecmanovic, 2010, 2014), which fundamentally perceives the literature review as a non-linear process of gradually developing an understanding of and insights into a domain of research. The approach consists of two intertwined hermeneutic circles, the *search* *and acquisition* circle and the *analysis and interpretation* circle (see figure 2). The steps in the circles are carried out in an iterative process, thereby approximating a deeper understanding of the area of interest.

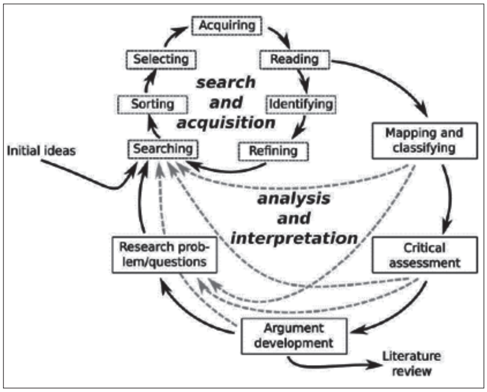

Figure 2. *The hermeneutic framework for the literature review process (Boell & Cecez-Kecmanovic, 2014, p.264)*

Applying the method

The i*nitial idea* was to operationalize RC into search words to cover central characteristics of the competency, and combine these with synonyms of dynamic geometry. The initial *search* was made in the MathEduc[2] and ERIC[3] databases, using the search words "dynamic geometry", "geometry software", "geometry technology", "interactive geometry" and "proof", "reasoning", "conjecture", and "justify", which, after *sorting* and checking for items occurring twice, gave a total of 151 items. 62 items were found to be irrelevant after studying their abstracts, giving a total of 89 items that were *selected* and *acquired* to be read. Furthermore, proceedings from the CERME[4] technology TWGs were searched. After *reading* literature acquired from the primary search, interesting references were *identified* and followed (citation tracking) (Boell & Cecez-Kecmanovic, 2014) and, if suitable, added to the review. In addition, after reading and gaining some insight into the area of interest (*mapping and classifying*), adjusted search words were used in focused searches, and the operationalization of RC was *refined* with the search words "counterexample", "argumentation", and "heuristic proof" in combination with synonyms of dynamic geometry. Other focused searches were related to theory, specifically "instrumental genesis" and "semiotic mediation" and "Hiele" combined with dynamic geometry. A total of 136 publications were included to be examined in the review. The definition of RC played a decisive role in the review process, as it influenced the choice of search words and the *sorting* and *selection* of literature, and was the perspective used in the *critical assessment* of the mapped literature, helping to decide which DGE potentials and dimensions of guidelines were relevant. The *argument development* which is the synthesizing result of the literature review is unfolded in the following chapter.

Results

The aim of the review process was to address two issues (corresponding to the two auxiliary research questions): (i) to find the potentials of DGEs in relation to RC, and (ii) to inform the development of guidelines for teaching, i.e. use the literature to understand what dimensions the guidelines should entail, including which theoretical constructs may prove useful for this endeavor.

In the following section, theoretical constructs are introduced that were identified in the review to be useful in sharpening the guidelines conceptually. Then the argument development leading to the potentials is presented, followed by review findings leading to the dimensions of the guidelines, which are unfolded subsequently. Figure 3 provides an overview of the structural development of the guidelines that will be presented in the following sections.

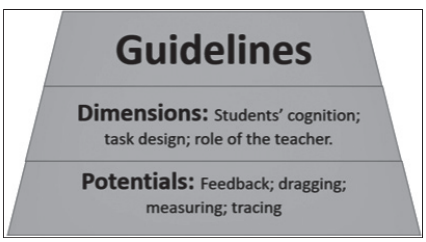

Figure 3. *Structural argument development of the guidelines*

Three theoretical findings to be used in the guidelines

Many studies from the review (e.g. Bretscher, 2009; Alqahtani & Powell, 2015; Gómez-Chacón, 2012; Gómez-Chacón et al., 2016) are embedded in the Instrumental Approach, which involves the process of instrumental genesis (Artigue, 2002; Guin & Trouche, 1999). According to this point of view, an artefact (such as DGEs) is not an instrument for a student from the outset, but becomes an instrument when the student can use the artefact in some meaningful way (Vérillon & Rabardel, 1995). Building on

Vergnaud's (2009) notion of schemes, instrumental genesis characterizes how subjects develop utilization schemes, which are cognitive schemes intertwining technical knowledge and mathematical knowledge. In a scheme-technique duality (Drijvers et al., 2013), these mental schemes evolve along with instrumented techniques for using the artefact to solve specific tasks. The instrumented techniques are the observable manifestation of the students' utilization schemes.

