Scrutinizing teacher-learner interactions on volume

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This study adds to research on volume and spatial reasoning by investigating teacher-learner interactions in the context of Lesson study. Our analysis illustrates how the mathematical object of volume is realized, and what metarules of discourse that can be observed over two iterations of a research lesson. The study unpacks the mathematical work of teaching volume in terms of discourse, and shows how an undesirable and unexpected result from the first research lesson can be attributed to the communicational work of teaching rather than to lack of skills among students.

”Why don’t students use their previous knowledge of equations to solve the problem?” This question came up as a group of Norwegian mathematics teachers discussed their observations from the first research lesson on volume. This took place when the group of teachers were engaged in a professional development project that was organized around the principles of Lesson study (hereafter LS). LS is a structured approach to professional development of teachers – originating in Japan, over hundred years ago (Stigler & Hiebert, 1999) – where teachers collaboratively investigate their own practice in order to improve student learning. This structured professional development is commonly organized around cycles of collaborative work on a so-called research lesson that includes planning the lesson, conducting the lesson, evaluating and refining the lesson, and sharing the results (e.g. Lewis & Hurd, 2011). The teachers in this LS group had formulated as a goal for the lesson that their students should learn to understand volume as the relationship between base area and height. The students were given three problems to work on in the research lesson, and the question occurred when discussing how the students approached one of these problems. The students tried to find the dimensions of a sandbox in order to fit a given

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Tyskerud, A. & Mosvold, R. (2018). Scrutinizing teacher-learner interactions on volume. *Nordic Studies in Mathematics Education*, 23 (2), 49–67.

amount of sand (500 liters), and the teachers had expected them to use the volume formula for a rectangular box. Although many students used the volume formula in the process, they did not seem to use their previous knowledge of equations to solve the problem. Instead, they randomly guessed three numbers where the product came close to 500.

The teachers were disappointed by the students' achievements, and they were perplexed by this gap between their own expectations and the students' performances. In their evaluation meeting after the first iteration of the research lesson, the teachers agreed that the students' inability to use their previous knowledge caused this. Instead of adjusting their presentation of the problem or scrutinize their own communication about volume, the teachers decided to adjust the problem in the next iteration of the research lesson. In this study, we analyze the teacher-learner interactions in the two research lessons in order to understand this perplexing situation. Contrary to the teachers' own conclusion, our hypothesis is that an explanation can be found in the teaching rather than in attributes of the students. We apply Sfard's (2008) commognitive theory as analytic framework, and we pay particular attention to the teachers' routines. Before elaborating on this framework and explicating our research questions, we frame the problem in light of previous research on teaching and learning of volume.

Introduction and theoretical framework

There is extensive research on spatial reasoning and students' understanding of volume (e.g. Assuah & Wiest, 2010; Battista & Clements, 1996; Gough, 2004; Miles, 2014; Obara, 2009; Tekin-Sitrava & Isiksal-Bostan, 2014), and several studies investigate the different approaches students take to solve problems on volume (e.g. Gough, 2004; Obara, 2009). Some studies focus on problems of maximizing volume (e.g. Miles, 2014), whereas others identify common student errors and misconceptions (e.g. Gough, 2008). Another group of studies concentrate on different tools that can be used to enhance students' learning of volume; many explore the use of various computer software (e.g. Purdy, 2000), and numerous papers promote the application of origami for exploring volume (e.g. Wares, 2011). Some studies investigate students' understanding of volume of rectangular prisms – often focusing on enumeration of unit cubes within the prisms (e.g. Battista & Clements, 1996; Tekin-Sitrava & Isiksal-Bostan, 2014). These studies seem to indicate a low level of understanding of volume among students. Further, Tekin-Sitrava and Isiksal-Bostan (2016) find that middle school teachers – at least in the context of their Turkish study – have weak understanding of volume of three-dimensional objects.

