University students' general and specific beliefs about infinity, division by zero and denseness of the number line

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A study of university students' beliefs about infinity and related concepts, e.g. division by zero and denseness of the number line, was conducted. The concepts were chosen for the students' proven cognitive challenge in coping with them, and part of the study was to analyze individual beliefs of the different concepts in relation to each other. A questionnaire was designed to discover relationships between pre-service teachers' and technology students' beliefs. Particular foci in the study were general and specific perspectives of the concepts and admission requirements for the programs. The results show incoherence with respect to general and specific representations of aspects concerning denseness of the number line, and also show that admission requirements are significant when it comes to validity of beliefs about division by zero.

Infinity is a concept humankind has struggled to cope with throughout history. Humans are better equipped to handle finite processes and objects, as infinite ones tend to cause contradictions (Fischbein, 2001). During my 20 years as a teacher educator, I have had many discussions with pre-service primary school teachers about division by zero and denseness of the real number line, connected to limits and infinity. The students were often unaware of the fact that it is impossible to divide a number by zero, and a common belief, in line with prior research results (e.g. Crespo & Nicol, 2006) was that it is equal to the number itself or zero. Division by natural numbers is an area that pre-service teachers are familiar with, but when it comes to division with zero, there are more complicated aspects to consider. When faced with questions about division by zero in class, the pre-service elementary school teachers in Crespo and Nicol's (2006) study were reluctant to find answers on their own. Finding

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Juter, K. (2019). University students' general and specific beliefs about infinity, division by zero and denseness of the number line. *Nordic Studies in Mathematics Education*, 24 (2), 69–88.

answers of this kind demands knowledge of related concepts. Aspects related to division by zero can be complex concepts to understand, as has been shown in several studies, e.g. Sbaragli (2006), Tall and Tirosh (2001) and Tsamir (2001) about infinity, Juter (2007, 2009) about limits and Katz and Katz (2010), Vamvakoussi and Vosniadou (2012) and Yopp, Burroughs and Lindaman (2011) about the denseness of the number line. A part of learning mathematics is to extend already known properties to a new mathematical context, which means that properties that are applicable in one context may need to be reassessed in the new context. Tsamir (1999) concluded, from prior research, that students have a tendency to adopt all properties from a specific domain of numbers when expanding to a more general domain without considering the consequences of the more general situation. Division by zero is an example of one such expansion from division by numbers larger than zero. Tsamir and Tirosh (2002) stated that many secondary school students thought that division by zero results in a number, as the pre-service teachers did in Crespo and Nicol's (2006) study. Tsamir and Tirosh's results also showed that the students' age (grades 9–11) did not improve their performance on tasks about division by zero. Another example of expansion is comparisons of finite sets expanded to comparisons of infinite sets, as discussed in Tsamir's (1999) and Sbaragli's (2006) papers. A majority of 16 primary school teachers studied in Sbaragli's research were inclined to believe that there are more natural numbers than either odd or even numbers, and only a few teachers stated that infinite sets cannot be compared. Yet another example is to go from finite to infinite decimal expansions. Yopp, Burroughs and Lindaman (2011) found that in-service elementary school teachers' understandings, but also misunderstandings, of repeated decimal expansions continued developing during their professional years, and they stressed the importance for elementary school teachers to understand concepts related to infinity. Elementary and primary school teachers work with mathematics in many areas related to infinity, e.g. division, the number line and various sets of numbers, albeit on a low level, but it requires a solid understanding to be able to provide a useful foundation for the children to develop their learning from.

The results reported raise more questions concerning students' beliefs about these concepts, particularly about relations between different types of representations of the concepts in the students' learning processes. Juter (2007) examined students' learning limits of functions and found that the students noticed different things from a general context about theory than from a specific context of problem-solving. Several students believed that limits are unattainable from interpreting a general limit definition wrongly, but when solving a problem with a limit of a specific linear function, they claimed that the function can attain that value, causing a cognitive conflict for them. The terms *general* and *specific* will be defined in the theoretical frame section and used through the present study. The objective is to add knowledge about students' beliefs about infinity, division by zero, and denseness of the number line in general and specific contexts, and relations between the beliefs. Pre-service teachers at various stages of their education on three different programs and technology students in one program were studied. The technology students and one of the pre-service teacher groups (studying to teach at secondary school level) were selected for the higher program admission requirements in mathematics as a factor for comparison. The research questions are:

1. What beliefs do students have about infinity, denseness of the number line and division by zero?
2. How do students' general and specific beliefs cohere in relation to aspects of infinity?
3. How do students on programs with different admission requirements in mathematics compare with respect to research questions 1 and 2?

