Algebra teachers' questions and quandaries – Swedish and Finnish algebra teachers discussing practice

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Taking the teachers' own practices as a point of departure, this study investigates what areas of mathematical knowledge algebra teachers brought up in collegial discussions and how they used their knowledge in acts of decompressing, trimming and bridging. The discussions centered around aspects of teaching and learning school algebra previously shown to be problematic, but gave rise to mathematical quandaries, revealing gaps in the teachers' own understanding of the mathematical content. The study implies that the ability to unpack a mathematical concept is essential in algebra teaching and that teachers may need external input concerning mathematical knowledge to enable development in pedagogical content knowledge.

Research into algebra, its teaching and its learning, shows that students around the world continue to encounter difficulties in algebra that teachers struggle to help them overcome (e.g. Bush & Karp, 2013; Kieran, 2018). According to Chick (2009), algebra is notorious among teachers for being difficult to teach. She points at aspects of algebra such as the level of abstraction, ideas of generality, and the use of a symbolic language as contributing to the challenges teachers experience. The depth and breadth of mathematics teachers' own content knowledge is shown to be of critical importance, but also teachers' knowledge of the ways students think and how the teachers use their knowledge in the classroom (Wasserman, 2015).

It has been shown that mathematics teachers can improve their teaching by exploring student contributions and reflecting on the subject matter of their teaching practices together with colleagues (e.g. Holmqvist & Wennås Brante, 2011; Kazemi & Franke, 2004; Röj-Lindberg, 2017). Nevertheless, Hodgen, Oldenburg and Strømskag (2018) point out

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that teaching interventions that are successful in the short term might still lead to misconceptions in the long run, in particular on a complex topic such as algebra. For instance, based on their findings, MacGregor and Stacey (1997) suggest that the root of students' misconceptions related to the concept of variable often lies in teachers' instructional choices. Treating variables as real-world objects only, is an example of such an inadequate choice (Chick, 2009). According to Carraher, Schliemann, Brizuela and Earnest (2006), students' misconceptions about the equal sign are a result of their early mathematics instruction that must later be "undone". It is evident that teachers of mathematics are better prepared when they are aware of the potential of the mathematics they are teaching, as well as of the range of algebra skills and associated misconceptions of students.

This paper is based on a qualitative study investigating what areas of knowledge teachers feel a need to develop when teaching algebra. Following others (e.g. Jacobs et al., 2007), we build on the claim that "effective professional learning is situated in practice" (Hunter, Anthony & Burghes, 2018, p. 381) to argue that professional development would benefit from being explicitly connected to the teachers' own work with students in their classrooms. In this study, video episodes selected by the teachers themselves from their own classrooms were used to structure focus-group discussions about the teaching and learning of algebra. The teachers taught introductory algebra in grades 6 and 7 (student aged 12–13) and are here referred to as "algebra teachers". The aim of the study was to identify aspects of algebra knowledge that algebra teachers grapple with, and their ways of "using" that knowledge.

Theoretical framework

With the ambition of understanding, developing and assessing mathematical knowledge essential for teachers, Ball, Thames and Phelps (2008) identified six mathematical content knowledge domains constituting a framework of *Mathematical knowledge for teaching* (MKT). Three of the domains are explicitly related to *Pedagogical content knowledge*, i.e. connecting content to students, to teaching or to curriculum, and three domains relate to distinct characteristics of *Subject matter knowledge*. Teachers draw on their mathematical knowledge when planning, enacting and reflecting on their teaching practice.

Several attempts have been made to further characterize mathematical knowledge unique to the teaching profession (e.g. McCrory et al., 2012; Silverman & Thompson, 2008). Ball and Bass (2000) introduced the notion of "unpacking" as one aspect of knowledge a teacher needs when planning a lesson, for example when making choices about what to vary and what to keep invariant in order to draw attention to essential features of an idea (Marton, 2015). A teacher also needs to make decisions about what to unpack during the lesson, i.e. what complexity to reveal and make students aware of, and, conversely, when and how to simplify and reduce complexity (Wasserman, 2015). When reflecting on teaching practice, a teacher needs to recognize what features of an idea her students grasped and what features were obscured or out of reach for them. Further, a main issue in the design of tasks, examples and arguments is to maintain mathematical integrity when simplifying and adjusting to specific groups of students.