An interesting example of research on students' understanding of volume is Assuah and Wiest's (2010) exploration of two middle school students' attempts to solve a problem on comparing the volumes of rectangular prisms without using the volume formula. One student solves the problem by using a measuring container, whereas the other solves it by counting the number of unit cubes that can be contained in the two boxes. The authors suggest that future research is necessary to uncover what solution strategies that are more common among students. Tekin-Sitrava and Isiksal-Bostan (2014) respond to this call as they uncover a list of common strategies that include counting, layer multiplication and use of the volume formula. They suggest that students who know the formula seem to use it automatically without considering other solution methods, and they further advocate that teaching of volume should be organized by first letting the students get experience with volume from exploring concrete materials before they learn the formula (cf. Tekin-Sitrava & Isiksal-Bostan, 2014).

Our study differs from the aforementioned studies in two significant ways. Firstly, whereas previous research on volume entails a predominant focus on the students and their understanding, our study focuses on teacher-learner interactions – with a focus on the teacher's communication. Secondly, in our analysis of teacher-learner interactions on volume, we adhere to a participationist rather than an acquisitionist view of learning (cf. Sfard, 1998). Both of these differences in perspectives rely on Sfard's (2008) theory of thinking as communicating – often referred to as a theory of commognition – and we elaborate on this theoretical framework in the following paragraphs.

Since our conceptualization of teaching rests on a particular theory of learning, we first present some foundational perspectives of learning that inform this study. Unlike the more traditional cognitively laden studies of volume that highlight students' lack of understanding or misconceptions (e.g. Gough, 2008; Tekin-Sitrava & Isiksal-Bostan, 2014), our study is framed in a participationist view of learning, in which teaching is regarded as a process of helping students become participants in a mathematical discourse. Sfard (2008) defines discourse as a special type of communication "that draws some individuals together while excluding some others" (p. 91), which is "made distinct by its repertoire of admissible actions and the way these actions are paired with reactions" (p. 297). In the process of becoming participants in the mathematical discourse, communicational gaps or discursive conflicts frequently occur.

Mathematical discourses are characterized by certain properties, often identified as word use, visual mediators, endorsed narratives, and routines. Word use relates to how the user defines and uses particular words. The process of developing word use in a mathematical discourse (individualization) is described in four stages: passive use, routine-driven use, phrase-driven use, and object-driven use. Passive use refers to hearing the word, without using it oneself, routine-driven use refers to using the word in one concrete situation, phrase-driven use relates to being able to use the word in similar situations. Object-driven use refers to "the users' awareness of the availability and contextual appropriateness of different realizations of the word" (Sfard, 2008, p. 182). Visual mediators are visible objects that are used in communication – for instance mathematical signs, symbols, tables and graphs – and narratives are defined as any sequence of utterances framed as a description of the object. Endorsed narratives are usually labeled as true. Routines are discursive metarules that define patterns in the activity of the participants of the discourse, in contrast to object-level rules that define regularities in the behavior of objects of the discourse. Sfard distinguishes between three types of routines – explorations, rituals and deeds – depending on the goal of the discursive actions. In an explorative routine, the goal is to produce endorsed narratives about the world, which can happen in three ways: by constructing, substantiating, or recalling narratives. In this study, our primary distinction is between ritual and explorative participation in the mathematical classroom discourse. Whereas explorative participation aims at producing endorsed mathema-tical narratives, ritual participation has the goal of alignment and social approval and often entails a focus on manipulating with mathematical symbols (Heyd-Metzuyanim, Tabach & Nachlieli, 2016).

Mathematical discourses center on mathematical objects. Sfard (2008, p. 172) defines mathematical objects as "abstract discursive objects with distinctly mathematical signifiers". She makes a distinction between concrete and abstract, discursive and primary objects, but these distinctions are not focused on in the present study. Instead, a mathematical object is considered as a signifier together with its realization tree (figure 3 is an example). The realization of a signifier can have different forms: visual or vocal. Visual realization can be divided into four subcategories: verbal (either written words or algebraic symbols), iconic, concrete and gestural. The discourse on human behavior and actions (a student has solved many of the tasks perfectly in the test) develop into an impersonal discourse on objects. Objectification is a metaphor of mathematical discourse development, a duplex process of reification and alienation. Whereas reification turns actions into objects (the student has developed a mathematical understanding of the subject), alienation separates objects from the discursants (and their mathematical understanding).