Theoretical frame

The notion belief is well known for its intangibility and many diverse definitions (Leder, Pehkonen & Törner, 2002). Beliefs are here defined as *subject-matter beliefs* (Törner, 2002), where the belief objects are specific to mathematics, e.g. infinity and division. Törner stated that all mathematical terms, objects and processes can be objects of beliefs, and students' beliefs about mathematics are in this way integrated, through experiences of all kinds, in their concept images (concept image as defined by Tall and Vinner, 1981). Students experience university mathematics in various programs with their particular conditions and Bingolbali and Monaghan (2008) found that students' concept images for derivatives developed differently depending on programs' different perspectives and teaching practices. General and particular or specific experiences can develop concept images in different ways, as reported in Juter (2007). *Specific* is here used in Mason and Pimm's (1984) sense. A specific case of functions can be a certain line, e.g. $y = 7x - 2$, in contrast to lines as particular functions, e.g. $y = kx + m$. The general perspective in this case includes all functions. The different learning experiences may result in inconsistencies or contradictory beliefs caused by *compartmentalization*

(Vinner, 1990). Compartmentalization occurs when an individual is expected to evoke a particular detail in a particular context relevant for that detail, but this does not happen. Vinner compares the phenomenon with forgetting in the sense that the mind is less efficient than it would be if all details relevant in the context were evoked.

Different views of infinity influence the use of the concept. Students can, for example, adopt various strategies for comparing infinite sets, used locally or globally. In this paper, a *local* use is when students focus on parts of sets in their comparison, e.g. deciding that there are more whole numbers than even numbers, since every other whole number is an even number. A *global* use is in line with Tsamir's (1999) description of a global view as including views of all infinite sets having the same number of elements or that infinite sets cannot be compared in the number of elements. Focusing on particular elements of an infinite set, and not on the whole set, can be seen as a way of reducing the abstraction level (Wijeratne & Zazkis, 2016). Fischbein (2001) described this type of divided focus as a dynamic kind of infinity with an endless number of processes, each finite, in a way to cope with what cannot be conceived of as a whole, e.g. the set of odd numbers where each odd number is followed by another odd number, i.e. every other number in the set of whole numbers. When a person handles very abstract concepts, they tend to use alternative representations or mental models, which are more accessible but may lead to contradictions or errors (Fischbein, 2001). Humans handle complexity, such as infinite contexts, by deliberately focusing on relevant information in smaller pieces (Tall, 2001). These results combined mean that infinity may prompt students to evoke different parts of their concept images, depending on contexts, only to consider local details and all within a model that is accessible but possibly not completely accurate.

The study

A pilot study with informal and formal interviews and questionnaires designed on the basis of prior research (e.g. Sbaragli, 2006 and Tsamir & Tirosh, 2002) and experience, as mentioned previously, was conducted in order to find relations in university students' beliefs about concepts related to infinity so as to outline further investigations. Incoherent concept images of the number line, where it is uncertain whether different numbers can be placed close together with no other numbers between them, came to the fore, as did division by zero. A study was then conducted at a Swedish university, based on the pilot study. Preservice teachers studying to teach school years F–3 (ages 6–9) and 4–6 (ages 10–12) were first chosen with students at different stages in their programs to reveal possible differences and similarities through the programs (this will not be addressed in this paper, since the beliefs were similar in the different years). All students in five mathematics classes, two at the beginning of the programs and three closer to the end, and all students writing their exam assignments in mathematics at the end were chosen for the study. These students had an admission requirement of mathematics B (henceforth abbreviated as MaB) from upper secondary school, which means that they had studied real numbers, algebra, functions, statistics and geometry at a basic level. The mathematics courses they attended at the university aimed at deepening their knowledge in these areas, and did not cover any differential calculus. These students' pending profession requires them to teach about division, rational numbers, the number line and other concepts related to the concepts studied here. Another group of students with higher admission requirements from upper secondary school, mathematics D (henceforth abbreviated as MaD), were added as a contrast to compare the first group with. The students were from a technology program (year 1 with 24 students of which 17 were international students and year 2 with 4 students) and a pre-service secondary school mathematics teacher program (year 1 with 4 students). All students in the classes were chosen for the data collection, and the students attending the specific lecture when the data collection was conducted took part. The students with MaD had studied trigonometry, more details about functions, and differential- and integral calculus at upper secondary school in addition to what the students with MaB had. In total, there were 137 students, with 105 in the first group of students (with MaB, henceforth labelled group B for *basic* experience of mathematics) and 32 in the second group (with MaD, henceforth labelled group D for *deeper* experience of mathematics). The aim was to discover different beliefs and relations between the beliefs, with respect to prior knowledge if applicable.

Rationale of the data collection and analysis

14 statements were used to collect data. The aim was to depict students' beliefs concerning a set of statements about concepts that are related to infinity. Students were asked to mark whether they agreed, disagreed or were unsure by underlining the corresponding words for each statement. The words *Agree*, *Do not know* and *Disagree* were listed below each statement. The statements were formulated based on experiences of common student beliefs from teaching, the pilot study with an earlier version of the questionnaire and interviews testing the questions. The focus was

not to have a mathematically exhaustive set of statements about the concepts, but statements likely to provoke and expose students' beliefs about the concepts. The statements used are the following.