Several DGE studies (e.g. Falcade et al., 2007; Mariotti, 2012; Ng & Sinclair, 2015) are anchored in the *Theory of semiotic mediation* (Bartolini-Bussi & Mariotti, 2008). Emerging from a Vygotskian (1978/1934) perspective, the theory describes how teachers can exploit the possible ways of using an artefact (such as DGEs). In the semiotic perspective, the teaching and learning process is characterized as an evolution of signs such as gestures, verbal utterances, or DGE-mediated actions. The theory addresses students' initial production of *situated signs* as the artefact is used and on the following evolution into *mathematical signs*, which can be mediated by the teacher through social interaction as the teacher connects mathematical meaning to the evoked signs. Bartolini-Bussi and Mariotti (2008) use the notion of the semiotic potential of an artefact to describe the duality of emergent personal meanings and the possible mathematical meanings evoked by using an artefact. Mariotti (2012) considers the analysis of the semiotic potential of an artefact to be the core of any teaching design, and exploiting the potential involves:

> the orchestration of didactic situations where students face designed tasks that are expected to mobilise specific schemes of utilisation [...] the orchestration of social interactions during collective activities, where the teacher has a key role in fostering the semiotic process required to help personal meanings, which have emerged during the artefact-centred activities, develop into the mathematical meanings that constitute the teaching objectives" (Mariotti, 2012, p. 170)

Similarly to other DGE research (e.g. Jones, 2000; Idris, 2009; Kaur, 2015; Forsythe, 2015), utility is found in the van Hiele model of levels in order to relate the students' cognitive progression to a model of mathematical thinking. Van Hiele (1986) outlined how students' progress through five levels of mathematical thinking. The hierarchical structure has been criticized, as studies have found that students can be at several van Hiele levels in different situations (Burger & Shaugnessy, 1986). However, the levels may be thought of as different modes of thinking (Papademetri-Kachrimani, 2012; Forsythe, 2015) that can be activated in different situations. The latter understanding of the levels is adopted in this article and used in relation to students' cognition. Levels 1–4, which are relevant for this article, are elaborated upon: (1) *Recognition*. Students visually recognize figures intuitively by their global appearance. (2) *Analysis*. Students can describe the properties of a figure, but do not interrelate properties of figures. (3) O*rdering*. Students order the properties of figures by short chains of deductions and understand the interrelationships between figures. (4) *Deduction*. Students understand deduction and the role of axioms, theorems, and proof.

Potentials of DGEs in relation to RC

As a result of the review, four types of DGE affordances were identified as potentials[5] regarding students' development of RC: feedback, dragging, measuring, and tracing. DGEs are designed to mimic theoretical systems, such as Euclidean geometry, essentially creating a microworld in which activities follow the theoretical system governing the environment (Balacheff & Kaput, 1997). This signifies the existence of an inherent *feedback* function in the environment, since only objects which are possible in Euclidean geometry can be constructed. In a pencil and paper environment, there is no control on behalf of the paper over impossible constructions, allowing for imprecision, for example in a triangle where the medians do not intersect in the same point. Furthermore, dynamic geometrical figures can be constructed in the environment, so that certain properties are conserved when the figure is manipulated by use of the *drag* mode. The relationship between the elements of the figure is locked in a hierarchy of dependencies determining the outcome of a dragging action (Hölzl et al., 1994). This allows students to explore the figure by dragging free points to discover invariant properties of the figure, i.e. properties that are conserved. In a "robust" construction, the properties are conserved when free points are dragged. On the contrary, in a "soft" construction, not all properties are conserved (Healy, 2000; Laborde, 2005a).

Types of invariants have been classified to elaborate their role in conjecturing and reasoning (e.g. Leung, 2015; Baccaglini-Frank & Mariotti, 2010). Baccaglini-Frank and Mariotti (2010) suggest discernment between direct invariants, which are invariants in the construction that are defined directly by DGE commands used to complete the construction, and indirect invariants, which are those that arise as a consequence of the theory of Euclidean geometry, which governs the DGE. If a student is aware of the direct invariants of a construction and through exploration discovers indirect invariants, the activity might lead the student to make a conjecture (this will be discussed further in the section on task design). Many DGEs contain *measuring* tools that allow students to take measurements of, for example, angles, lengths, areas, and perimeters of constructions. If free points of the construction are dragged, causing the measures to change, the measurements are updated instantly and continuously. Therefore, it is possible for the students to discover invariant relationships between measures (Olivero & Robutti, 2007). In addition, many DGEs contain the possibility of *tracing* an object, so that the path can be visualized from a dragging action. In this way, tracing combined with dragging can be used to discover underlying invariant relationships (Baccaglini-Frank & Mariotti, 2010; Leung, Baccaglini-Frank & Mariotti, 2013). The affordance of visually representing geometric invariants when using the drag mode is considered a key feature of DGEs in relation to the development of mathematical reasoning, the ability to generalize results, and conjecturing in geometry (e.g. Arzarello et al., 2002; Laborde, 2001; Leung, 2015; Baccaglini & Mariotti, 2010; Edwards et al., 2014), which are some of the characteristics of RC.

