# Development of algebraic thinking: opportunities offered by the Swedish curriculum and elementary mathematics textbooks

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In search of the reasons for Swedish students' low achievement in algebra in international and national evaluations, we investigate how the development of algebraic thinking is addressed in the Swedish national mathematics curriculum and two widely used mathematics textbook series for grades 1–6 in Sweden. The analytical tool used is based on the classification of "big ideas" which research has shown as important for developing pupils' algebraic understanding in early school grades. The results show that functional thinking, expressions, and equations are well represented topics both in the curriculum and the textbooks; however generalized arithmetic is a topic that is poorly developed in both the curriculum and the textbooks.

In the past two decades, the research field of *early* algebra has emerged. The assumption that young children would not be capable of thinking algebraically has been challenged by several mathematics educators (Blanton et al., 2015). Also, the idea that students' development of algebraic concepts would be reflected in the historical development of algebra (Sfard, 1995; Katz & Barton, 2007) has been questioned (Bråting & Pejlare, 2015), and recent studies show that it is possible and even beneficial to start working with algebraic ideas and generalizations in parallel with arithmetic in early grades (Cai et al., 2005; Carraher, Schliemann, Brizuela & Earnest, 2006). The focus of these studies is on the nature of algebraic thinking and how it can be developed for children in primary and lower secondary school. At the heart of the matter are activities such as mathematical structure (Blanton et al., 2015), the usage of numbers and words in general terms (Britt & Irwin, 2011), and reasoning that expresses relationships between numbers and quantities (Carraher & Schliemann, 2015). In turn, these activities include processes such as noticing,

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conjecturing, generalizing, representing, justifying, and communicating (Kieran, Pang, Schifter & Ng, 2016).

Similarly, the emergence of early algebra as a research field has influenced school mathematics. Several countries, including Sweden, have revised their curricula in order to incorporate algebra in primary school (NCTM, 2000, 2006). Nonetheless, Swedish students' results in algebra in national and international evaluations have remained poor. Algebra has turned out to be a part of mathematics that has caused problems for Swedish school students; since the 1960s, Swedish results in international evaluations have always been below average in algebra (Murray & Liljefors, 1983; Skolverket, 2008, 2012, 2016). Even in the 1995 Trends in international mathematics and science study (TIMSS) evaluation, where its overall result was the best ever for Sweden, Swedish results in algebra were still below the international average.

The study considered in the present paper is part of an ongoing research project aiming at characterizing Swedish school algebra (Hemmi et al., 2017). Both diachronic and synchronic studies are being conducted focusing on both formulation and realization arenas (Lindensjö & Lundgren, 2000) to identify the specific teaching tradition developed in Swedish school algebra. The formulation arenas refer to steering documents and curriculum materials, and the realization arenas to schools and teachers who develop and maintain their own more or less tacit traditions. The project aims to find reasons for the failure to raise the quality of algebra teaching in Sweden, but also to find possible ways to improve the situation. Also, we find the Swedish case interesting from an international point of view as several countries are struggling to implement algebra in their school curricula.

In the study reported in this paper, we analyze the algebraic content in the current Swedish mathematics curriculum and in textbooks for grades 1–6. The curriculum can only be regarded as a framework for school mathematics in Sweden since it does not support teachers with instructional materials, teaching methods, lesson plans, or tests (c.f. Hemmi, Lepik & Viholainen, 2013). In general, textbooks are important artefacts in the teaching and learning of mathematics as they often serve as the primary source for teachers when planning their teaching, not only in Sweden (Johansson, 2006) but also in other countries (e.g. Stein, Remillard & Smith, 2007). Furthermore, in Sweden, the role of students' textbooks is especially important as one typical method used by teachers to individualize teaching has been to let students to work using their textbooks at their own pace (Neuman, Hemmi, Ryve & Wiberg, 2015). Therefore, studying mathematics textbooks helps us to discover what kind of learning opportunities (cf. Hiebert & Grouws, 2007) students have to develop their algebraic thinking. Previous textbook studies have mostly been focused on content analysis and analyses of problems presented in textbooks (e.g. Yang, Tseng & Wang, 2017) sometimes with comparisons between textbooks from different countries (e.g. Hong & Choi, 2014). However, what has been less studied is how the expected student progression within a certain mathematical area is reflected in mathematics textbooks covering several grade levels. In this study, we refer to progression as moving from informal and concrete algebraic activities to a more formal study of algebra (Cai et al., 2005; Cai & Knuth, 2011), but also as a way of widening the algebraic knowledge in terms of becoming acquainted with several topics in algebra, regardless of the difficulty level.

The aim of the present study is to obtain an overview of the algebraic content in the Swedish curriculum and textbooks for grades 1–6 in light of the results of current research on the development of algebraic thinking. In order to characterize the algebraic content, we have used Blanton's et al. (2015) so called ”big ideas” as a base for an analytical tool. Specifically, we aim to answer the following question,

> Based on Blanton's et al. (2015) big ideas, what are the characteristics of the algebraic content in the current Swedish curriculum and mathematics textbooks for grades 1–6, and what progression can be found within each part of the algebraic content?