In an attempt to specifically map the territory of *Knowledge of algebra for teaching* (KAT), McCrory et al. (2012) extended the ideas from MKT by developing a framework including three types of algebra knowledge and three ways of ”using” that knowledge. While the types of knowledge described in KAT roughly correspond to the subject matter knowledge domains of MKT, the three ”uses of knowledge” try to capture ways in which teachers ”use their knowledge in the ebb and flow of practice, responding quickly and accurately to student thinking, to unexpected mathematics, and to materials that may not be quite right for the circumstances in the classroom” (ibid., p. 601). The three different ways teachers use their algebra knowledge, here referred to as three teaching practices, labelled *decompressing*, *trimming* and *bridging*, are described below.

The original purpose of KAT was to inform teacher education by developing assessment measures for teacher knowledge (McCrory et al., 2012). However, Wassermann (2015) extends the use of the framework outside the domain of algebra and for a wider purpose than that of assessment. He suggests that studies of what mathematical knowledge teachers draw upon when planning, teaching and reflecting on their teaching, have implications for teacher preparation and professional development. In this study, we use the KAT framework to analyze teachers' reflections on their practice.

Decompressing, trimming and bridging

Decompressing highlights complexity, either by unpacking and making explicit hidden features of a mathematical idea, or by introducing complexity with the purpose of foreshadowing more advanced mathematical ideas. Over time, symbolic algebra has become more and more compressed and encapsulated into generalized forms that need to be unpacked to be fully understood. Consequently, decompressing involves attaching fundamental meaning to algebraic symbols, algorithms and representations (McCrory et al., 2012).

Trimming, on the other hand, is the act of removing complexity by simplifying to gain short-term clarity, either by concealing local complexities to help students discern specific features of an idea, or by intentionally adding or removing details to help focus attention. Essentially, McCrory et al. (2012) describe trimming as the careful attention that teachers must give to making sure that the ideas discussed "are correct for the current content but [also do not] lead to problems in more advanced mathematical work" (p. 606). Examples of trimming described by McCrory et al. (2012) include modifying examples, scaling up or down and adjusting to students' thinking. Although Bruner (1960) claimed that any concept could be taught to any group of learners in some intellectually honest way, removing complexity while maintaining integrity can be a challenge. An example of inadequate trimming described by Wassermann (2015) is when young students are taught that "multiplication makes bigger". While this may be true for all multiplications the students encounter at the time, it will not always hold true, for example when multiplying by 0 or 1, or when dealing with multiplicative relationships. Hence, such inadequate trimming could cause confusion rather than clarification. Another kind of trimming, referred to by McCrory et al. (2012) as "trimming up", is when the teacher recognizes more complex mathematics that is within the student's reach and adjusts the level of instruction accordingly. In addition, Wasserman & Stockton (2014) point to the importance of using accurate terminology and being careful with presentation and symbols when trimming.

Bridging is described as connecting and linking mathematics across topics, courses, concepts, representations and goals, or as connecting current algebraic ideas with more advanced ones. According to McCrory et al. (2012), bridging can be seen as a manifestation of the teacher's understanding of the intersection of algebraic concepts within algebra and with other mathematical domains, and involves "keeping a range of ideas in play in the classroom and presenting mathematics as a coherent, connected endeavor" (p. 608).

The two practices decompressing and trimming both relate to teachers' awareness of mathematical complexity, with the pedagogical purpose of making mathematics more comprehensible while maintaining mathematical integrity (Wassermann, 2015). The purpose of bridging is to help students make mathematical connections.

Methodology

The reported study is located within a tradition of participatory research and with a multi-method research design where the mode of participation of researchers and teachers is collaborative. Ultimately, participatory research is about respecting and understanding the people with whom researchers work (Cornwall & Jewkes, 1995). The data reported in this paper is based on video-recorded focus-group discussions within the VIDEOMAT[1] project, an international research project about introductory algebra in grades 6 and 7 (students aged 12–13).