Dimensions of the guidelines

The review showed that since its introduction, DGE research has had shifts in focus (see for example Jones, 2005; Mariotti, 2006; Laborde et al., 2006; Hollebrands et al., 2008; Olive et al., 2009; Sinclair & Robutti, 2013). In broad strokes, three dimensions of research could be identified. Initially, research focused on the learner, with some early contributions addressing student cognition (e.g. Arzarello et al., 2002; Hölzl et al., 1994). More recently, focus has shifted to design of adequate tasks to meet learning aims (e.g. Lin et al., 2012; Komatsu & Jones, 2018; Fahlgren & Brunström, 2014), as well as to the role of the teacher (e.g. Mariotti, 2006; Bartolini-Bussi & Mariotti, 2008). Sinclair et al. (2016) state that although research on DGE affordances is vast, task design and teacher practice remain understudied, a statement echoed by Komatsu and Jones (2018).

Findings from all three dimensions are relevant in relation to developing guidelines for teaching. Consequently, it was decided that the research-based guidelines should encompass findings regarding students' cognition, task design, and the role of the teacher.

Students' cognition

Several studies on students' cognition in DGE-related work are embedded in instrumental genesis (e.g. Leung et al., 2006; Bretscher, 2009; Baccaglini-Frank & Mariotti, 2010; Hegedus & Moreno-Armella, 2010; Gómez-Chacón, 2012). From this point of view, it may be described that the students need to develop instrumented techniques and utilization schemes with the DGE, in order for it to become a personalized instrument where exploration for invariants can occur in, primarily, conjecturing activities (e.g. Baccaglini-Frank & Mariotti, 2010). What do such utilization schemes (and corresponding instrumented techniques) entail?

The technique of exploring figures for invariants by dragging presumes that the students are aware of the relationship between the elements of a figure which determine the outcome of a dragging action, corresponding to van Hiele levels 2–3 (vH lvls 2–3) (Hölzl et al., 1994).

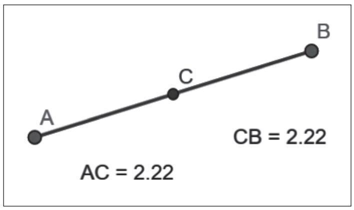

Figure 4. *Midpoint C is locked in the relationship AC = CB*

For example, understanding why the midpoint C in line segment AB will move when the free points A or B are dragged in figure 4, and why it is not possible to drag point C. This requires an awareness of the theoretical properties of figures which are mediated perceptually by the DGE (vH lvl 2). Therefore, the dragging technique/scheme to explore for invariants involves moving between the spatiographical and theoretical levels (Laborde, 2005b) which students need to coordinate. The spatiographical level refers to the perceptual appearance of a figure, while the theoretical level refers to the theoretical properties of a figure. Arzarello et al. (2002) describe how a DGE can potentially link the spatiographical level to the theoretical level in ascending and descending processes (vH lvl 1–2). Ascending happens when the students shift from the perceptual level to the theoretical level, while descending happens when the students shift from theory to perception. For example, if a student has made a conjecture about the theoretical property AC = CB in the construction in figure 4, she might validate it by a dragging test. The theoretical conjecture is confirmed perceptually in a descending process. On the other hand, if a student is unaware of the theoretical properties, the activity of dragging may prompt a shift towards awareness of the theoretical properties, since the properties are mediated perceptually by the DGE in the form of invariants (the property AC = CB remains). The perceptual output may result in theoretical awareness in an ascending process. Arzarello and colleagues (2002) found that students exploring geometrical figures by dragging in DGEs shift back and forth between empirical and deductive reasoning in ascending and descending processes. Many studies have found that DGEs can support students in connecting empirical and theoretical mathematics (e.g. Lachmy & Koichu, 2014, Baccaglini-Frank & Mariotti, 2010; Mariotti, 2006; Guven et al., 2010; Hadas et al., 2000; Jones, 2000; Laborde, 2005b). Even though DGEs can present opportunities for students working on the proving process (Laborde 2000; Olivero, 2002; de Villiers 2004; Sinclair & Robutti 2013), particularly in the production of conjectures, DGEs can also present challenges because of the strong link to the spatiographical level (Sinclair & Robutti, 2013). From the perspective of the Theory of semiotic mediation, the students' personal meanings underlying the initial situated signs stemming from the DGE activity do not necessarily relate to the theoretical aspects of the DGE constructions. However, the teacher can mediate the evolution of mathematical meanings and mathematical signs, which is needed if the students are to notice theoretical relationships in the DGE activity (vH lvl 2–3).