## Earlier research

There has been an explosion of research regarding the possibility of incorporating algebra in school curricula already from early grades (for an overview, see Cai & Knuth, 2011). The idea of an early introduction to algebra is to facilitate students' progression towards understanding more formal algebra. Scholars agree that algebraic thinking in early grades should reach beyond arithmetic and computational fluency, ”to attend the deeper underlying structure of mathematics” (Blanton & Kaput, 2011, p. 6). Recent studies show, for example, that nine and ten year old students can learn structural, algebraic strategies to solve problems (Blanton et al., 2015) and even develop fluency with complex formal expressions involving all four arithmetic operations (Hewitt, 2014).

Blanton et al. (2015) have been studying the impact of a sustained, comprehensive early algebra intervention in third grade. The term ”sustained” refers to instruction that spans over a significant amount of time, at least one year. ”Comprehensive” early algebra instruction refers to an approach that ”[…] intentionally integrates early algebraic practices into the elementary school curriculum across different conceptual domains that are recognized as important entry points to algebraic thinking" (Carraher & Schliemann, 2007, p. 675). The concepts and practices should be taught in a way that makes them accessible to students on multiple levels of thinking (Blanton et al., 2015, p. 42). For instance, the algebraic thinking practice of "representing generalizations" can be found in various conceptual domains. Generalized arithmetic is central in the process of students' development of algebraic thinking according to several researchers (e.g. Kieran, 2007). Some researchers also stress that a progression in "algebra as generalized arithmetic" throughout compulsory school is necessary to help pupils master algebraic manipulations (e.g. Hewitt, 2014; Fujii, 2003). More explicitly, it is important for pupils to start working with structures and generalizations in arithmetic as soon as possible in order to gain a progression from arithmetic to algebra. The most recognizable tool to represent generalizations is variable notation, which, traditionally, has been introduced as a fixed unknown quantity or a "place-holder", an approach that may be difficult for young students to understand. A comprehensive approach to introducing variables to young students would be to consider variables from other aspects, such as a varying quantity, a generalized number, or even a parameter (Blanton, et al., 2015, p. 42). Blanton's et al. study was based on Kaput's (2008) theoretical framework of algebraic thinking. Kaput (2008) suggests that algebraic reasoning occurs in three different strands: algebra as generalized arithmetic and quantitative reasoning, algebra as the study of functions, relationships, and joint variations, and algebra as a cluster of modeling languages. By analyzing Kaput's framework and research on the language of learning progression, Blanton et al. (2015) identified five so called *big ideas* that were represented in Kaput's content strands. These five big ideas consist of 1) Equivalence, expressions, equations and inequalities, 2) Generalized arithmetic, 3) Functional thinking, 4) Variables, and 5) Proportional reasoning. From now on, the following abbreviations are used for the big ideas: EEEI = Equivalence, expressions, equations, and inequalities, GA = Generalized arithmetic, FT = Functional thinking and VAR = variable. In our study, as reported in this paper, we have applied Blanton's et al. big ideas as an analytical tool in order to categorize the algebraic content in Swedish school algebra. A more detailed description of these big ideas is given in the methodology section below.

None of the Nordic countries show good results within the area of algebra in international evaluations (Hansen et al., 2014). There are only a few studies in these countries regarding how goals, contents, and the progression of algebra are formulated on an institutional level, such as in national steering documents and textbooks. In Norway, Kongelf (2015) has analyzed how algebra is introduced in Norwegian textbooks at the lower secondary school level. The results revealed that the introduction of letters as symbols for variables was not clear and varied with respect to grade, quantity, and context (Kongelf, 2015). According to Kongelf, the textbooks hardly took the opportunity to build algebra on arithmetic.

In Sweden, Jakobsson-Åhl (2008) conducted an historical study regarding the development of algebraic content in upper secondary school Swedish textbooks from 1960–2000. The results show that the algebraic content has changed from being dominated by algebraic manipulations and expressions to becoming more integrated with other school subjects. The level of complexity of algebraic expressions in textbook exercises has decreased over the years, and now algebra is more often considered as a tool for solving practical and everyday problems (Jakobsson-Åhl, 2008). Furthermore, Lundberg (2011) compared Swedish textbooks and national examinations with respect to proportional reasoning. The results showed that although a quarter of the tasks in both national examinations and textbooks were related to proportional thinking, there was a little variety in the types of tasks.

There are studies indicating that there is a complex relationship between teacher and curriculum materials (textbooks and teacher guides). A case study by Kilhamn (2014) shows that two Swedish grade 6 teachers using the same textbooks introduced variables in very different ways. The differences mainly depended on the two teachers' different views of the meaning of the variable concept, but also the meaning of algebra. Moreover, research shows that the relation between mathematics textbooks and students' learning is complex (Stein et al., 2007). Hence, our results show only the opportunities for students to learn algebra when working with their textbooks, which is important because of the usual working manner in Swedish mathematics classrooms.

## Methodology

We have conducted a qualitative content analysis with some quantitative elements (see Bryman, 2012) of the current Swedish national curriculum in mathematics and two textbook series for grades 1–6. In this section, we first briefly characterize the Swedish national curriculum and the two textbook series. Then we describe the analytical tool, and the procedures for the data analysis.