1C	There are twice as many integers as even numbers.
2C	You cannot compare the number of even numbers with the number of integers since there are infinitely many.
3C	You can compare the number of even numbers and integers since you know that every other integer is an even number and every other one is an odd number throughout.
4DZ	Two divided by zero is a number.
5DZ	Two divided by zero is equal to infinity.
6DZ	You cannot calculate two divided by zero.
7DNL	There is an infinite number of numbers between any two different numbers.
8DNL	There is at least one number between any two different numbers.
9DNL	Two different numbers may lie close together without any other number between them.
10DNL	The number 0.99... , where the number of nines after the comma is infinite, is equal to 1 since there is no number between 0.99... and 1.
11DNL	There are no numbers between 0.99... and 1, but 0.99... and 1 are different numbers anyway.
12DNL	There are numbers between 0.99... and 1, so they are different numbers.
13	The universe is infinite.
14	Infinity is a human construct and does not exist in the real world.

The letters after the numbers of statements 1–12 are abbreviations of topics in the statements, where C stands for cardinality, DZ for division by zero and DNL for denseness of the number line. Statements 1–3C were designed to reveal students' beliefs of infinity as relative or not, i.e. whether they believed the sizes of infinite sets can be compared. The statements were used to indicate whether students had a local or global view on infinity. A global view is where infinite sets are regarded as whole (Tsamir, 1999), and a local view is when students zoom in and locally regard the pattern of every other number as being even and use that as a means to compare infinite sets in a one-to-one manner. As examples, agreeing with statement 2C implies a global view and agreeing with statement 3C implies a local view. One aim with statements 1–3C was to be able to compare students' beliefs about comparing infinite sets to their responses about division by zero and denseness of the number line.

Statement 4DZ is an amalgam of two common answers to the specific problem ”two divided by zero” from pre-service teachers when they come to teacher education, namely zero and two. The aim was to see whether students had concept images allowing treatment of division by zero in the same way as division by any other number without recognizing the crucial differences in the situations. The following two statements (5DZ and 6DZ) were to reveal how infinity was linked to division by zero, i.e. whether they thought something could be said to equal infinity, or whether it is possible to perform the division at all.

Statements 7–9DNL were generally (as defined by Mason and Pimm, 1984) formulated to show the students' beliefs of denseness of the number line. A possible weakness of the formulations is that the number line or real numbers were not mentioned, as a complement to the formulation ”any two different numbers”, to avoid misunderstandings about what type of numbers was meant. In an attempt to keep the questionnaire as simple as possible, without losing clarity, the formulation ”any two different numbers” (”vilka två olika tal som helst” in Swedish) was chosen over ”two different numbers” (”två olika tal” in Swedish) to emphasize that the statements were about all numbers, not just for example integers. The pre-tests of the statements did not show any confusion among the participants and the formulation was kept with no complement. The following statements (10–12DNL) address the same issues in a specific case. Statement 10DNL addresses beliefs about the specific numbers 0.99... and 1 being equal, statement 11DNL about the numbers being different, but close together, and statement 12DNL about the numbers being different with other numbers coming between them. The aim was to see how the general and specific parts of the students' concept images cohered when evoked on the same occasion. Agreeing to both statements 7DNL and 11DNL, as an example, implies a cognitive conflict between a general and a specific case. The study mentioned previously (Juter, 2007) showed students' contradictory concept images of limits of functions when dealing with a general definition and a specific example and one aim here is to discover whether there are similar contradictory beliefs linked to students' beliefs of denseness. One result of the pilot study was that the students did not regard the representations 0.99... and 1 as representations of the same number, and that it is possible to add another 9 in the infinite line of nines.

Finally, statements 13–14 were added so as to include more everyday beliefs about infinity. The analysis of these statements is not included in this paper. They are merely included to give a complete description of the questionnaire.

The students filled out the questionnaire anonymously to prevent them from feeling constrained by disclosing their names. There were no time limits, and it took about ten minutes in general for the students to complete it. The students were not asked to give reasons for their responses in the questionnaire as it was designed to be short and not overwhelming. Interviews with 8 students were conducted as a complement. The results from them are not included in this paper since the aim does not cover the reasons for the students' responses.

Analysis and results

The analysis was first done with the students categorized by program and semester in several groups in order to determine any overall differences or similarities in the answers, aided by the computer software system IBM SPSS Statistics. The first analysis was also done to roughly determine whether there were any trends in relation to program admission demands in mathematics for the programs. The answers were quite diverse for some statements, but there were similarities among the students in group B and group D respectively. The sets of student groups were compared using SPSS for cross tabulation and clustering to unveil relationships between students' answers to the questions. The analysis was done with respect to various aspects, e.g. admission requirements, local and global perspectives on infinity and beliefs from general and specific contexts.