Within this theoretical framework, teaching can be regarded as a communicational activity that aims at bringing students' mathematical discourse closer to the canonical discourse of mathematics (Tabach & Nachlieli, 2016). In our efforts to understand why the students were not able to understand volume in a LS cycle, we thereby focus on the teacher-learner interactions, and in particular on how the mathematical object of volume is constructed in the teacher's discourse, and what kind of discursive routines the students are invited into. We approach the following research questions.

– How is the mathematical object of volume realized in the teacher-learner interactions of the research lesson?

– What metarules can be observed in the various realizations of volume in the research lesson?

Method

This study is part of a larger ongoing project that investigates teacher learning in LS. Teacher learning is defined as a change in teachers' discourse on teaching – either in their discourse on student learning, or in their routines in the classroom. This study focuses on the latter when it investigates the discursive routines of teachers when communicating about the mathematical object of volume over two iterations of a research lesson.

Participants and design

In 2016, a lower secondary school in Norway implemented LS as their school development project. As part of this project, the first author of this paper followed a group of mathematics teachers as an external expert (Takahashi, 2013). Throughout three LS cycles, she interacted with the group as a participant observer (Sfard, 2008; Wadel, 1991). The other group members were four mathematics teachers (one male and three females), and a group leader (from the school administration). None of the members had any previous experience with LS before the project, but they were all motivated to experience LS as professional development.

A LS cycle consists of four main steps (see figure 1). In the first step, it is important for the teachers to set goals for their own learning (Olson, White & Sparrow, 2011), and to formulate their own research question(s) (Chokshi & Fernandez, 2004). In the second step, teachers develop a detailed plan for a "research lesson" (Fujii, 2014, 2016). Prediction and observation are core elements of this step (Bjuland & Mosvold, 2015; Munthe & Postholm, 2012). In the third step, one teacher teaches the research lesson, while the other group members observe. The observation

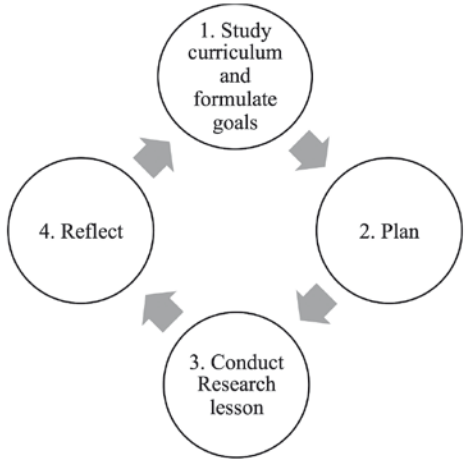

Figure 1. *The steps in a LS cycle (Lewis & Hurd, 2011, p. 2)*

involves structured collection of data, where the observers typically identify incidents that stimulate students' learning. Using the collected data as empirical evidence, the LS group attempts to answer their research question(s). In the last step, reflection on the observations is crucial. The teachers may decide to stop and write a report of what they experienced, or they may decide to refine their lesson plan and carry out a revised version of the research lesson in another class. In the latter case, they return to step three and complete another cycle (or even more) before they stop. To complete the LS process, the teachers share their experiences with others (Lewis & Hurd, 2011). The LS group in this study completed a cycle that involved two iterations of a research lesson.

Procedures for data collection

Table 1 provides an overview of the data collection, including video recordings from the teachers' meetings and the research lessons in two rounds of the first LS cycle. The first author transcribed all recordings for further analysis. Since our focus is on the teachers' discursive routines in the classroom, data analysis in this study concentrates on the two research lessons (bolded in table 1). We have used the data from planning and reflection meetings to describe and clarify the context of our analysis. The mathematical object of study was volume of three-dimensional shapes.