Procedures

Prior to focus-group discussions, authentic algebra lessons on the introduction to variables were recorded. The participating teachers were then asked to individually watch the videos from their own lessons, select episodes they wanted to discuss with peers, and pose questions related to the episodes concerning the teaching and learning of algebra. After three weeks of individual preparation, the teachers met in three focus groups, two in Sweden and one in Finland, for a session of 2–3 hours. In turn, each teacher presented his/her episode and related question/questions, initiating a discussion. Two cameras recorded the discussions, one facing the group and one facing the screen and whiteboard. The group members were seated with a clear view of each other, including a member of the research team who acted as facilitator. The facilitator assisted with the video playback of episodes, made sure everyone was involved in the conversation, and, if needed, redirected the discussion towards the original question or the algebra content. While the facilitator attempted to direct the participants towards each other, there were occasions, mainly following a lengthy discussion on something without reaching closure, when the facilitator introduced elements of algebra knowledge or opinions about the issues brought up. For this paper, such instances were excluded from analyses, although we are aware that they had an impact on the flow of the discussion. During the discussion, one page summaries of each lesson, so-called lesson graphs, were available for the teachers to look at. These gave everyone insight into how a chosen episode fitted into the flow of the lesson and were sometimes referred to in the discussion. Altogether, fourteen classroom episodes, each between one and five minutes, were discussed during the three focus-group sessions, with sixteen pre-prepared questions posed by the teachers in relation to the episodes.

Participants

We recruited the algebra teachers for the study using our networks as teacher educators, promoting their participation as an opportunity for professional development. Informed consent was collected from all the participating teachers, as well as the students and their parents. All participating teachers (8 in Sweden and 4 in Finland) have the required teaching credentials, and between 3 and 22 years of teaching experience. In both countries, the grade 6 teachers were generalist teachers with a primary school teacher exam, whereas the grade 7 teachers were subject-specific mathematics teachers with more university courses in mathematics. Since there was a majority of female teachers in the study (9 out of 12), we refer to all participating teachers using female pronouns in this paper for the sake of anonymity.

Four of the Swedish teachers taught grade 6 in four different classrooms in socioeconomically different areas in or near Gothenburg. Four consecutive lessons of 40–60 minutes were recorded in each classroom. Four of the teachers taught grade 7 in a smaller town, constituting a teacher team engaged in collaboratively developing the introductory algebra lessons together, and then co-teaching in pairs, so that four teachers were active in two different classrooms. In one of these classrooms, four consecutive lessons were recorded, and in the other only the first lesson was recorded. Each focus group involved teachers from both grade levels. In Finland, three teachers taught grade 6, where four consecutive 45-minute lessons were recorded, and one teacher taught grade 7, where three consecutive lessons of 70 minutes were recorded. All the Finnish teachers taught in Swedish-speaking schools in or near Vasa.

Method of analysis

The analysis was conducted in two steps. First, we analyzed the chosen classroom episodes and related questions presented by each teacher. The algebra content of the episodes was subdivided into four topics based on the learning objectives in the episodes, and then each teacher question was categorized according to what aspects of that content the question was about. These aspects were generated from the empirical material. In the second step, a directed content analysis approach (Hsiu-Fang & Shannon, 2005) was used to analyze the focus-group discussions. This approach entails reading textual data and highlighting those parts of the text that, on first impression, appear to be related to the predetermined codes dictated by the theoretical framework, in this case the KAT framework (McCrory et al., 2012). Initially, each researcher immersed herself in the video material from the discussions she had facilitated. By making chronological notes of what was discussed, she turned the verbal discussion into textual data. In these notes, all issues connected to the algebra content were summarized, interleaved with verbatim quotes. In a second viewing of the material, the notes were condensed and the content of the discussion coded according to which of three teaching practices (decompressing, trimming or bridging) was discussed. After an initial trial, the two researchers met to validate the coding, refining the operational definitions and discriminating between the practices. The chronological coding was then restructured by compiling all instances of each practice to look for similarities across the different instances.

As the focus was directed toward teachers' mathematical knowledge in this study, discussions about other issues were not included, e.g. discussions of lesson structure or teacher identity, which were quite frequent in the Finnish data. Throughout the study, the teachers were asked to focus on the algebra content, both when choosing the episodes and in the discussions. Consequently, we cannot draw any conclusions about how content focused these teachers are when planning and reflecting on their teaching in other settings. Both Nyman and Kilhamn (2014) and Holmqvist and Wennås Brante (2011) have argued that teachers in Sweden tend to separate activities from content and attend to the activities and to social features of the classroom more than the content. Therefore, we specifically asked the teachers to focus on the content in their choice of episodes and in their discussions of these.