### *The data: the curriculum and the textbook series*

The current Swedish mathematics curriculum for compulsory school (Skolverket, 2011) consists of three sections: 1) Introduction to the subject, 2) Central content, and 3) Knowledge demands. Section 1 is the same for all grades 1–9, while sections 2 and 3 are split between grades 1–3, grades 4–6 and grades 7–9. The present study is limited to including grades 1–3 and 4–6. We have considered all three sections, but only section 2 (Central content) is included in our present investigation.

The following textbook series have been analyzed in this study:

1. Eldorado; *Matte eldorado*, grades 1–6, published by Natur och Kultur.

2. Direkt; *Matte direkt safari*, grades 1–3 and *Matte direkt borgen*, grades 4–6, published by Sanoma Utbildning.

There are no statistics available regarding the popularity of various textbook series in Sweden in general. Therefore, our choice of textbook series is based on the following. Firstly, these were the most popular series in schools in two large municipalities that were focusing on textbook research (Neuman et al., 2015). Secondly, these series represent different approaches to how teaching is organized; teachers who use *Matte direkt* allow their students to work individually in their textbooks or use other materials more often than teachers who use *Matte eldorado* (Neuman et al., 2015). Thirdly, *Matte direkt safari* was first published 2004 and *Matte direkt borgen* in 2007 while *Matte eldorado* was first published in 2008 (grades 1–3) and 2011 (grades 4–6). This means that the first editions of *Matte direkt* were based on the 1994 national curriculum while *Matte eldorado* for grades 1–3 was based on the revised 1994 national curriculum from 2008 and *Matte eldorado* for grades 4–6 was based on the current national curriculum from 2011. The older textbooks have been adapted to the current curriculum. By taking these reasons into account, we have guaranteed that the textbooks studied here are frequently used and, at the same time, we have increased the possibility of finding varying views on algebraic teaching and learning.

In addition to the textbooks, both series consist of teacher guides, homework books, and extra material. However, in our study, we have focused on the textbooks since textbooks seem to be the most important artefacts in Swedish mathematics classrooms (Neuman et al., 2015). The two series consist of twelve textbooks each (two for each grade), that is, the total number of analyzed textbooks within the study is 24 (2 x 12). The overall structure of the textbooks is quite similar; both *Direkt* and *Eldorado* split each book into 5 or 6 chapters and each chapter includes one common component for all students. This is followed by two different tracks; a basic one and an advanced one where the latter contains more challenging tasks. In both textbook series, each chapter alternates between short introductions or descriptions of specific concepts, worked out examples, and tasks for the students to solve.

### *Data analysis*

In order to characterize the algebraic content in the material, we used Blanton's et al. (2015) big ideas as a base for an analytical tool. These are the areas that should be developed through the grades as students develop their algebraic thinking. Next, we describe the categories and how we interpreted them in our analysis.

1 Equivalence, expressions, equations, and inequalities include relational understanding of the equal sign, representing and reasoning with expressions and equations, and relationships between and among generalized quantities (Blanton et al., 2015, p. 43). An example of a task within this category is the solving of the open number sentence: $8 + 5 = \_\_ + 4$ and being able to reason based on the structural relationship in the equation. Number sentences such as $8 + 5 = \_\_$ have not been included in this category since this kind of tasks consider the ability to calculate.

2 Generalized arithmetic has emerged from that part of algebraic thinking that considers the study of structures and relationships arising in arithmetic (Kaput, 2008). Previously, generalized arithmetic has been associated with letter-symbolic algebra, with its equations and unknowns (Kieran et al., 2016, p. 19). However, through the years, and within the field of early algebra, the term has acquired a much broader sense in that the relationships and properties inherent to arithmetical operations are explored and seen by students as being generalizable, without necessarily involving alphanumeric symbols (Kieran et al., 2016, p. 19). For instance, consider the equality $78 - 49 + 49 = 78$. By letting young pupils discover that the equality is true regardless of what number is subtracted and then added back, or that 78 can be any number provided that the number subtracted and then added back is the same, they can learn to think algebraically without using the general equality $a - b + b = a$. Instead the focus should be directed to the relation between the different numbers as well as the relation between the numbers and the operations. This approach is sometimes referred to as the bridge between arithmetic and algebra (Fujii, 2003). A more detailed description of generalized arithmetic can be found in Bråting, Hemmi and Madej (2018).

3 Functional thinking involves generalizations of relationships between co-varying quantities, and representations and reasoning with relationships through natural language, algebraic (symbolic) notation, tables, and graphs (Blanton et al., 2015, p. 43). For instance, this can mean generating linear data and organizing it in a table, identifying recursive patterns and function rules and describing them in words and using variables, and using a function rule to predict far function values.

4 Variable refers to ”symbolic notation as a linguistic tool for representing mathematical ideas in succinct ways and includes the different roles variable plays in different mathematical contexts” (Blanton et al., 2015, p. 43). One typical example within this category is the ability to use variables in order to represent arithmetic generalizations.