Table 1. *An overview of the data collection*

Part of cycle	Video recordings
Planning meeting 1	154 min.
Planning meeting 2	162 min.
Research lesson 1	**70 min.**
Reflection meeting 1	55 min.
Planning research lesson 2	88 min.
Research lesson 2	**70 min.**

When studying discourse, it is important to have a focus on communication – including cadences, body language and gestures. A crucial part of the analysis is "mapping the intricate relations between things said and deeds performed is the principal focus of this researchers' attention" (Sfard, 2008, p. 278).

Analysis of data

Based on the theory of thinking as communicating (Sfard, 2008), our hypothesis is that the explanation to the students' seeming inability to use their previous knowledge of equations to solve the problems of volume may be found in the teaching. To answer our research questions, we have analyzed teacher-learner interactions from the two research lessons. The transcribed video-recordings were foundational in the analysis process. We focus on teacher-learner interactions in the beginning of the lessons, when the tasks were presented and discussed, and towards the end of the lesson, when the lesson was summarized in plenary. We have included the discourse of both students and teachers, but we focus mainly on how the teacher introduced the problems and tasks to the students, how the students responded, and how the teachers followed up in the dialogue. We analyze how the mathematical object of volume is realized in the discourse, and we investigate the metarules that govern the mathematical discourse between students and teacher, and we discuss if the discourse invites to explorative or ritual mathematical discourse. A realization tree was created by identifying the different signifiers used to realize the mathematical object of volume, and then consider the realizations of these realizations (see figure 3). The metadiscursive rules were identified by careful considerations of observable patterns in the communication about the mathematical object of volume (Sfard, 2008).

Findings

Learning can be considered in terms of participation in discourse (Sfard, 2008), and the kind of discourse students are invited into as well as the given metarules within this discourse are therefore decisive. Before presenting results from the analysis of the classroom discourse, we provide some results from our analysis of the teachers' discussions in the planning and reflection meetings to provide some context.

In the teachers' conversations from the first planning meeting, they talk about students' participation and learning. The teachers aim at creating a lesson that provides opportunities for dialogue and discussion among students, and they want to facilitate situations where students

Table 2. *The given problems and tasks in the research lessons*

Problems	Tasks for students to do	Signifier and its realization
1. Introduction problem		
Starts with a question: Which of these glasses will accommodate most water? Follows up by a task for the students to calculate: How much sand will Sara need for her doll's sandbox? The length is 40 cm The width is 30 cm The height is 20 cm	Argue why one of the other have a bigger volume. Make the students focus on volume. To discuss how much is 24 000 cm^3?	Volume as amount of water RL 1: Picture – iconic visual mediator RL 2: Two mugs – concrete visual mediator Amount of sand. Measurement unit. RL 1: Bucket of sand – concrete visual mediator.
2. The sandbox		
This pile of sand has the volume 500 liters. Can you build different sandboxes that would fit this amount of sand? Make a drawing of the sandbox and mark the side's sizes.	Expectation: the students will construct different shapes of the sandboxes. Triangular prism, rectangular prism, cylinder and a composed figure. Decide the size of the sandbox' height and use equations to calculate possible base areas.	Visual mediator: Iconic (picture) and verbal (written words on the blackboard) Vocal: The teachers read the task to the students.
3. Folding a sheet of paper		
You will get two different cylinders if you fold a sheet of A4 paper from corner to corner. Which one will have the biggest volume? Alternatively, will the volumes be equal?	Argue why the one cylinder has bigger volume than the other, and discover the relation between base area and height. Make a hypothesis, then fill the cylinders with puffed rice and compare the amount. Do the calculation and find the exact volume of each cylinder.	Visual mediator: concrete (folding two papers; two cylinders, amount of puffed rice) Vocal: Summarize the task at the end of the lesson.

must explain their thinking and argue for their answers and calculations. From our analysis of the discussions in the planning meetings, we notice that the teachers want to observe how the students are thinking when they solve problem-oriented tasks. They develop three problems; two of them are open-ended and chosen for observation (for an overview, see table 2), the first problem serves as an introduction to the research lesson.