Additionally, comprehension of direct and indirect invariants is required. The students need to understand the difference between invariants caused by the construction and invariants caused by the rules of Euclidean geometry in order to investigate a construction to make conjectures (see example in next section) (vH lvl 3). Furthermore, exploration of invariants requires capacity regarding certain dragging techniques/schemes. Research on ways of dragging in DGEs has resulted in a classification of several dragging modalities, which can broadly be divided into two categories (Hölzl, 2001; Leung, 2015): (1) Dragging for searching/discovering, containing dragging modalities where the student drags in order to explore the figure for new properties. For example: *wandering dragging* – dragging randomly to try to discover regularities or interesting configurations; *guided dragging* – dragging basic points to make a particular shape; *maintaining dragging* – realizing an interesting configuration and trying to keep the specific property invariant while dragging (noticing a soft invariant); and (2) Dragging for testing, encompassing the dragging modalities in which the students drag to test an expected reaction from the construction. For example, *the dragging test* – dragging objects in order to see if the construction maintains desired properties, i.e. if it is robust; *the soft dragging test* – testing a conjecture about a soft invariant (Arzarello et al., 2002; Baccaglini-Frank & Mariotti, 2010). Similarly, measuring modalities for searching and testing have been classified into two broad categories: measuring for discovery – wandering measuring, guided measuring, perceptual measuring; and measuring for testing – validation measuring, proof measuring (Olivero & Robutti, 2007). Students' development of instrumented techniques and utilization schemes for dragging and measuring to explore, develop and test conjectures is a prerequisite for working on tasks that can support students' progression of RC. In addition, the development of schemes and techniques for utilizing tracing and the feedback function of DGEs can be valuable in order to work on tasks that may mobilize students' RC, which will be explained in the next section.

Task design

The literature review revealed several types of task design in DGEs. Some studies report on task design principles or models for task design (e.g. Lin et al., 2012; Komatsu & Jones, 2018; Fahlgren & Brunström, 2014; Olsson, 2017), while some have developed models to assess task quality (e.g. Trocki, 2014; Trocki & Hollebrands, 2018) in relation to DGEs in general, and with focus on reasoning and proof (Baccaglini-Frank et al., 2013, 2017, 2018; Leung, 2011; Sinclair, 2003). Models for task design will not be introduced in this article, but using the perspective of RC, five types of task design were identified as having the potential of mobilizing students' development of different characteristics of RC, thereby potentially increasing students' *degree of coverage* of RC.

Construction tasks. (1) The students can be supported in creating and justifying mathematical claims in general by offering tasks similar to what Mariotti (2012) coined "construction tasks", which require the students to construct robust figures with specified invariants using limited construction commands. Such a task could involve constructing a robust square using only construction commands such as points, line segments, lines, perpendicular lines (some might prefer not to allow this command), circles, and intersection points (see figure 5). The students have to describe the procedure and explain why the figure remains invariant, which is to create and justify a mathematical claim in terms of the RC. Dragging is an instrument to confirm the validity of the construction. This type of

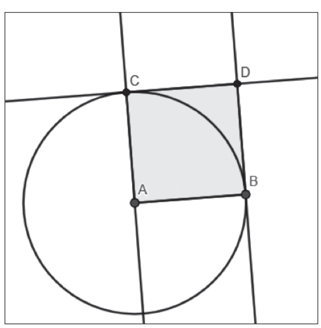

Figure 5. *Constructing a robust square using specified construction commands*

task can help students to develop an awareness of direct invariants, and is therefore a valuable prelude to working on conjecturing open tasks, which is described next.

Direct indirect invariants. (2) If a student has developed an awareness of the constructed direct invariants, then the discovery of indirect invariants through dragging can lead to conditional "if-then" conjectures, with the direct invariants being the premise for the indirect invariants (Baccaglini-Frank & Mariotti, 2010, Lachmy & Koichu, 2014). For example, a conjecturing open task for students could be to construct ΔABC, the midpoints of two of the sides, and to draw a line segment connecting these midpoints (figure 6).

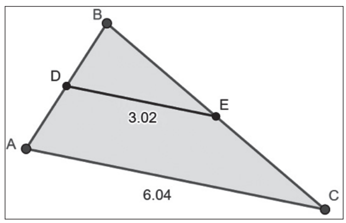

Figure 6. *Direct and indirect invariants in a robust construction*

In exploring the construction by dragging free points, students can discover the invariant parallelism of DE and AC, which was not a property of the initial construction. The potential of measuring can also be used to discover the invariant relationship of length DE being half of length AC. These conjectures can, with guidance from the teacher (discussed later), lead to a proof process of the midpoint theorem. In this case, dragging and measuring are tools to investigate the theoretical properties of the figure.

Maintaining dragging. (3) The usage of robust constructions has been prevalent in DGE teaching (Ruthven et al., 2005), but in a slightly different type of task, the students can be prompted to make a soft construction and to try to discover the conditions for which some property is maintained, using the maintaining dragging modality (Baccaglini-Frank & Mariotti, 2010). For example, a simple task for students could be to construct line segments AB and BC and look for the positions of B which satisfy $AB = BC$, using trace activated on point B. By interpreting the trace path shown in figure 7, the students can discover and perhaps conjecture that points which are equidistant from two given points would all lie on the perpendicular bisector of the line segment joining the two points. In this case, the potential of tracing is utilized to unveil an underlying invariant.