5 Finally, the big idea of Proportional reasoning refers to opportunities for reasoning algebraically about two generalized quantities that are related in such a way that the ratio of one quantity to the other is invariant (Blanton et al., 2015, p. 43).

In Blanton's et al. (2015) study of third graders, the big idea of proportional reasoning was not included. In our study, reported in this paper, we have also decided not to include proportional reasoning as a separate big idea. The reasons for this are as follow. Firstly, the big idea of proportional reasoning refers to algebraic reasoning about two generalized quantities. Since our study only includes grades 1–6, relationships between two generalized quantities never occur in Swedish contexts, such relationships first appear in grades 7–9. In order to better understand this, let us consider a typical example, ”Anne has 5 pennies and Tom has twice as many, how many pennies has Tom?”. Here, there is a proportional relationship between a generalized quantity (the requested solution) and a known quantity (the 5 pennies). Thus, there is only a single generalized quantity in this example which disqualifies it from being included in the big idea of proportional reasoning. Secondly, the treatment of proportional reasoning that appears in grades 1–6 is well covered within the big idea of functional thinking. This is especially prevalent in these grades as only linear functions are considered and are, in many cases, expressed in proportional terms. Hence, in our study we consider the appearance of proportional reasoning as part of the big idea of functional thinking.

We agree with Blanton et al. (2015) in that the big ideas cannot be considered as totally separate from each other. For instance, the big idea of variable is difficult to separate from the other big ideas since a variable can be viewed both as a varying unknown quantity in connection with functional relationships and as a generalized number when examining fundamental properties. Nonetheless, because of its important role within algebraic thinking, Blanton et al. (2015) decided to identify variable as a distinct big idea. However, in our analysis of the textbook series, we decided not to use variable as a separate big idea. In our pilot study (see below), we discovered that in the textbooks for grades 1–6, it is almost impossible to separate the big idea of variable from the other big ideas. In fact, variable always appears together with some other big idea and, as a result, we decided to not include it as a big idea in our analysis of the textbooks. This was, however, not a problem when analyzing the curriculum where "unknown numbers" appeared explicitly in the central content, which is why we kept it as a big idea there.

Furthermore, we have looked at the total amount of algebra in each textbook series (table 2) in order to be able to spot differences in how well algebra is represented in the textbooks. However, even if there are differences between the amount of algebra in the textbook series, the main purpose of this study is to look at the distribution of the different big ideas in the algebra content (table 3) as well as the progression within the big ideas. We have applied two different views of progression in our analysis: 1) Progression that *deepens* the students' algebraic know-ledge within a big idea (or a specific topic within a big idea). For instance, within the big idea *Equivalence*, *expressions*, *equations & inequalities* the students practice the structural meaning of the equal sign in grades 1–3 and then learn to formally solve equations in grades 4–6. 2) Progression that *widens* the students' algebraic knowledge within a big idea. For instance, within *Functional thinking* the students learn the two topics proportional relationships and patterns in grades 1–3. These topics are not necessarily dependent of each other and therefore widen the students' algebraic knowledge. Next, we describe how we analyzed progression within the big ideas.

We analyzed the algebraic content that was already divided into the different big ideas in the curriculum and in the textbooks, respectively. For each separate big idea, we compared the content in all different grades. In the textbooks, we found the one topic "equations and algebraic expressions" within EEEI. In FT the topics consisted of "proportional relationships", "patterns", and "the coordinate system". Finally, GA consisted of the one topic "transition from arithmetic expressions to algebraic expressions". If a certain topic changed from being studied informally to a more formal study of algebra or if the level of difficulty increased between the grades, it was marked as progression that deepens the students' knowledge. One example is when the students practice the meaning of the equal sign in grades 1–3 and formally solve equations in grades 4–6 (that was just mentioned in the above paragraph). On the other hand, if a topic only appeared in grades 4–6 it was marked as progression that widens the students' knowledge.

### *Unit of analysis and the classification process*

We first conducted a pilot study where we defined the unit of analysis and the interpretation of the categories based on the big ideas (Bråting, Hemmi, Madej & Röj-Lindberg, 2016). When deciding on the unit of analysis, we mainly had two aspects to take into consideration. Firstly, the structure of the textbook series: Since each chapter of the textbooks includes short introductions or descriptions of concepts, worked examples, and tasks for the students to solve, the result of counting tasks would exclude all introductions, descriptions and examples from our investigation. We wanted to analyze the tasks against the relevant background and also analyze the worked examples. Moreover, the pages consisting of tasks for the students to solve contain many similar tasks, which imply that the result of counting the number of tasks belonging to a certain big idea would probably not differ much from the result obtained by counting whole pages. Secondly, generalized arithmetic differs somewhat from the other big ideas as it contains tasks that might be difficult to express in a short exercise; for example "[a]nalyze information to conjecture an arithmetic relationship", to "[e]xpress the conjecture in words and/or variables" and to "[j]ustify an arithmetic generalization using either empirical arguments or representation-based arguments; and examine limitations of empirical arguments" (Blanton et al., 2015, p. 45). Taking this and the qualitative emphasis of our study into consideration, using pages as our unit of analysis is more suitable than using the number of exercises or square centimeters of page (Valverde et al., 2002).