Results

This section first describes the algebra content in the chosen episodes and gives examples of questions teachers asked about their algebra teaching. The content is summarized in Table 1, where the vertical dimension shows what algebraic ideas are present in the episodes and the horizontal dimension describes aspects of those ideas that were focused on in the question posed. Most of the questions concerned aspects of variables known to be problematic from previous research (Bush & Karp, 2013).

While the use of different representations and the meaning of variable were aspects present in several of the Swedish episodes, the equation-solving procedure was mainly present in Finnish episodes. These results mirror the differences between the two countries' curricula, where Finnish textbooks approach variables by solving equations with one unknown, whereas Swedish textbooks tend to start with creating and interpreting variable expressions, saving methods for equation solving until later. While in previous work we have described how these introductions to variables were enacted in the grade 6 classrooms (Kilhamn, 2014; Röj-Lindberg, Partanen & Hemmi, 2017), we here present results from our analysis of the focus-group discussions around the episodes and questions described above. Using the KAT framework (McCrory et al., 2012), we attempt to describe examples of decompressing, trimming and bridging discussed by algebra teachers.

Table 1. *Examples of teachers' questions, with rows showing different algebraic ideas and columns showing what aspect of the algebraic ideas is in focus in the questions.*

	Representation	Procedure	Choice of examples	Meaning of variable	Teaching approach
Variable expressions	Using symbols to write expressions: What is the student's problem here?		Does this pattern help students to understand what a variable is?	What does the statement "*x* can be anything" mean, and is such a statement true?	Could you start doing operations with letters in a different way in order to promote students' understanding?
Equation/ Equivalence	How could an equation like 9=12–*x* be represented using the manipulatives?			What understanding could this student-created equation enhance? △ +◆+◆+☺= 80	
Equation solving		How can the different ways of solving equations be made comprehensible to students?	Is 5*x*+2=18 too difficult for the students at that stage? How can this "mistake" be utilized?		How can the teacher capitalize on working with student-generated examples?
Algebra as a field of knowledge					Can we increase students' interest in mathematics by referring to the history of mathematics?

Decompressing

Large parts of the discussions concerned unpacking the meaning of algebraic notation, procedures and concepts. Sometimes the practice of decompressing in class was discussed, for example how to unpack the meaning encapsulated in the expression 3*x* and how it could be interpreted by the students. Sometimes, however, the discussion turned into the activity of decompressing itself, when the teachers started unpacking the meaning of something for themselves. Five different aspects of algebra were being unpacked in the focus-group discussions, described below. The *unpacking of an equation-solving procedure* is elaborated in detail as an example of decompressing both in the actual teaching context, and as a reflective practice of unpacking for oneself.

1. Unpacking the concept of variable
The teachers discuss whether a variable always varies, whether defining a variable as ”something that varies” is correct, and whether an unknown in a simple equation like $3x + 1 = 7$ really is a variable.

2. Unpacking the expression $3x$ (referred to in Finland as a monomial)
Both Swedish and Finnish teachers discuss how different contexts can generate an understanding of $3x$ as $3 \cdot x$ or ”three x's”, as repeated addition or as ”groups of x”, as well as the problems involved in seeing x as an object, with $3x$ meaning 3 *and* x, so that subtracting x from $3x$ would only leave 3, and the difficulty in not seeing x as $1 \cdot x$. They also discuss the commutative property where $3 \cdot x = x \cdot 3$, which is not apparent in contexts where 3 and x belong to different measure spaces and play different roles as multiplier and multiplicand.

3. Unpacking the idea of ”difference”
When representing a difference algebraically, the ideas of absolute difference and directed difference became relevant. In relation to two Swedish episodes, the teachers discuss verbal expressions for difference such as ”younger than” and ”older than” and how these connect to addition (counting up) and subtraction (counting down).

4. Dealing with the process/object dilemma of the minus sign
This dilemma appears several times, for example in relation to a Finnish episode (grade 6) when the teachers focus on different ways to solve the equation $20\,\text{kg} - x = 11\,\text{kg}$. It also came up in a discussion about the use of manipulatives (Sweden, grade 6), where the equation $12 - x = 9$ turned out to be impossible to visualize using the manipulatives unless $-x$ was interpreted as ”adding a box with an unknown number of things missing, i.e. adding a negative number”. This is further discussed below, as an instance of trimming.