Pseudo-objects. (4) By offering tasks instigating students to construct non-constructible pseudo-objects (Baccaglini-Frank et al., 2013, 2017

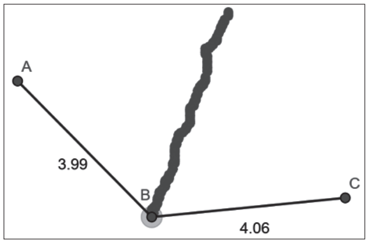

Figure 7. *Maintaining the property AB=BC in a soft construction with trace activated*

2018), abilities in proving by means of contradiction can evolve. A pseudo-object contains contradictory properties with regard to Euclidean theory, and it is therefore not possible to construct in a DGE. As students attempt to construct the pseudo-object, (e.g. figure 8 or 9) the feedback affordance provided by DGEs can assist the student in realizing the impossibility of the construction. Designing such tasks involves identifying proto-pseudo objects, which are objects that have the potential of becoming pseudo-objects for the students, for example a triangle in which two angle bisectors are perpendicular (Baccaglini-Frank et al., 2013, 2017, 2018).

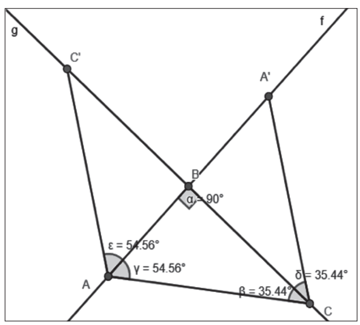

Figure 8. *Starting from two perpendicular lines g and f (the angle bisectors) and reflecting AC in them*

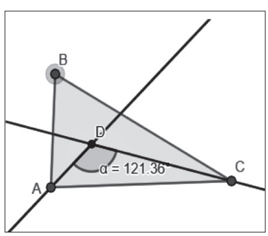

Figure 9. *Investigating the pseudo-object by constructing the triangle first and then dragging and measuring*

Hidden conditions. (5) Another aspect of RC is to understand the role of counterexamples. To this end, the potential of dragging can be utilized by designing tasks, which prompts students to discover counterexamples to conjectures as they manipulate constructions. The conjectures may be given in the task description or discovered by the students themselves. Komatsu and Jones (2018) suggest that such heuristic refutation tasks could include ambiguous diagrams with ”hidden conditions”, exemplified in figure 10 with the accompanying task: ”there are four points A, B, C, and D on circle O. Draw lines AC and BD, and let point P be the intersection point of the lines. What relationship holds between ΔPAB and ΔPDC? Write your conjecture. (2) Prove your conjecture.” (Komatsu & Jones, 2018, p. 9). The students might argue that $\angle$ BPA = $\angle$ DPC (vertical angles are equal) and that $\angle$ ABP = $\angle$ PCD (inscribed angle theorem), hence $\angle$ PAB ~ $\angle$ PDC.

But when the students are prompted to drag points A, B, C and D, they might discover local counterexamples to the conjecture, such as figure 11, and be motivated to revise their conjecture.

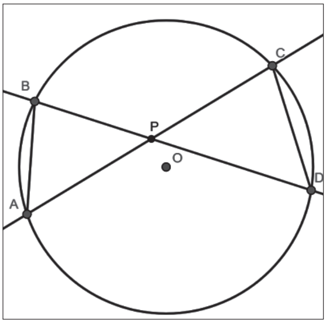

Figure 10. *Diagram with ”hidden conditions,” inviting insufficient conjectures*

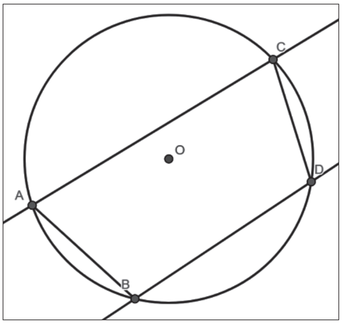

Figure 11. *Counter-example to the conjecture, AC parallel to BD*

4.6 Role of the teacher

The teacher plays an important role in helping students to transform personal meanings into mathematical meanings as described by Bartolini-Bussi and Mariotti (2008). To do so, the teacher should be conscious of the mathematical goal of the DGE class activity, and should use opportunities to emphasize mathematical meanings of the DGE-mediated signs. When teaching, discussing or giving feedback, the teacher should identify emerging signs, and try to inject mathematical meaning into these signs (Bartolini-Bussi & Mariotti, 2008). The teacher can guide the students to understand the mathematical implications of the feedback provided by the DGEs as well as the mathematical meaning of dragging, measuring, and tracing. As the students produce initial situated signs and develop personal meanings in relation to feedback, dragging, measuring, and tracing, the teacher can mediate the evolution of mathematical meanings and signs. RC includes the ability to present formal arguments and to develop arguments based on heuristics into formal proofs (Niss & Højgaard, 2011). In order to further progress the students' *degree of coverage*, the teacher needs to engage the students in proving their conjectures. This, however, does not happen automatically. Some studies (e.g. Marrades & Gutiérrez, 2000; Connor et al., 2007) even suggest that exploratory use of DGEs can inhibit the progression of students' deductive proving, since the students are empirically convinced that a fact is evident, and do not see the point of having to prove it (again). However, studies also show that DGE exploration does not have to jeopardize this progression (Lachmy & Koichu, 2014; Sinclair & Robutti, 2013). Researchers highlight the important role of the teacher in guiding and motivating the students towards justifying their conjectures and towards applying a theoretical approach (e.g. Mariotti, 2012; Arzarello et al., 2002). De Villiers (2007) argues against questioning the conviction that empirical methods give, or trying to convince students to undertake theoretical verification to further verify what they already find evident, but stresses that instead the teacher should motivate the students by asking "why" the fact is evident. Trocki (2014) argues that motivation and guidance for students to connect informal exploration and conjecturing to theoretical justification might also be incorporated into the DGE task itself, for instance by including questions that prompt students to justify their conjectures.