In the reflection meeting after the first research lesson, the teachers realize that the students have not discussed their answers nor argued mathematically. Attempting to invite the students into more explorative routines, the teachers decide to include certain types of questions in the second research lesson – like "What do you think is a reason for that? What decides how big the volume might be? What do we seek for when we want to know how much water the glass accommodates?" By asking such questions, they hope to engage the students in mathematical thinking and argumentation. To answer these kind of questions, the students are expected to produce endorsed narratives through engagement in explorative routines. Although they add such questions in the second iteration, the two research lessons develop similarly in terms of students' participation. In addition to ask the questions differently, the teachers decided to give the students verbal guidelines if they struggle working on the tasks.

In the following, we provide some illustrative examples of our analysis of teacher-learner interactions in both iterations of the research lesson, to better understand how the mathematical object of volume is realized and what kind of routines that seem to govern the mathematical discourse that the students are invited into. Lines 1–14[1] are from the first research lesson, whereas line 15 and 16 are from the second – which illustrate one of the adjustment the teachers made.

Realization of the mathematical object of volume

One of the teachers' metadiscursive rules – related to teaching practice – is to always present the learning aims at the beginning of the lesson. The following utterance from the first research lesson illustrates this.

[1] T: The learning aim for this lesson is for you to understand what volume is. You are supposed to calculate the volume of different geometric shapes and reflect upon your answers. We want you to engage in the tasks and cooperate well.

This statement (line 1) communicates something important about volume. Firstly, when presenting the aim as understanding "what volume is", the teacher indicates that volume is an object. The statement thus indicates an intent to engage students in the discursive construction of mathematical objects. Secondly, the teacher's statement (line 1) represents a reification of volume as something we can find by calculation. Volume is not described as the act of calculating something, but as an object that is related to the product of a calculation process. After this presentation of the learning aims, volume is introduced in the lesson by the example of different glasses filled with water (figure 2). The picture serves as an iconic mediated artefact in the discourse of volume.

Figure 2. *Iconic visual mediated artefact*

The verbal discourse continues like this.

[2] T: Glasses of water. How does this relate to volume?

[3] S: How much water that fits into the glass.

[4] T: Yes, and if the height of water is equal in all glasses, which one has the biggest volume? How can we tell? What do you think?

With her first question (line 2), the teacher indicates that the iconic artefact is related to the mathematical object of volume, and the student responds by realizing volume as the amount of water a glass contains (line 3). The student's response indicates an objectified discourse of volume as quantity. In the continued discourse, when the teacher presents the first problem, volume is realized in three ways: firstly, as amount of sand (line 5 and 6), secondly, as number of buckets (line 8), and, thirdly, as a measurement unit (line 7 and 8).

[5] T: This is a sandbox [shows a picture of a sandbox at the blackboard, another iconic visual mediator]. A girl has built a sandbox for her doll. The sandbox is given this size: length 40 centimeters, width 30 centimeters and height 20 centimeters. How much sand is necessarily needed to fill the whole sandbox? You can discuss your answers in pairs.

[6] T: [repeat the question] How much sand?

[7] S: 24000 square centimeters [cm^3]

[8] T: 24000 square centimeters [cm^3]. How much is that? Is it easy to imagine how much sand that is? For instance, how many buckets of sand does the girl in the task need? [points at and picks up a garbage can]. Is it possible to find another measurement unit? One that makes more sense according to this amount of sand?

The students suggest both cubic meters and cubic decimeters. The teacher asks if it is possible to measure the amount of sand in liters – indicating a saming[2] of the signifiers ”cubic decimeter” and ”liter”. A student quickly responds by asking, ”isn't one cubic decimeter one liter?” The teacher confirms that this is correct, and she states: ”Now it is easier to know how much sand is needed.” This last utterance indicates a colloquial discourse.

The signifier ”amount of sand”, originating from the first task, implies the use of the volume formula. To find how much sand that would fit the sandbox, the students have to plug in the numbers given and get the volume. This is an example of a ritual routine, which is restricted and has a situated procedure. The signifier is not regarded as an equation that can be solved for different unknowns, but rather as a formula where you calculate something, and the point is the answer rather than the equality.