The students' overall beliefs

The overall results, as depicted in appendix, show that a majority of the students thought that you can compare the number of even numbers with the number of integers since every other integer is an even number (agree with statement 3C). The majority also thought that two divided by zero is a number (agree with statement 4DZ) that you can calculate (disagree with statement 6DZ). A vast majority believed that there are numbers between any two different numbers (agree with statements 7DNL and 8DNL), and a majority also stated that two different numbers cannot lie close together without any other number between them (disagree with statement 9DNL). Opposing the results on statements 7–9DNL, a majority did not think that 0.99... is equal to 1 in view of the fact that there are no numbers between them (disagree with statement 10DNL). A majority thought that there are numbers between 0.99... and 1 and that they are hence different numbers (agree with statement 12DNL). The students' responses to statements 1C, 2C and 11DNL were more evenly distributed across the answers *Agree* and *Disagree*, showing ambivalence in the group as a whole. A deeper analysis was conducted to further penetrate the data set for statements 1–12.

The students' beliefs about statements 1–6

The overall result for statement 1C, ”There are twice as many integers as even numbers”, was quite even as table 3 in appendix shows, with 39 % agreeing and 38 % disagreeing. A majority (59 %) thought that the numbers of even numbers and whole numbers are comparable, since every other number is even and the rest odd (statement 3C), implying a local view of infinity, as defined previously. 40 of the 53 students (75 % of the 53) who agreed with statement 1C also agreed with statement 3C, and did not agree with statement 2C (38 disagreed, 2 did not know), which is a further indication of a local view of infinity among these students (29 % of the entire group answered this way). In statement 2C, ”You cannot compare the number of even numbers with the number of integers since there are infinitely many”, agreeing implies a global view of infinity which 47 or 34 % of the students had. 23 of these 47 students (49 % of the 47) did not agree with statements 1C and 3C, further implying a global view. The students' responses to the three statements, classified as indicators of local or global views of infinity if possible, were compared to responses to the rest of the questions.

The analysis of statements 4–6DZ showed significant differences with respect to admission requirements. The results are hence presented in the following section covering that aspect.

The students' beliefs about statements 7–12

Statements 7–9DNL are general statements about the proximity of real numbers. Most students agreed with statements 7DNL and 8DNL, stating that there are numbers between any two different numbers, as depicted in appendix. Despite this, 43 students (31 %) agreed with statement 9DNL, i.e. ”Two different numbers may lie close together without any other number between them”. 3 students agreed with statement 9DNL and disagreed with both statements 7DNL and 8DNL, i.e. coherently stating that different numbers may lie close together, leaving 40 students (29 %) revealing a contradictory position about whether or not different numbers can lie close together. All 3 students disagreeing with both statements 7DNL and 8DNL were from the beginning of the lower primary school teacher program and the result may come from the vagueness of the statements in the questionnaire, i.e. *numbers* may have been

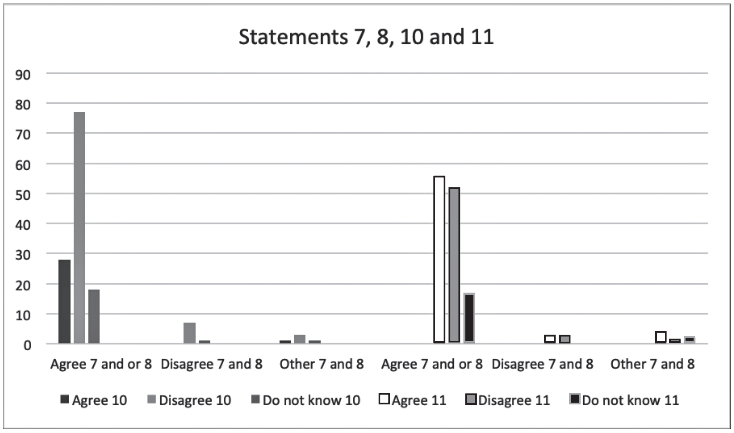

Figure 1. *Numbers of responses to statements 7DNL and 8DNL compared to responses to statements 10DNL (three first sets of bars) and 11DNL (three last sets of bars)*

interpreted as *integers*, a specific type of numbers these students often work with.

From statement 10DNL, 29 of the 137 students thought that 0.99... and 1 are equal (figure 1), since there is no other number between the two. All but one of the 29 students had agreed with statement 7DNL and/or 8DNL, stating that they believed that any two different numbers have at least one other number between them. General and specific situations were hence coherent in this respect, but only 3 of the 29 students disagreed with both statements 11DNL and 12DNL (both states that 0.99... and 1 are different) which reveals incoherent beliefs among the remaining 26 students when it comes to what constitutes different numbers on the real number line.