5. Unpacking an equation-solving procedure
This example of decompressing concerns an episode in a Finnish class (grade 6) where the teacher tried to unpack the meaning of the procedure described in the episode as ”moving the seven to the other side and making it minus”.

$$7 + x = 12$$
$$x = 12 - 7$$

The teacher justified this procedure by saying that she ”takes away 7 from both sides”, but at the end of the episode she said ”I made this very difficult because I wanted to explain why we can do this". This is an example of trying to highlight an underlying complexity in a simple procedure. However, when showing the episode, the teacher explains that she did not think the students understood, and she wonders if the justification procedure was necessary. Another teacher in the group then suggests that she could represent "subtracting the number from both sides" symbolically as well, so that the students could see all the steps. She writes on the board:

$$7 + x - 7 = 12 - 7$$
$$x = 12 - 7$$

At this point, there is a switch in the discussion from unpacking in the classroom to unpacking among themselves. With this notation, the teachers notice new difficulties: students are used to counting from left to right, and since they do not know the value of x they get stuck at $7 + x$. The group does not find a mathematical justification for why it is possible to subtract the seven. One suggestion is to interpret subtraction literally as "take away" by rubbing out the seven. The arithmetical interpretation of the equal sign as "showing the answer" interferes with the necessity of looking at the structure of the problem. Another teacher suggests starting with the variable:

$$x + 7 = 12$$
$$x + 7 - 7 = 12 - 7$$
$$x = 12 - 7$$

The teacher who posed the original question agrees to this idea, but also says she is not "a mathematician" and cannot be expected to explain every rule. She admits that a year ago she did not understand this herself, she simply executed the procedure. She says that it was "a great aha moment" when she did understand that what you really do is "take the seven away on one side and then take it away on the other side". In spite of this, she rounds up the conversation with the words: "Maybe we should just look at it from a long-term perspective. Let the students learn the technique and then perhaps understanding will come later. Learn a simple version first." In this conversation, we notice how the teachers struggle with their own understanding and use of appropriate mathematical language to justify why $7 + x - 7 = x$.

Trimming

When talking about trimming, the teachers take the integrity of mathematics seriously and discuss the consequences of simplifications made for the sake of clarity. They discuss, for example, how a specific representation or context could cause problems on a more abstract level. This often seems to turn the discussion into a decompressing activity, unpacking the deeper mathematical ideas involved. The use of proper terminology comes up several times, indicating an awareness of its importance along with the challenge of managing it. One Swedish grade 6 teacher explicitly points out that she is often unsure about the use and accurate meaning of algebraic terms such as equality, equation, formula, expression, and variable, that appear in her lesson. As a consequence, she is vague and inconsistent in her use of them. She says: "As a grade 6 teacher I am not very experienced in teaching algebra, so there are many concepts and words that are new to me". Four instances of trimming were identified in the discussions.

1. Making use of student contributions

Several questions concern the use of student contributions and the complexity they could introduce that you may want to avoid or that you do not know how to deal with. This is an example of teachers discussing the need for trimming in the moment – trimming down when what a student suggests is too advanced to be fully explored, or trimming up when a student contribution opens up new connections and interesting ideas. The teachers express that they find trimming through adjusting to what comes up in the lesson a demanding practice.

2. Using manipulatives or contexts

Manipulatives such as boxes and beans for visualizing simple equations [2], or contexts such as a balance scale or a function machine, can make some things clear but may limit the mathematics and entail problems further on. One Swedish grade 6 teacher describes how her students had spent two lessons working with boxes and beans to construct and solve equations like $3x + 6 = 15$ and $5x + 2 = 3x + 6$. On a worksheet she provided for the third lesson, one task was to solve the equation $12 - x = 9$. Since this equation was impossible to represent with boxes and beans, several students failed to solve it. From that example the teachers go on to discuss the possibility of representing negative numbers as boxes "missing a certain number of beans", which would have presumed interpreting the equation $12 - x = 9$ as $12 + (-x) = 9$. This trimming of the representation is discarded as complicated and counter-intuitive. The first teacher says: "I feel I simply get stuck somehow, in this manipulative swamp. I don't go on. That's what I feel when I see this – I'm stuck in those damn boxes the whole time." We see this as an example of how the use of manipulatives entails a risk of inadequate trimming.