It is important to address some of the consequences of using pages as a unit of analysis. First of all, there are pages that not only consist of algebra. In such cases, we have decided that a page should only be counted if more than half the page can be classified as one or more of the big ideas. Secondly, when a unit is as big as a whole page, there may be more than one big idea present on the page. This causes the sum of the number of pages of the different big ideas to be greater than the total number of pages containing algebra (table 3).

Author 1 and 2 together conducted the final classifications in the study. These two authors first classified two chapters independently of each other and compared the results. The results were almost the same by both authors, they only differed by one unit at most at any category.

The differences were discussed and a common view were agreed on. After this, and at any other point of uncertainty that occurred during the classification process, authors 1 and 2 asked for a second independent opinion from author 3. In all cases where this occurred, the three authors came to the same conclusion regarding the classification of the page in question.

## Results

We first report the results of the classifications of the big ideas in the content description of the national curriculum. Thereafter, we show the proportion of algebraic content and the units categorized into the big ideas of the two textbook series as well as presenting the progression within each big idea.

### *Algebra in the Swedish national curriculum for grades 1–6*

Table 1 shows the classification of algebraic content with respect to the big ideas in the section Central content in the Swedish mathematics curriculum. Grades 1–3 and 4–6 respectively, can be found on the horizontal axis and Blanton's et al. (2015) big ideas on the vertical axis.

Table 1 reveals that the big idea FT is indeed covered in the Swedish curriculum, and so is EEEI. The big idea, FT, consists of three topics

Table 1. *Classification of algebraic content in the curriculum*

| Big ideas | Grades 1–3 | Grades 4–6 |
|---|---|---|
| EEEI | Mathematical similarities and the importance of the equals sign. | Simple algebraic expressions and equations in situations relevant to the pupils.<br>Methods of solving simple equations. |
| GA | | |
| FT | Different proportional relationships, including doubling and halving.<br>How simple patterns in number sequences and simple geometrical forms can be constructed, described, and expressed. | Proportionality and percentages and their relationship.<br>How patterns in number sequences and geometrical patterns can be constructed, described, and expressed.<br>Graphs for expressing different types of proportional relationships in simple investigations.<br>The coordinate system and strategies for scaling coordinate axes. |
| VAR | | Unknown numbers and their properties and also situations where there is a need to represent an unknown number using a symbol. |

for grades 1–6: proportional relationships, constructions of patterns, and graphs and the coordinate system. The first, proportional relationships, includes fundamental proportions such as doubling, and halving (grades 1–3), followed by percentages, the relationships between proportions, and percentages and the usage of graphs for expressing proportional relationships (grades 4–6), see table 1. Clearly, there is a progression based on different proportional relationships between grades 1–3 and 4–6 that deepens the students' knowledge. The second topic within FT is constructions of patterns, which includes constructions of number sequences and geometrical forms in both grades 1–3 and 4–6. Here, the difference between grades 1–3 and 4–6 consists merely of the word "simple"; in grades 1–3, the text refers to "simple patterns", while in grades 4–6, the text refers to "patterns"; that is, the progression is based on the patterns' levels of difficulty (an example of a task in grade 6 is given in figure 2) and therefore deepens the students' knowledge. The third topic within FT is graphs and the coordinate system, which first appear in grades 4–6. As we can see, strategies for scaling the coordinate axis are emphasized as is the ability to use "graphs for expressing proportional relationships in simple investigations." That is, there is a progression within grades 4–6 that widens the students' knowledge regarding graphs and coordinate systems. It is noticeable that the term "function" is not applied in the Swedish curriculum for grades 1–6. The corresponding term used is (proportional) relationships.

The big idea EEEI is, as mentioned above, also represented in the Swedish curriculum. In grades 1–3, EEEI-categorized content consists of mathematical similarities, especially the equal sign, and in grades 4–6, it consists of equations and algebraic expressions. The curriculum for grades 4–6 emphasizes that equations and algebraic expressions should be connected to situations that are relevant for the students (table 1). The progression between the grade levels 1–3 and 4–6 primarily deepens the students' knowledge.

The big idea GA is not represented in the content component of the Swedish curriculum for grades 1–3 or 4–6 (table 1). In fact, neither the word "generalize" nor the word "generalized" is included in any of the three curriculum components for grades 1–3, 4–6 or 7–9.

Although the term "variable" does not appear in the mathematics curriculum for grades 1–3 or 4–6, we classified the sentence "unknown numbers and their properties as well as representations of unknown numbers with symbols" as VAR (table 1). In the Swedish curriculum, the term "variable" first appears in the curriculum for grades 7–9, which is not included in this study.

### *Algebra in the textbooks*

The total number of pages in the two textbook series is approximately the same; Eldorado consists of 866 pages for grades 1–3 and 884 pages for grades 4–6; and Direkt consists of 834 pages for grades 1–3 and 912 pages for grades 4–6 (table 2). However, the proportion of algebraic content based on the big ideas differs between the two textbook series. Eldorado contains 11 % algebra for grades 1–3 and 12 % for grades 4–6; while Direkt contains 6 % algebra for both grades 1–3 and grades 4–6 (table 2). That is, based on our investigation, Eldorado contains more algebra than Direkt.