The possession and development of RC in relation to DGEs

Based on the review results presented in chapter four, we can analyze how the development of RC may be supported with DGEs using the terminology in the KOM framework concerning a person's attainment of a competency. As previously mentioned, the attainment is described in three dimensions: *degree of coverage*, *radius of action*, and *technical level*.

Degree of coverage

It is evident from the findings that the development of students' possession of various characteristics of the RC can be supported by utilizing DGE potentials (task types 1–5), thereby increasing the students' *degree of coverage* of RC, in particular the ability to create and present informal arguments, because the dragging arguments are empirically based on invariant relationships discovered in many examples. Hence, focus is primarily on the exploration and conjecturing phase of the proof process. For example, the direct/indirect invariants task (task type 2) illustrates how dragging and measuring might be utilized in the conjecturing phase of conditional statements. Similarly, the soft invariant task illustrates how dragging, measuring, and tracing can be utilized in exploring and conjecturing about equidistant points and the perpendicular bisector of the line segment joining them. However, with guidance from the teacher, the students' abilities in relation to the second phase of the proof process – validating the conjecture – can also evolve, thereby expanding the students' *degree of coverage* of RC. While validating the conjecture, the students might primarily work outside the DGE, but they are likely to return to verify/refute their progress as found by Olivero and Robutti (2007). As mentioned, RC consists not only of being able to reason yourself, but also of being able to follow and evaluate reasoning put forward by others (Niss & Højgaard, 2011). This may be incorporated in the teaching design, in order to support the students' development of their *degree of coverage* of RC, by organizing the teaching environment in a way that encourages the students to collaborate (e.g. to work in pairs). It can also be explicitly addressed in the tasks themselves, e.g. by asking the students to explain their reasoning to each other, in pairs and in class discussions. Additionally, they can be required to evaluate the reasoning put forward by others. When the students manage to validate their conjectures deductively, the teacher (or perhaps prompts in the task) should highlight the difference between the theoretical proof and the conjecture to foster an understanding of the difference between the two – which is a characteristic of the RC.

Radius of action

As the students work on the aforementioned types of tasks (1–5), their *radius of action* regarding RC may expand, because they can progressively activate their RC in an increasing amount of subject matter. Covering a variety of areas in geometry in the task design will further this agenda. Furthermore, the students' radius of action may be promoted, since they develop the ability to activate their RC in the context of using a DGE for this purpose.

Technical level

The students' progression within the *technical level* of RC can also be supported, particularly if consecutive tasks demand higher levels of reasoning from the students, for example by successively increasing the number of steps needed in the chain of reasoning to solve the task or the number of presuppositions needed to prove a conjecture. As previously mentioned, the degree of difficulty regarding the technical level of RC involved in solving a certain task has a subjective character, as it depends on the educational level of the students trying to solve the task. The guidelines do not address a particular educational level; however, if we take Danish lower secondary school as an example and look at the task example used to characterize task type 3, then we can imagine that the task is not too demanding with regard to students' *technical level* of RC, while the example in task 2 is more complicated, even though the conjecturing phase of the example in task 2 may be relatively simple, proving the conjecture involves a few steps and demands a higher degree of *technical level* of RC.

Formulating guidelines for teaching with DGEs to support RC

Based on the findings from the review on DGE literature and the subsequent analysis (auxiliary questions (i) and (ii)), guidelines are suggested in table 1 in appendix A. In essence, the guidelines are an analysis of the semiotic potential of DGEs when the educational aim is to support students' development of RC, with the analysis building on previous research in the field.

There are six columns in the table; the first column holds steps of progression 0–4, in which steps 2–3 have a subset of steps. The van Hiele levels of mathematical thinking are also indicated in the first column. Although the guidelines are presented in steps of a hierarchical nature, and van Hiele's levels are used, they are not considered discrete or clearly continuous, which can be seen with some overlapping descriptions in steps 1 and 2 (a, b, c, and d). In addition, it may well be that the development of several steps can occur at the same time, for example developing an understanding of free and locked objects (step 1) at the same time as developing basic DGE proficiency (step 0). Columns two and three address the dimension of students' cognition using the notions of instrumented techniques and utilization schemes[6]. Column four indicates what kind of tasks might mobilize the desired techniques and schemes, while column five describes the role of the teacher in facilitating the process. Column six describes which characteristics of the RC the DGE-mediated activity is expected to mobilize.