Statement 11DNL states that the numbers 0.99... and 1 are different and lie close together without any other numbers between them. The right-hand side of figure 1 again shows that almost all students generally thought that there have to be numbers between two different numbers (from statements 7DNL and 8DNL), but a slight majority also agreed with statement 11DNL (56 students) in conflict with their answers to statements 7DNL and 8DNL. Of the total of 63 students agreeing with statement 11DNL, 7 did not agree with statement 7DNL or 8DNL and 19 agreed with both statements 10DNL and 11DNL. The latter shows a contradiction within a specific example, i.e. there are no numbers between 0.99... and 1 but the numbers are equal in statement 10DNL and not equal in statement 11DNL.

Answers to statement 11DNL and 12DNL were compared to show possible coherence within the statements about the numbers 0.99... and

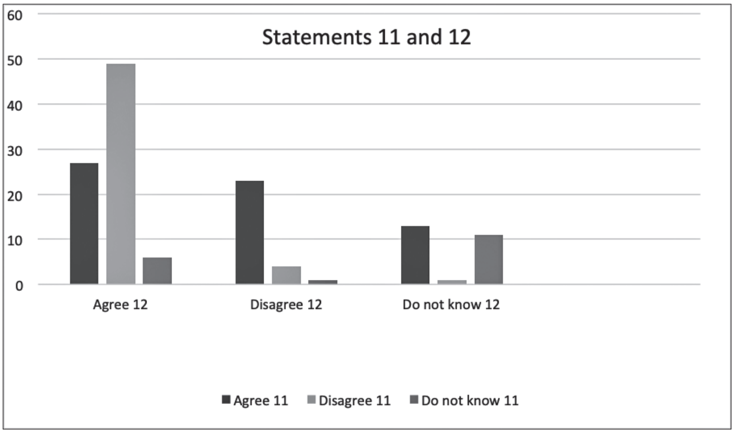

Figure 2. *Numbers of responses to statements 11DNL and 12DNL compared*

1 being different and the general views shown in answers to statements 7DNL and 8DNL. Most students (72) answered either *Agree* or *Disagree* to statement 11DNL and the opposite of the two answers to statement 12DNL (see figure 2), implying that they think the numbers are different and that they take a stance on why. This is further emphasized by the fact that 59 of the 72 students disagreed with statement 10DNL (9 agreed and 4 did not know). A majority (of the 72), 49 students, agreed with statement 12DNL and disagreed with statement 11DNL, i.e. they thought there are numbers between 0.99... and 1. It is not revealed in this study what type of numbers the students are thinking of, but the pilot study had examples of individuals stating that ”you can always add a 9”. 39 % of the 49 students had a local view on infinity compared to 22 % among the remaining 88 students or 28 % in the whole group of students. The local perspective can make the students focus on the numbers they think are at the end of the line of digits, even though it is infinite, rather than on the number 0.99... as a whole compared to 1. On the other hand, 23 students agreed with statement 11DNL and disagreed with statement 12DNL, i.e. they believed that there are no numbers between 0.99... and 1, but they are not equal. This contradicts the dominant general belief that there is at least one number between any two different numbers (21 of the 23 students had agreed with statement 7DNL and/or 8DNL, 1 had disagreed with both statements and 1 was unsure on both).

To sum up, almost all students who claimed that there are no numbers between 0.99... and 1, and that they are different numbers, also stated that there are numbers between any two different numbers (45 students of 50). They had contradictory concept images when it comes to general and specific cases dealing with the number line. On the other hand, almost all students who claimed that 0.99... is equal to 1 also stated that any two different numbers are separated by other numbers (28 of 29), showing traces of a coherent concept image in that perspective.

Results from an admission requirements perspective

When the students were divided according to admission requirements, some interesting patterns became visible (see tables 4 and 5 in appendix for answer frequencies in the two groups separated). Cluster analyses in SPSS showed that there were differences in the students' responses to statements 1C, 4DZ, 5DZ, 6DZ and 12DNL, with major differences for statements 4DZ and 6DZ. The distribution of responses for statement 1C was skewed towards *Agree* for the students in group D (the higher admission requirement) and towards *Disagree* for the students in group B. Statements 2C and 3C were not as skewed with respect to admission requirements. The distribution of local and global views, based on all three statements, in the different admission groups is depicted in table 1, and shows that the distribution was rather even for the students in group B, whereas the students in group D tended to have a local view of infinity.

Table 1. *Distribution of students' local and global views of infinity based on statements 1–3C, depending on admission requirements, in numbers (and percent)*

	Local	Global
Group B (n = 105)	24 (23 %)	20 (19 %)
Group D (n = 32)	16 (50 %)	3 (9.4 %)

The set of clusters in table 2 shows the major differences in the students' responses to statement 4DZ, "Two divided by zero is a number", and statement 6DZ, "You cannot calculate two divided by zero". The clusters had a good cluster quality with a silhouette measure of cohesion and separation of 0.6 on a scale from -1 to 1.