Table 2. *Number of pages with algebraic content in relation to total number of pages*

| Textbook | | Pages with algebra | Total number of pages | |
|---|---|---|---|---|
| Eldorado | grades 1–3 | 97 | 866 | 11 % |
| | grades 4–6 | 108 | 884 | 12 % |
| Direkt | grades 1–3 | 53 | 834 | 6 % |
| | grades 4–6 | 54 | 912 | 6 % |

### *The progression within the big ideas*

We will now describe the algebraic content and the progression in the textbooks on the basis of each big idea. Table 3 shows the classification of the algebraic content with respect to the big ideas in the two textbook series. Each cell shows the proportion between the number of pages belonging to a specific big idea and the total number of pages of algebraic content. For instance, in the cell showing the proportion of EEEI for grades 1–3 in Eldorado, 44/97 means that 44 pages of the total 97 pages of algebraic content are classified as EEEI, and 44/97 = 45 %.

Table 3. *Classification of the algebraic content in the textbook series*

| Textbook | | EEEI | GA | FT |
|---|---|---|---|---|
| Eldorado | grades 1–3 | 44/97 (45 %) | 14/97 (14 %) | 50/97 (52 %) |
| | grades 4–6 | 55/108 (51 %) | 14/108 (13 %) | 52/108 (48 %) |
| Direkt | grades 1–3 | 43/53 (81 %) | 1/53 (2 %) | 9/53 (17 %) |
| | grades 4–6 | 26/54 (48 %) | 1/54 (2 %) | 27/54 (50 %) |

### *Equivalence, expressions, equations and inequalities*

Table 3 reveals that EEEI is highly represented in both textbook series, especially in Direkt where 81 % of the algebraic content for grades 1–3 is EEEI. However, for grades 4–6, the amount of EEEI decreases to 48 % of the total algebraic content. In Eldorado, the proportion of EEEI is 45 % for grades 1–3 and 51 % for grades 4–6. Hence, the proportion of EEEI is more stable between the grades in Eldorado when compared to Direkt.

In both textbook series, there is a relatively clear progression that deepens the students' knowledge within EEEI: The stages of the progression consist of: 1) Mathematical similarities and the importance of the equals sign, 2) Open number sentences, and 3) Algebraic expressions and equations. A typical example of the first stage is when the students are first introduced to the meaning of the equal sign. The authors of both textbook series use toys such as balloons and flags before they use numbers. The students are to put an equal sign between two groups of toys if they both contain the same number of toys. The second stage consists of open number sentences such as: $8 + 5 = \_ + 4$. This is a dominant topic, especially in Direkt. The third stage, algebraic expressions and equations, is represented in figure 1. We believe the authors' purpose in this example is to teach the students how to set up an equation based on a given problem and then solve the equation.

In figure 1 the authors connect arithmetic expressions to algebraic expressions by first letting the students express in words what the numerical expression can be interpreted as and, a few tasks later, by doing exactly

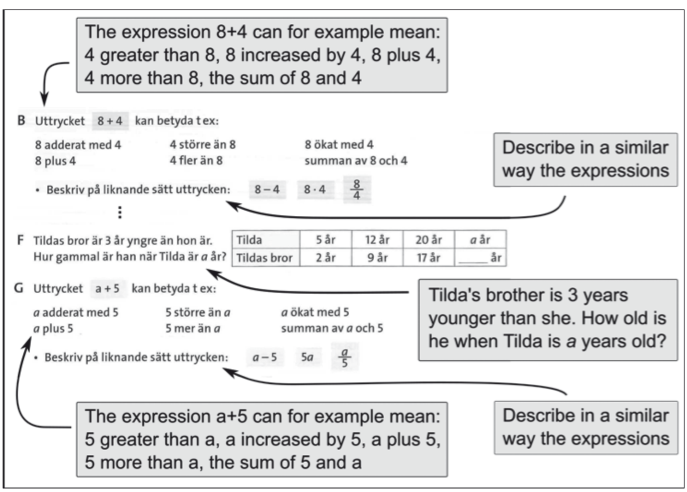



Figure 1. *Example of algebraic expressions and equations (Eldorado 6A, pp. 8–9)*

the same thing, but now using an algebraic expression. This is an example of how the categories intertwine because the development between tasks B and G can also be connected to GA, which is the focus of the next section. Furthermore, the table in Task F tries to use an arithmetic relationship to make the students create an algebraic expression.