Below, some comments are added to each step of progression.

0 Basic DGE proficiency regarding commands of construction and measuring is needed to work on tasks which can support the development of RC.

1 Being able to discern between free and locked objects requires that the students are aware of the theoretical properties of figures (vH lvl 2). Additionally, an awareness of theoretical relationships between the elements of a figure or between figures is required (vH lvl 3).

2 (a, b, c and d) Awareness of the hierarchy of dependencies which determine the dragging outcome is necessary. This covers an understanding of free points, direct invariants, and robust and soft constructions (vH lvl 2–3). Furthermore, the ability to discern between direct and indirect invariants is required in, for example, conditional "if-then" conjecturing (vH lvl 3–4). Comprehension of certain dragging modalities and measuring modalities can support the exploration for conjectures. Tasks which support this cognitive development and the *degree of coverage* of RC include: construction tasks that encourage creation and assessment of mathematical claims, as well as understanding of direct invariants (vH lvl 2–3); conjecture open tasks which encourage construction (robust and soft) of direct invariants that bring on indirect invariants and allow for exploratory work in order to support the development of the first phase of the proof process (vH lvl 3–4).

3 (a and b) Understanding and being able to exploit the feedback function inherit in the DGE to investigate the construction of non-constructible pseudo-objects in order to foster abilities in proving by means of contradiction (vH lvl 3–4); understanding and being able to exploit the feedback function inherit in the DGE to find counterexamples to conjectures about diagrams with hidden conditions (vH lvl 3–4).

4 Being able to prove the conjectures. The role of the teacher is important in motivating the students towards theoretical validation of their conjectures to develop the second phase of the proof process (vH lvl 4). The teacher can encourage the students to return to the DGE in order to verify/refute their progress as they are proving their conjectures.

Concluding remarks

The process of answering the research questions comprised of searching the literature for DGE affordances that are considered potentials in relation to supporting students' development of RC, and of identifying which dimensions the guidelines should entail. Four DGE potentials were identified: feedback, dragging, measuring, and tracing. The utilization of these was described in three dimensions of the guidelines: students' cognition, task design, and role of the teacher. The guidelines in Appendix A contain five steps of progression, in which the dimensions are addressed and the expected mobilization of RC is described.

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Notes

1 The English translation of the acronym is *Competencies and mathematical learning*. The KOM framework was published in Danish in 2002 and in English in 2011. In 2019, an updated version of the framework was proposed by the authors (Niss & Højgaard, 2019).

2 https://www.zentralblatt-math.org/matheduc/

3 https://eric.ed.gov/

4 Congress of the European Society for Research in Mathematics Education.

5 To clarify, the notion of "potential" is used here regarding affordances of DGEs which are not easily available in other common mathematics education mediums, in particular the pencil and paper environment. It could be argued that the possibility of constructing, for example, a circle or a regular polygon is an affordance of DGEs, but it is not a major addition compared to the pencil and paper environment, and therefore not considered a potential in this review.

6 The table provides a broad description. A fine-grained analysis is needed at the level of schemes, e.g. rules of action, operational invariants etc. (Vergnaud, 2009).

Appendix A

Table 1. *Research-based guidelines for mathematics teaching with dynamic geometry environments to support students' development of reasoning competency.*