Math discourse on volume and equations, and their metarules

After this introduction, the second sandbox problem is presented verbally.

[9] T: The sandbox needs to fit 500 liters, 500 liters of sand. You have to decide the shape of your sandbox and you might need some formula. You can locate the formulas either in your textbooks or on the Internet. You are going to build two sandboxes and [you are] supposed to make a drawing, write down the measurement units and then calculate the volume. Afterwards we want you to explain to your fellow students how you were thinking. If you find mathematics difficult, you can select a simple shape. If you think that is too easy, select a more complex shape.

Considering the presentation of the problem (line 9). By telling the students that they have to decide a shape, locate formulas, make a drawing, write down measurements and calculate, the teacher emphasizes human actions on mathematical symbols. This indicates that the students are invited into ritual rather than explorative participation in the mathematical discourse (Heyd-Metzuyanim et al., 2016). There are no degrees of freedom in the course of actions, but the students are presented with a list of steps they have to carry out in order to solve the problem. The last assignment differs somewhat from the one described above, as it invites the students to construct and sustain new or endorsed narratives

– as in an explorative routine. The students work on the tasks for about 20 minutes, and then some of the groups display their solutions on the blackboard. Two of the students' (S1 and S2) responses are reported.

[10] S1: I took 500 and divided by 10. And then divided by 10 again, and then 5. No, I am not sure what I have done.

[11] T: You thought the other way around.

[12] S1: I knew it was decimeter. If you take 10 times 10 times 5, it becomes 500 liters or 500 decimeters [dm^3] which was our given answer.

[13] S2: Firstly, we took 50 times 5 which is 250. Then we took times 4 which is 1000. That is a rectangle. And then we divided it in half, then it became 500 liters. Then we have a cylinder. We took radius, that was 3. You must take pi times squared radius, then it is 3 times 3 which is nine ...

When presenting their solution to the problem, we notice how students focus on the actions they perform. They use phrases like "we took [...] and then we took", and "you must take [...] and then it is". The utterances can be described as processual and personal rather than structural (Sfard, 2008). The students appear to use the volume formula by plugging in numbers to get an answer (metarule). The discourse is also characterized by a lack of objectification. Neither of the students say, "the volume *is* 500 liters," but they describe the process and conclude that, "it becomes 500 liters" (lines 12 and 13). The students' discourse illustrates a typical phrase-driven word use, as they adopt the use of the formula from a comparable situation – the former problem. In the first problem, the visual mediators serve as both iconic (picture of the sandbox) and verbal signifiers, written words (the sides of the sandbox are described as "40 cm long, 30 cm wide, and 20 cm high"). In the second problem, the signifiers are realized both visually (iconic, picture of a rectangular sandbox) and verbally (the students' and teachers' discourse, lines 9–14).

Three observations can be made about how students appear to take more advantage of saming the signifiers that are visually rather than verbally realized. Firstly, most of the students select a shape of the sandbox that is similar to the one in the picture at the blackboard, even though the teacher encourages them to find other shapes. Secondly, they use the same formula as the one displayed. Thirdly, the students follow the same metarules when they multiply three numbers to get a given volume. The students turn to the procedures they have been introduced to instead of constructing new endorsed narratives. The lack of reification and alienation is reflected in the teacher's final comment.

[14] T: None of you used equations. It is a quite simple task if we use equations. Let me show you [writing on the board: $V = l \cdot w \cdot h$] [...] Length times width times height equals 500. If you have found "length times width", if you

have written down these sizes. Let me pick a number. For instance, 100. Then we can think of h as an x, an unknown [writes: $100 \cdot h = 500$]. Now you can divide 100 here [writes a fraction bar under 100, followed by 100]. Then you can cross out these [points at the fraction 100/100], and you have to divide 100 here as well [writing a new fraction bar under 500]. Now you can find what you have been looking for, [writing $h =$] in this case, equals 5. Equations is something you can use to solve almost any task.