Cluster 2 comprises 100 % students from group B. All believed that two divided by zero can be calculated and that the result is a number. The majority of the students in cluster 1 were from group D and the trend among their beliefs contradicted the beliefs in cluster 2 for statements 4DZ and 6DZ. A closer look at the results for statements 4–6DZ shows that the students in group D tended to think that division of two by zero is not a number. They did not agree on whether it is equal to infinity or not, but they tended to think that you cannot divide two by zero (see figure 3). The students in group B showed an almost opposite pattern

Table 2. *Clusters based on admission requirements and statements 4DZ and 6DZ*

Cluster inputs	Cluster 1 (73 students)	Cluster 2 (64 students)
MaB/MaD	65 % MaD	100 % MaB
Statement 4DZ	67 % disagree	100 % agree
Statement 6DZ	63 % agree	100 % disagree

in their responses. Most of them thought that two divided by zero is a number, they did not think it is equal to infinity, but many did not know, and a majority believed that you can divide two by zero. Beliefs that two divided by zero can be equal to infinity may stem from experiences from examples tending to infinity when dividing numbers with numbers close to zero. Students with a global view of infinity were slightly more inclined to think that two divided by zero is equal to infinity as 17 % of the students with global view and 7.5 % of the students with a local view agreed with statement 5DZ.

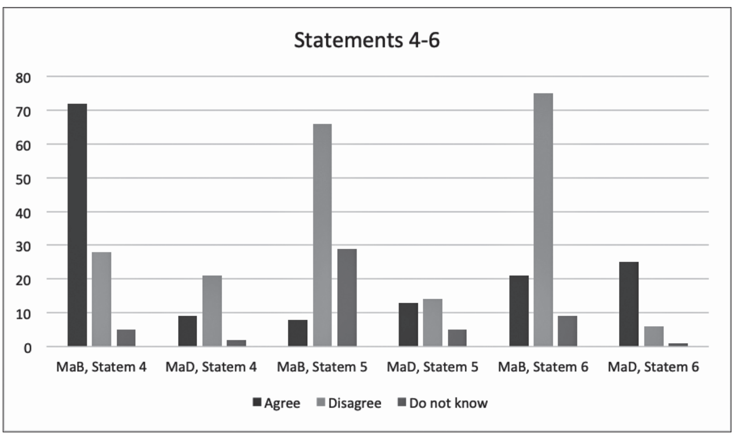

Figure 3. *Numbers of responses to statements 4–6DZ with respect to admission requirements*

The results for statements 7–11DNL were not very different with respect to admission groups. For statement 12DNL there was no significant difference in distribution when it came to how many agreed, but 16 % of the students in group B disagreed and 24 % did not know, whereas 31 % disagreed and 3 % did not know in group D, showing more certainty (see tables 4 and 5 in appendix). 30 students answered either *Agree* or *Disagree* to both statements 11DNL and 12DNL. 27 students agreed and hence claimed that there are numbers between 0.99... and 1, and also that there are not, i.e. 17 % of the students in group B and 28 % of the students in group D agreed with both statements. 24 students of the 27 had agreed with statement 7DNL and/or 8DNL stating that they believe that, in general, any two different numbers have at least one other number between them. Only 3 students (1 in group B and 2 in group D) disagreed with both statements and all 3 agreed with statement 10DNL (that the numbers are equal) implying coherent beliefs regarding these statements.

Discussion and conclusions

The topics in research question 1 are discussed with aspects of the following two research questions, when applicable, throughout, i.e. general and specific beliefs and admission requirements.

Infinity

The distribution of students agreeing and disagreeing with statement 1C (There are twice as many integers as even numbers) was even in the entire group, but skewed towards *Agree* for students with the higher admission requirement, group D, and *Disagree* for students with the lower requirement, group B, i.e. the pre-service primary school teachers. This stands in contrast to Sbaragli's (2006) results, where a majority thought that there are more natural numbers than odd or even numbers. The contradictory results may be due to differences in different countries' educational systems. The differences in the two groups with different admission requirements in the present study was not expected. More than half of the students in group D were from other countries, so no conclusions about the Swedish school system before university level can be derived from the study in that respect. Table 1 shows that students in group D were more inclined to have a local view on infinity, whereas students in group B were more evenly divided on local and global views on infinity when statements 1–3C were used to classify them. These results on infinity hence add to Bingolbali and Monaghan's (2008) conclusion that students develop concept images of derivatives differently in different programs. The clearest result from this separation was that students with a local view were more inclined to think that there are numbers between 0.99... and 1, which will be discussed further in the next section.