3. Being careful with the use of terminology when aligning mathematics and context

This aspect of trimming emerges in several discussions. In one episode (Sweden, grade 7) the teacher asks ”How do I express what I will buy?” when she wants the students to write $3x$ for the price of x apples at a cost of 3 crowns each. In the discussion this is highlighted as not being mathematically correct. To avoid misunderstanding, another teacher suggests: ”How do I express *the cost* when I buy x apples?”. While watching the episodes with boxes and beans (Sweden, grade 6), the difference between interpreting the expression $2x$ as ”two boxes ” or ”two times the number of beans in each box” is discussed. The teachers agree that, in school mathematics, a variable is always a number, and that this may not be clear to students in early algebra and needs to be made explicit. Several times the choice of letter is discussed, identifying the risk of interpreting the variable as an object. One teacher says: ”These things are mere details, but I think such details can play an immense role in the learning process”. Being sloppy with vocabulary or inconsistent with symbolic notation might be a result of teachers' wish to avoid complexity but has the opposite effect, and could well be described as a case of inadequate trimming.

4. When the textbook reduces complexity

The Finnish focus group discusses pros and cons of following the textbook in the introduction of different kinds of equations one at a time (from simple addition, such as $x + 2 = 6$, via subtraction, then multiplication and division and finally reaching two-step equations such as $4x + 8 = 56$), versus starting with an equation the students actually cannot solve. One teacher strongly advocates against the latter. According to her it is ”really wrong” to introduce mathematics above the students' current level of understanding. In one episode (Finland, grade 6) the teacher mistakenly introduced the equation $5x + 2 = 18$ instead of the intended $5x + 2 = 17$. She describes the didactic dilemma that appeared when the solution did not stay within the domain of whole numbers. Several approaches are suggested in the discussion, such as exploring possible solution strategies in small groups or comparing the two equations to notice differences and similarities. In contrast to the way the textbook introduces one idea at a time, the discussion brought up the potential of keeping a range of ideas in play in the classroom.

Bridging

In the discussions, there are some examples of bridging on a small, local scale, but none of bridging as making connections between algebra and other topics. However, in one of the groups, the teachers discuss mathematics in general and algebra in particular in connection to mathematics as a science that develops over time and across cultures. They point out that for students it is all about "calculating and getting correct answers" because, to the students, mathematics represents a fixed and stable science. We interpret this as an awareness among the teachers of the necessity of thinking broadly about algebra, connecting it to other aspects of mathematics. The few instances of bridging that were identified in the data were mainly about connecting different representations and keeping a range of ideas in play at the same time.

Discussion

In this study, the teachers discussed questions closely related to their own teaching. We could see that this approach provided relevant questions and rich discussions, and characterize it as an example of professional learning situated in practice, as suggested by Hunter, Anthony and Burghes (2018). We can see that the collegial discussions provided answers and deeper understanding about algebraic issues that are known to be difficult to teach and learn. As Chick (2009) noted, the topic of algebra is found to be challenging, which was true for these teachers as well. In our study, the teachers answer their own and each other's questions about algebra teaching by making use of their subject matter knowledge. However, although the content discussed can be considered as very basic algebra, some comments were mathematically inaccurate. Some discussions led to new insights for the participants in the group, whereas other discussions did not. Several issues were left hanging in the air when no one in the group could offer more input. At times, a discussion ended in mathematically incorrect or ambiguous statements, for example when the discussion about unpacking the equation-solving procedure resulted in justifying subtraction by literally and visually taking away rather than referring to a mathematical argument. Another difficult aspect where no closure was achieved was the process/object dilemma of negative numbers which was discussed at some length in relation to equations and to the use of manipulatives.

While common content knowledge gives teachers the ability to solve relevant equations, decompressing and trimming in the practice of teaching about equations requires more advanced mathematical knowledge such as deep understanding of algebraic structures and mathematical properties. In line with Kieran (2018), we argue that a stronger focus on structure is an important aspect of early algebra that needs to be addressed, not only in schools but also in teacher education.

A general feature of the focus-group discussions is that the grade 7 teachers use more mathematical vocabulary, use it more freely and bring in more specific examples than the grade 6 teachers. This difference could be a result of the way the school system is structured, where, up to grade 6, a generalist teacher typically teaches all subjects, following one student group for several years, which means that grade 6 algebra is taught only once every three years or so. In addition, teacher training for this level, although substantial in pedagogical and didactic areas, does not include university mathematics. The use of terminology in a consistent and adequate way is something that teachers who revisit algebraic equations only once every three years may not be able to develop.