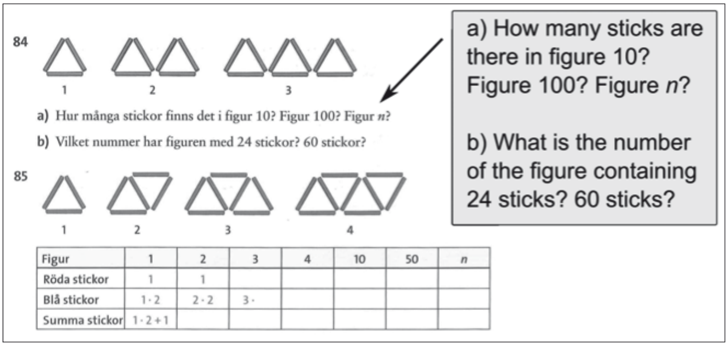
### *Generalized arithmetic*

The big idea GA is clearly the least represented idea in both textbook series; in Direkt only 2 % of the algebra content for grades 1–3 and grades 4–6 pertains to GA. In Eldorado, 14 % of the algebra content for grades 1–3 and 13 % of the algebra content for grades 4–6 pertains to GA (table 3). Furthermore, when compared with EEEI and FT, GA often appears in conjunction with another big idea, which is exemplified in figure 1. It should be noted that figure 1 is one of the very few examples in the textbooks that contained generalized arithmetic.

### *Functional thinking*

Like EEEI, the big idea FT is well represented in both textbook series, especially in Eldorado. In Eldorado, 52 % of the algebraic content for grades 1–3 and 48 % of the algebraic content for grades 4–6 is FT. The corresponding amounts for Direkt are 17 % FT for grades 1–3 and 50 % FT for grades 4–6 (table 3). That is, the tendency in Direkt is for the amount of FT to increase when going from grades 1–3 to grades 4–6, while the proportion of FT in Eldorado is stable between the grades. The topics for grades 1–3 are basic proportional relationships, such as doubling and halving, and patterns and number sequences. For grades 4–6, the coordinate system and graphs are the dominant topics, but patterns and number sequences are also represented.

The progression in this category cannot be described as one single progression that deepens the students' knowledge across the grades 1–6 as in the case of EEEI. This mainly depends on that there is a greater variation of topics within FT compared to EEEI and some of them are considered only in grades 1–3 while others only in grades 4–6. However, covering different topics is a typical example of a progression that is widening the students' knowledge. In connection with the topic patterns, there is also progression that deepens the students' knowledge between grades 1–3 and 4–6. In grades 1–3, the students shall verbally describe and construct simple patterns in number sequences and simple geometrical forms and in grades 4–6, the patterns become more difficult and should be described by means of symbolic rules (see figure 2).




| Figur | 1 | 2 | 3 | 4 | 10 | 50 | n |
|---|---|---|---|---|---|---|---|
| Röda stickor | 1 | 1 | | | | | |
| Blå stickor | 1 · 2 | 2 · 2 | 3 · | | | | |
| Summa stickor | 1 · 2 + 1 | | | | | | |

Figure 2. *Example of patterns in grade 6 (Eldorado 6B, p.98)*

In task 84 in figure 2 the students are asked to predict far (function) values for the number of sticks in figures 10 and 100. They are also asked to give an expression for the number of sticks in figure *n*, which gives them a function rule for how the number of sticks depends on the variable *n*. When trying to predict these values and finding the function rule, the students could use proportionality. Task 85 shows the same type of reasoning, where the different figures have been divided into a red component and a blue component. It should also be added that this page is also an example of GA, since the students are guided from numerical to algebraic expressions.

## Discussion

Next, we discuss the algebraic content in the curriculum and in the textbooks as well as the progression within different big ideas. We conclude by discussing the analytical framework.

### *Content and progression*

It is not possible to conduct a deeper comparison between the content of the curriculum documents and the textbooks because they are written at a different level of generality. However, it is noticeable that in the curriculum document the big idea FT has the largest number of items. One reason to this might be that ”Relationships and change” constitutes a new, separate category of mathematical content in the current Swedish national curriculum. In previous Swedish curricula, the content in ”Relationships and change” was distributed among the different topics, especially algebra. The emphasis on ”Relationships and change” is probably an effect of a recent international trend where ”change and relationships” has been identified as one of the four broad mathematical content categories in the PISA framework for school mathematics (OECD, 2010).

Regarding the distribution of the algebraic content there is a difference between the two textbook series. In Eldorado, the proportions of EEEI and FT are relatively stable between the grades 1–3 and 4–6. Meanwhile, in Direkt EEEI dominates the content, especially in grades 1–3 (table 3). The latter may depend on that Direkt was first introduced before the recent 2011 curriculum reform and has since then been adjusted to fit the new curriculum. Eldorado, on the other hand, is based on the current 2011 curriculum from the beginning.

Although the research emphasizes the importance of fostering algebraic reasoning in early grades to better prepare students for the eventual transition to algebra (Blanton et al., 2015; Cai et al., 2005; Carraher et al., 2006) our results show that the amount of algebra in both textbook series are very low (table 2). In fact, the low amount of early algebra in Swedish mathematics textbooks may be a reason for the lack of an effect on Swedish students' transition to algebra. It is noticeable that in Eldorado algebra covers twice as much of the total content as in Direkt. One reason for this is that Eldorado has longer introductions to new subjects, containing pages where the students are expected to explore and discover new topics.