Denseness of the number line

Most students thought that in general there is at least one number, and hence infinitely many numbers, between any two different numbers (figure 1). In the specific case of comparing 0.99.... and 1, that rule was not applied by individuals stating that they are different numbers but there are no numbers between them (agreed with statement 11DNL), i.e. a possible case of compartmentalization (Vinner, 1990). Even though the students' concept images were evoked from a general and a specific perspective within a short time span, they answered contradictorily in the two situations. This distinction between general and specific cases is similar to students' beliefs of attainability of limits of functions in Juter's (2007) study. Several students in the study claimed that the general epsilon-delta definition of limits of functions means that all limits are unattainable for all functions, while at the same occasion stating that a specific function, e.g. a linear function, can attain a limit value. Different parts of students' concept images may be evoked by the general and specific contexts in this case as well. The two can co-exist if the students handle the complexity of the situation by focusing on smaller pieces as Tall (2001) describes. Students may fail to attend to important links between properties of the concepts in such a cognitive conflict, i.e. the evoked part of the concept image is too narrow. The general beliefs were overridden in both cases in favor of the specific cases. The general belief was, however, wrong in the case of limits and correct in the case of numbers on the number line. The students who thought that there are numbers between 0.99... and 1 (agreed with statement 12DNL) kept true to their general beliefs, but the specific case was not correctly understood. Figure 2 shows that more students agreed with statement 12DNL and disagreed with statement 11DNL than the other way around. The majority hence believed that there are numbers between 0.99... and 1. The infinite line of digits in 0.99... was seen as something that eventually ends and to which another digit can be added, in a local interpretation of infinity in the sense that the students focused only on a part of the listing of nines. This may be due to transfer of behavior they are used to from prior experiences of finding numbers between different numbers on a number line. If they have always been able to find numbers to place between different numbers in the past, they might expect there to be so in this case too, without considering the particularities of the situations and if any possible such numbers exist. The behavior is then a form of adoption of properties from a familiar situation to a situation where the students do not recognize the new aspect, i.e. there are no numbers to find between the numbers since there is an infinite number of nines. This is in line with Tsamir's (1999) report of students' tendencies to adopt all properties from a specific domain when expanding to a more general domain without considering all the consequences. The fact that the representations 1 and 0.99... differ so much in appearance may add to the cognitive conflict. It may be easier to accept for example that 1/3 and 0.33... are equal since neither of them are integers and one representation does not seem to be smaller than the other as 0.99... does compared to 1.

Division by zero

One hypothesis about the statements about division by zero was that the results from the pre-service primary school teachers (with MaB) in their first year would be similar to those from Crespo and Nicol's (2006) and to Tsamir and Tirosh's (2002) studies, where several pre-service elementary school teachers and secondary school students respectively believed that division by zero is a number. One reason is that they typically did not change beliefs about this in secondary school, according to Tsamir and Tirosh (2002). The students in group B had less prior mathematics experience than the students in group D. The first group would hence have a background in mathematics that is more similar to that of the secondary school students. The hypothesis turned out to be partly true, as table 2 and figure 3 so clearly show. The hypothesis was for pre-service primary school teachers in their first year, but this was true of all years for these students. Division by zero turned out to be the area with the largest student belief differences, with respect to admission requirements, of the areas studied here. A majority of the students with deeper experiences of mathematics (MaD) answered correctly to statements 4DZ and 6DZ, which could be expected. The results from the pre-service primary school teachers about division by zero implies a need for a more meticulous treatment of these concepts in the programs to rise the pre-service teachers' awareness and prevent such misconceptions from spreading to pupils in schools.

Conclusions

To conclude, the pre-service primary school teachers had serious misconceptions about division by zero, and they did not show any trends in preference about a local or global view on infinity. The students in group D had more accurate beliefs about division by zero, and they tended to have a local, rather than a global, view of infinity. Both groups showed inconsistencies about denseness of the number line from general and specific contexts, and one group's beliefs did not significantly differ from the other. The misconceptions and inconsistencies in the pre-service teachers' beliefs may come to interfere with their teaching about for example division or the number line. Questions from pupils may be answered inaccurately and learning possibilities may be overlooked.

The students in group B had not studied much calculus and were less mathematically experienced than the students in group D. This may have resulted in difficulties for the students in group B to understand some of the statements in the questionnaire. The statements were formulated to be as simple and clear as possible to adapt to the different levels of experience the students had, with the vagueness mentioned in the methods section as a result, i.e. both questionnaire and variation in student experiences need to be considered in the results of the study. This study cannot be expanded to investigate aspects of development of particular individuals' beliefs, since the participants were answering anonymously. Such investigations would be possible steps to take further in the pursuit of understanding students' beliefs of infinity related to other concepts. Another area for further research is the effects on beliefs and understanding from activities designed to promote learning in these areas. The results of this study, along with the reluctance among students to explore properties about division by zero found in Crespo and Nicol's (2006) study, implies a need for a more exploratory approach in teacher education mathematics courses. An awareness of the own limitations in mathematics and confidence to investigate the unknown are important means to find answers to all types of questions pupils may ask, e.g. "What is two divided by zero?" or "Are there fewer odd numbers than integers?".