In our analyses of the focus-group discussions using a framework of knowledge of algebra for teaching (McCrory et al., 2012), we gained some insight into the use of algebra knowledge in teacher practices. Some ambiguities and difficulties described by Wassermann (2015) are apparent also in this study, where the demarcation between decompressing and trimming was not always apparent. Our interpretation of these ambiguities is that teachers rarely do either of these things in isolation. We found several instances when a discussion about trimming involved the decompressing of an idea. On the whole, we conclude that decompressing in the sense of unpacking the meaning of an idea for oneself, is an activity teachers engage in whenever they reflect on what they are doing. When, for example, carefully constructing expressions or equations to present students with clear examples free from disturbing complexities, subtle but important differences that emerge as a result of the choices made need to be noticed. Wassermann argues that both decompressing and trimming are about handling complexity: decompressing aims at highlighting it and trimming at reducing it. We agree with this and extend the argument by suggesting that the ability to handle mathematical complexity in the classroom, both in highlighting and reducing it, depends on the ability to decompress the mathematical idea in question.

One noteworthy aspect of the discussions is the teachers' insightful reflections about inadequate trimming. There were several instances in all three focus groups when the teachers critically examined things they did during their lessons, acknowledging that complexity had been removed as a result of the sequencing present in the textbook, the choice of manipulatives, or the variation present in the examples. In all these situations, the mathematical integrity was questioned as the teachers identified potential difficulties or limitations in the students' opportunities for further development. The discussions revealed that these teachers are aware of many of the well-known pitfalls described in the literature (e.g. Bush & Karp, 2013) and raised questions about how to deal with them.

The video-based approach to the study proved immensely valuable for several reasons. Firstly, the videos supported the teachers in their preparation by making them observers of their own classroom, facilitating self-reflection. As one teacher put it: "Being filmed [and watching the film] was very exciting because I realized I did a lot of things in the wrong order in the whole-class instruction." Secondly, watching the chosen episodes brought the other teachers right into the classroom, so that everyone in the conversation was clear about what was going on. In this way, the teachers could make use of each other to better understand what was happening. One teacher explicitly acknowledged this opportunity when introducing her episode, which showed how a student was struggling with variable expressions describing age relations, by saying:

> I think we should watch this to see what I miss, what could have been fruitful but where I don't catch on. [...] I think I make a lot of mistakes, [the student] really offers me these golden opportunities but I don't take them.

As a result of that introduction, the ensuing discussion involved both decompressing the embedded ideas to gain insights into what the student was actually asking questions about, and discussions about trimming by making use of what the student was saying and being very careful and consistent in the use of terminology. Furthermore, one teacher pointed out the difference between a first impression of a lesson and the impression that emerged after watching and reflecting. Given these points, and the results presented in this article, we conclude that the use of video from a teacher's own classroom to support reflection can induce productive discussions about teaching practices.

Conclusion and implications

As shown above, teachers are proficient in collaboratively reflecting on matters of content in relation to students and/or teaching, and make use of their knowledge of algebra when doing so. They engage in decompressing, trimming and bridging practices both in their daily work and in the focus-group discussions. However, we could also see how their algebra knowledge was challenged when they grappled with finding algebraic justifications for arithmetic operations, making use of distinct terminology and uncovering the deeper meanings of algebraic concepts and symbols. Consequently, we urge teacher educators to emphasize such aspects of algebra knowledge in both pre-service and in-service training. In line with van Bommel (2014), we argue that a focus on pedagogical content knowledge in teacher education and professional development is insufficient unless the teachers' subject matter knowledge is simultaneously taken into account.

The results presented above indicate that teachers can identify important issues when they start reflecting on specific content in authentic teaching situations, and that video recordings are useful tools to initiate fruitful discussions. We argue that questions brought up by teachers as relevant to discuss should be taken seriously within teacher education and professional development. The teachers in this study initially asked questions related to their students and teaching practice, but in the discussions, gaps in their subject matter knowledge emerged as they started to unpack the meaning of algebraic concepts and procedures for themselves. An implication for teacher education and in-service training is that teachers may need external input concerning subject matter knowledge to enable further development in pedagogical content knowledge.

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Notes

1 See Kilhamn & Säljö (2019) for a detailed description of the project as a whole.

2 See Rystedt, Helenius & Kilhamn (2016) for more details.

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