These utterances (line 14) indicate a procedural rather than an objectified discourse. An indication of this is when the teacher presents a possible solution by explaining what the students should do – using phrases like ”*you* can cross out”, and ”*you* have to divide”. This word use focuses on actions performed rather than on mathematical objects. In addition to this lack of reification, we also observe that the agent of these actions is highly visible in the discourse. This lack of alienation is visible from the teacher's use of pronouns: ”you can cross out”, and ”you have to divide”. The focus is thus on manipulation of objects in order to find the volume, rather than in engaging the students in a discussion of volume as a mathematical object.

Between the first and the second research lesson, the teachers make some changes to the structure of the lesson. One example is that they decide to help those students who are struggling, and guide them through the problems. To invite the students into a mathematical discourse on equations, they maintain a focus on what to do with the numbers to get the given volume.

[15] T: If you have decided that the height is 40 cm, what do you do to find the size of the two other sides? You know that length times width times height is 500 000. Three numbers multiplied, and one number is given. You are supposed to find the two others.

Even towards the end of the second research lesson, as the teacher tries to summarize the content and the aim of the lesson, the routines still focus on procedures.

[16] T: This one (holding a sheet of paper folded like a cylinder) has a bigger base area, yes. Moreover, when we calculate the area of a circle, we take radius, times radius times phi. The radius twice. This counts more than the height, because you just multiply this once.

To sum up our analysis of the discourse from the two research lessons, we discovered four main signifiers from the data: quantity, measurement units, figures and formula (see figure 3).

The signifiers are realized mainly through ritual routines, and the metarule in the discourse is to use formulas to get a product, to find the

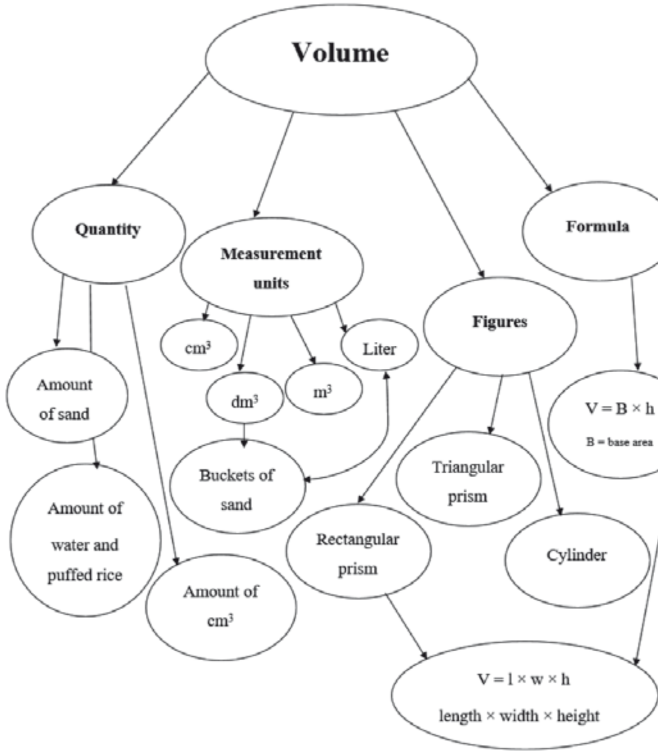

Figure 3. *Realization tree of the mathematical object volume*

volume. Our findings illustrate how the students adapt the teacher's discourse. They participate in the discourse they are invited into, which in this case was more ritual than explorative. In addition, our analysis has revealed that it seems to be a gap between the discourse (including the metarules and routines) the teachers want to invite the students into and the discourse they are practicing; Heyd-Metzuyanim et al. (2016) report on a similar gap in their study.

Concluding discussion

Previous studies on volume of three-dimensional solids tend to focus on students' strategies, understandings or misconceptions (e.g. Battista & Clements, 1996; Gough, 2004; Tekin-Sitrava & Isiksal-Bostan, 2014). Among the few studies that focus on the teacher, the majority seems to concentrate on attributes of teachers, in particular their knowledge (e.g.