Acknowledgements

I would like to thank Kristianstad University for funding the study and the reviewers of this paper for their valuable comments.

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Appendix

Table 3. *Overall results for statements 1–14 in numbers (and percent) for the 137 students (group B and group D). Percentages from 50 and above in bold*

Statement	Agree	Disagree	Do not know	No answer
1C	53 (39 %)	52 (38 %)	29 (21 %)	3 (2.2 %)
2C	47 (34 %)	60 (44 %)	29 (21 %)	1 (0.7 %)
3C	**81 (59 %)**	28 (20 %)	25 (18 %)	3 (2.2 %)
4DZ	**81 (59 %)**	49 (36 %)	7 (5.1 %)	0
5DZ	21 (15 %)	**80 (58 %)**	34 (25 %)	2 (1.5 %)
6DZ	46 (34 %)	**81 (59 %)**	10 (7.3 %)	0
7DNL	**105 (77 %)**	18 (13 %)	12 (8.8 %)	2 (1.5 %)
8DNL	**119 (87 %)**	11 (8.0 %)	7 (5.1 %)	0
9DNL	43 (31 %)	**76 (55 %)**	17 (12 %)	1 (0.7 %)
10DNL	29 (21 %)	**87 (64 %)**	20 (15 %)	1 (0.7 %)
11DNL	63 (46 %)	53 (39 %)	19 (14 %)	2 (1.5 %)
12DNL	**82 (60 %)**	27 (20 %)	26 (19 %)	2 (1.5 %)
13	**95 (69 %)**	11 (8.0 %)	31 (23 %)	0
14	29 (21 %)	**69 (50 %)**	38 (28 %)	1 (0.7 %)

Table 4. *Overall results for statements 1–14 in numbers (and percent) for the 105 students in group B. Percentages from 50 and above in bold*

Statement	Agree	Disagree	Do not know	No answer
1C	33 (31 %)	43 (41 %)	27 (26 %)	2 (1.9 %)
2C	37 (35 %)	43 (41 %)	24 (23 %)	1 (1.0 %)
3C	**57 (54 %)**	21 (20 %)	25 (24 %)	2 (1.9 %)
4DZ	**72 (69 %)**	28 (27 %)	5 (4.8 %)	0
5DZ	8 (7.6 %)	**66 (63 %)**	29 (28 %)	2 (1.9 %)
6DZ	21 (20 %)	**75 (71 %)**	9 (8.6 %)	0
7DNL	**80 (76 %)**	13 (12 %)	11 (10 %)	1 (1.0 %)
8DNL	**89 (85 %)**	9 (8.6 %)	7 (6.7 %)	0
9DNL	35 (33 %)	**53 (50 %)**	17 (16 %)	0
10DNL	22 (21 %)	**63 (60 %)**	19 (18 %)	1 (1.0 %)
11DNL	45 (43 %)	41 (39 %)	17 (16 %)	2 (1.9 %)
12DNL	**62 (59 %)**	17 (16 %)	25 (24 %)	1 (1.0 %)
13	**75 (71 %)**	5 (4.8 %)	25 (24 %)	0
14	16 (15 %)	**58 (55 %)**	30 (29 %)	1 (1.0 %)

Table 5. *Overall results for statements 1–14 in numbers (and percent) for the 32 students in group D. Percentages from 50 and above in bold*

Statement	Agree	Disagree	Do not know	No answer
1C	**20 (63 %)**	9 (28 %)	2 (6.3 %)	1 (3.1 %)
2C	10 (31 %)	**17 (53 %)**	5 (16 %)	0
3C	**24 (75 %)**	7 (22 %)	0	1 (3.1 %)
4DZ	9 (28 %)	**21 (66 %)**	2 (6.3 %)	0
5DZ	13 (41 %)	14 (44 %)	5 (16 %)	0
6DZ	**25 (78 %)**	6 (19 %)	1 (3.1 %)	0
7DNL	**25 (78 %)**	5 (16 %)	1 (3.1 %)	1 (3.1 %)
8DNL	**30 (94 %)**	2 (6.3 %)	0	0
9DNL	8 (25 %)	**23 (72 %)**	0	1 (3.1 %)
10DNL	7 (22 %)	**24 (75 %)**	1 (3.1 %)	0
11DNL	**18 (56 %)**	12 (38 %)	2 (6.3 %)	0
12DNL	**20 (63 %)**	10 (31 %)	1 (3.1 %)	1 (3.1 %)
13	**20 (63 %)**	6 (19 %)	6 (19 %)	0
14	13 (41 %)	11 (34 %)	8 (25 %)	0

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