Steps of progression	Guidelines				Aim
	Students' cognition		Tasks	Teacher's role	Reasoning competency
	Instrumented techniques	Utilization schemes		(during instruction, discussions and feedback) *Generally*: Identifying emerging signs and progressing towards mathematical meanings.	*Degree of coverage*: which characteristic is activated.
0. Basic DGE proficiency, van Hiele levels 1–2.	Students need to develop technical proficiency with the DGE. They need to be able to use the different commands for constructing, measuring, and dragging.	Technical and mathematical knowledge about the commands of constructing, measuring, and dragging.	Tasks which require the students to use the commands of the DGE. E.g. constructing points, lines, line segments, parallel lines and circles, finding intersection points, measuring angles, lengths, areas etc. and dragging these objects.	Highlighting the mathematical meaning in DGE-mediated activities, e.g. constructing a circle with two mouse clicks (definition of a circle), or the mathematical meaning of dragging as a way of varying the existing construction, such as varying coordinates of a point.	Prerequisites for working on conjecturing tasks and for justifying mathematical claims.
1. Free and locked objects, van Hiele levels 2–3.	Making constructions with free and locked objects. Dragging/measuring to investigate the construction.	Understanding the difference between free and locked objects. This requires attention to and comprehension of the theoretical properties of constructions.	Construction tasks which highlight the difference between free and locked objects. E.g. constructing two points and a midpoint between them. Describe why you can/cannot drag some points.	Focus on the mathematical relationship between elements of the construction which determine whether the objects are free or locked, which means focus on the theoretical aspects of figures, e.g. the mathematical meaning of a midpoint.	Justify mathematical claims. Prerequisite for working on conjecturing tasks.
2a. Robust constructions, van Hiele levels 2–3.	Making robust constructions with dependencies between elements in the construction, so that some desired properties remain invariant when free objects are dragged. Dragging/measuring to investigate the construction.	Understanding that direct invariants occur because of the theoretical properties induced in the construction.	Tasks that require the student to construct figures in which some properties remain invariant during dragging. "Construction tasks" (Mariotti, 2012). E.g. constructing a quadrilateral with one right angle.	Highlighting and encouraging student focus on the theoretical properties of the figure. E.g. right angles require perpendicular line segments. Invariants occur because of the mathematcal relationship between elements of the figure.	Justify mathematical claims. Prerequisite for working on conjecturing tasks. Exploring and conjecturing.
2b. Soft constructions, van Hiele levels 2–3.	Making soft constructions with non-dependencies between some elements in the construction, so that some desired properties remain invariant only when certain conditions are satisfied. Dragging/measuring and tracing to investigate the construction.	Understanding that direct non-invariants occur because of the lack of dependencies between theoretical properties induced in the construction.	Tasks which require the student to construct figures with soft invariants, in which some properties can be maintained only under certain conditions. E.g. finding the positions of point B for AB = BC.	Highlighting and encouraging student focus on the theoretical properties of the figure. E.g. meaning of perpendicular bisector. Non-invariants occur because of the mathematical relationship between elements of the figure.	Justify mathematical claims. Exploring and conjecturing.

2c. Dragging/ measuring modalities for exploration, van Hiele levels 2–3.	Being proficient in different dragging/measuring modalities: Dragging for searching/dragging for testing; measuring for searching/ measuring for testing.	Understanding fruitful dragging and measuring modalities in order to explore constructions to unveil indirect invariants and make conjectures. Including the "maintaining dragging" modality for soft invariants.	Tasks requiring the students to make conjectures by dragging and measuring in certain ways in order to notice indirect invariants. This includes tasks with soft invariants.	Explaining and illustrating how to drag free points with different aims: Randomly, looking for invariance of properties and measures, maintaining a property (maintaining dragging).	Justify mathematical claims. Prerequisite for working on conjecturing tasks. Exploring and conjecturing.
2d. Direct and indirect invariants, van Hiele level 3.	Constructing direct invariants which induce indirect invariants because of Euclidean theory. Dragging/measuring to investigate the construction.	Understanding the difference between direct and indirect invariants, and the connection between them.	Tasks requiring the students to make constructions with direct invariants, where dragging free points in the construction also unveils (surprising) indirect invariants.	Explaining and highlighting that direct invariants can induce indirect invariants because of the "rules of Euclidian geometry". E.g. lines perpendicular to parallel lines are parallel. Introducing the "if-then" relationship between direct and indirect invariants. Stressing the empirical nature of the conjecture.	Justify mathematical claims. Exploring and conjecturing.
3a. Feedback: non-constructible pseudo objects, van Hiele levels 3–4.	Constructing, measuring and dragging to test the possibility of an object.	Understanding the feedback function inherent in the DGE, and thereby the possibility of exploring whether objects can be constructed.	Tasks instigating the students to construct non-constructible pseudo-objects.	Highlighting and encouraging student focus on the theoretical properties of the non- constructible figure. Injecting mathematical meaning into the students' evolving signs of conflictions regarding the object.	Exploring and conjecturing. Abilities in proving by means of contradiction can be developed.
3b. Feedback: Counterexamples to conjectures, van Hiele levels 3–4.	Constructing figures and exploring (dragging, measuring) in order to find counterexamples to conjectures.	Understanding the feedback function inherent in the DGE. Understanding how a counterexample forfeits the conjecture.	Tasks which prompt students to discover counterexamples to conjectures as they manipulate constructions. Such tasks could include diagrams with hidden conditions, and the tasks should explicitly prompt the students to find counterexamples.	Highlighting and encouraging student focus on the theoretical properties of the figure which underlie the conjecture to which counterexamples are to be found. Injecting mathematical meaning into the students' evolving signs regarding the object.	Exploring and conjecturing. Understanding the meaning and role of counterexamples.
4. Proving the conjectures from steps 2b, 2c, 2d, 3 and 4, van Hiele level 4.	Using DGEs to verify/refute progress on proving the conjectures.	Understanding the feedback function inherent in the DGE, and thereby the possibility of verifying/ refuting conjectures.	Follow-up tasks requiring the students to prove their conjectures. Students can work outside the DGE, but return to verify/refute their progress.	Motivate the students to undertake theoretical verification by asking "why" their conjecture is true instead of dismissing the empirical evidence provided by the DGE (De Villiers, 2007).	Abilities in proving; developing an argument based on heuristics into formal proof. The difference between a proof and other forms of mathematical reasoning such as explanations based on examples.

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