Tekin-Sitrava & Isiksal-Bostan, 2016), rather than on the actual work of teaching. This corresponds with a more general tendency in the mathematics education literature (e.g. Hoover, Mosvold, Ball & Lai, 2016). When discussing students' learning of volume in their research lesson, the teachers in the LS group were surprised to observe that the students did not solve the sandbox problem by using their previous knowledge of equations. They concluded that the students' understanding was too weak, and they decided to change the problem to make it easier for the students. Through our analysis, we propose a different interpretation by focusing on the work of teaching instead – here seen in terms of the communicational activity of the teacher in teacher-learner interactions (cf. Tabach & Nachlieli, 2016).

The teachers talk about the students' inability to apply their previous knowledge of equations, and they thus apply an acquisitionist metaphor of learning (Sfard, 1998) – as if knowledge is an actual object that can be acquired and transferred for use in different contexts. Adhering to Sfard's (2008) commognitive theory, we suggest switching the perspective and instead consider students as participants in various mathematical discourses – each of which is typically considered by its participants as distinct. The students have previously engaged in a discourse of equations, and they have solved equations for an unknown – typically signified by the letter x. When analyzing teachers' mathematical discourse in two iterations of a research lesson, we notice that they never use a word like "equation" and only once use the word "unknown". In fact, there seems to be less focus on creating narratives about mathematical objects like these, and more focus on inviting the students to perform actions on mathematical symbols. This is a common characteristic of ritual discourse (Heyd-Metzuyanim et al., 2016; Sfard, 2008). Even though the teachers aim at facilitating an explorative discourse, our analysis indicates that the mathematical discourse of students as well as the teacher in the research lessons is predominantly ritual. By stating this, we do not intend to imply that a discourse being ritual is negative. We simply argue that the characteristics of the mathematical discourse that these students are invited into are in line with how ritual and deobjectified discourses are described in the literature (e.g. Heyd-Metzuyanim et al., 2016; Sfard, 2008), and we suggest that the students' apparent lack of understanding might be explained by these attributes of the discourse. In addition, the teachers do not seem to make a clear connection between the discourse of volume and the discourse of equations. The metarules governing the discourse of the volume formula differ from the metarules that govern the discourse of equations. Instead of talking about equality and performing the same operations on both sides of the equals sign in order to find the unknown, the volume formula is always communicated as a recipe for finding the volume. Thereby, the teachers do not signal to the students that the present discourse of volume is connected to the previous discourse of equations.

Our study contributes to the field by identifying some important aspects in the communicational work of teaching volume. Where earlier research tended to focus mostly on the students, our study focuses on how teachers realize volume, how they communicate metarules of the discourse on volume, and how these metarules are connected (or not) with metarules of discourses on other mathematical objects. From our analysis, we have indicated some connections in the teacher-learner interactions that might be relevant to investigate further. On a meta level, our study represents an attempt to conceptualize the teaching of volume in terms of communication, and we believe that it thereby also has potential to influence the ongoing efforts towards developing a theory of teaching as communicating (cf. Mosvold, 2016; Sæbbe & Mosvold, 2016; Tabach & Nachlieli, 2016).

We suggest that Sfard's (2008) theory of commognition can be a useful theory in the context of LS – not only for analyzing data, but potentially also for informing the actual conduct of LS. It has been observed that studies on LS tend to be vague about observation and learning (Larsen et al., 2018). The commognitive theory provides a definition of learning in terms of observable communication that might be useful for teachers who engage in LS, since they often seem to struggle in observing student learning. This potential use draws upon the strength of the commognitive theory for analyzing local discourses in more detail. This does not imply, however, that the larger context is not important. In fact, there might be a tendency of teachers in the Western world to interpret students' actions in the local context, rather than in terms of a more coherent curriculum (e.g. Fujii, 2014, 2016).

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Notes

1 The numbered lines are not related to the transcripts. Their function is to make it easier for the readers to follow the analysis.

2 "The process of saming can be seen as the act of calling different things the same name" (Sfard, 2008, p. 170).

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