Although Direkt emphasizes EEEI more compared to Eldorado, both textbook series show a similar kind of progression through the grades 1–6 within EEEI. However, an interesting result was that the big idea EEEI consisted of only one single progression in both textbook series, while FT consisted of several progression lines. One of them deepened the students' knowledge and the rest of them were widening the students' knowledge. Again, this may reflect the emphasis of "change and relationships" in the PISA framework for school mathematics (OECD, 2010) in the sense that there are more topics within FT than EEEI in the Swedish mathematics curriculum. It is noticeable that we found progression that deepens the students' knowledge regarding different proportional relationships in the curriculum but not explicitly in the textbooks. Otherwise, the topics and progression identified in the textbooks were quite similar to the curriculum.

Perhaps the most striking result of this study is how weakly GA is addressed in the Swedish curriculum and the two textbook series. GA is not represented at all in the content description of the mathematics curriculum for grades 1–6, which is probably one reason for its low representation in the textbooks. As already mentioned, generalized arithmetic is seen as one of the most important parts of school algebra by several researchers (Blanton et al., 2015; Hewitt, 2014; Kieran et al., 2016).

Sometimes generalized arithmetic is considered as a bridge between arithmetic and algebraic thinking (Fujii, 2003), that is, as a development of "algebra as generalized arithmetic" throughout compulsory school. In order to help pupils master algebraic manipulations such a development is necessary (Hewitt, 2014). However, as with Kongelf's (2015) study regarding algebraic content in Norway, we cannot find any notion of building a bridge between arithmetic and algebra in the Swedish national curriculum and very little in the textbook series. In the example of algebraic expressions and equations from Eldorado (figure 1), we can see a tendency of building a bridge from arithmetic to algebra. Generalized arithmetic is also involved in learning proof and proving (Kieran, 2007). Our results confirm the results of a study analyzing proof-related competences presented by Hemmi et al. (2013) showing that the aspect of generalization was barely visible in the Swedish mathematics curriculum.

As mentioned above, this study is limited to only identifying algebraic content in the part of the curriculum, Central content, which presents mathematical content. The other two parts of the curriculum, *Introduction to the subject* and *Knowledge demands*, are not analyzed. This may, of course, have affected the results since we may have missed implicit algebraic content, such as generalized arithmetic, in those parts of the curriculum. A similar problem occurred in Kongelf's (2015) study of algebraic content in the Norwegian mathematics curriculum. One way to detect implicit algebraic content in the curriculum would be to investigate how teachers interpret the algebraic content in the curriculum in the light of the general mathematical abilities (included in the first part of the curriculum) and the knowledge demands, that is, within the realization arena (Lindensjö & Lundgren, 2000).

### *Big ideas as an analytical framework*

The framework we have used highlights important aspects of algebra, but there are other aspects not covered by this framework. For example, the operational component of algebra, with important skills like the manipulation of algebraic expressions, is not included in the framework. To be able to solve equations in a more formal way, as is included in the big idea, EEEI, one needs to be able to manipulate algebraic expressions (Kieran, 2007). Even though it might not be possible to consider this type of algebraic skill as a big idea, it is important not to forget it as an aspect of algebra.

We believe that generalized arithmetic differs from the other big ideas as it is broader and easier to look at it as a platform for algebra as a field rather than a sub-idea of algebra. GA includes aspects such as looking at a numerical relationship or law and then generalizing it to an algebraic context. For example, if students are able to perform mental arithmetic like $23 \cdot 8$ by splitting the calculation in two parts, $20 \cdot 8 + 3 \cdot 8$, as is done in grades 4–6, they have used the distributive law. This technique is later used when expanding algebraic expressions such as $3(x + 1)$ or factorizing $3x + 3$, but it might not be that obvious to the students that it is exactly the same technique. Even though this study does not cover secondary school and, therefore, is unable to look at examples like this, discovering this connection between arithmetic and algebra is an example of the big idea of GA.

In summary, the framework worked well to start with, but the more we worked with it, the more aspects of algebra we found which cannot be easily categorized as a topic of only one big idea. Similarly, the big ideas are very general and include ambiguous notions (see, for example, patterns in McGarvey, 2012) and, as a result, we now need to continue with a more fine-grained analysis of the topics within the big ideas. As we have mentioned, the big idea of variable was difficult to distinguish from the other big ideas in the textbooks and therefore we decided to remove variable from our analytical tool in the textbook analysis. However, we agree with Blanton et al. (2015) that the role of variables is of the utmost importance in algebra and should, therefore, constitute a distinct big idea. Even though we did not analyze variables explicitly in this textbook analysis there are other ways of doing that. One way is to focus only on variables by means of the different ways of using variables in the textbooks; as representing unknown numbers, solutions to equations, arguments of functions, varying quantities, parameters, etcetera. The result of such an analysis could perhaps tell us something more about what perspective on algebra learning that reflects Swedish school algebra.

The framework could also be combined with other aspects of curriculum orientations, for example, representation forms, contextual factors, and response types (Yang et al., 2017). In the future, studying the balance between pure mathematics and practical and everyday mathematics could be of interest since practical and everyday mathematics were strongly emphasized both in the Swedish national curriculum and in the two textbook series within this study.

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