Oral presentations as a tool for promoting metacognitive regulation in real analysis

MARGRETHE NAALSUND AND JOAKIM SKOGHOLT

Real Analysis is for many students their first proof-based mathematics course, and many find it challenging. This paper studies how oral presentations of mathematical problems for peers can contribute to students' metacognitive reflections. The paper discusses several aspects tied to preparing for, and carrying out, oral presentations, that seem to spur important sub-components of metacognitive regulation such as planning, monitoring, and evaluating. Thoughtful guidance from an expert encouraged the students to further monitor their cognition, and evaluate their arguments and cognitive processes when expressing their reasoning to their peers.

The transition from calculation-based mathematics courses to proof-based courses is difficult for many students, particularly when it comes to understanding proofs (Fuller et al., 2014; Selden & Selden, 2003; Weber & Mejia-Ramos, 2014). Most proof-oriented university mathematics courses are taught in a "definition-theorem-proof" format where the lectures consist of professors presenting proofs to the students (Fuller et al., 2014; Weber, 2004). A deductive, linear way of presenting proofs can make difficult the process of seeing the structure of the proof or the overarching method being applied, thus hindering comprehension (Fuller et al., 2014). Studies show that students may learn little from studying proofs (Weber & Mejia-Ramos, 2014) and that they are confused by the proofs that they read (Cowen, 1991; Fuller et al., 2014), particularly regarding underlying argumentation (Alcock & Weber, 2005; Selden & Selden, 2003). Students spend little time reading each others' proofs in most classes (Weber & Mejia-Ramos, 2014), an activity that may be beneficial for a better understanding of proofs (Cowen, 1991).

Margrethe Naalsund, *Norwegian University of Life Sciences*
Joakim Skogholt, *Norwegian University of Life Sciences*

Naalsund, M. & Skogholt, J. (2017). Oral presentations as a tool for promoting metacognitive regulation in real analysis. *Nordic Studies in Mathematics Education*, 22 (4), 103–119.

Alcock and Weber (2005) studied the skills needed to determine whether a proof is valid in real analysis. They argue that in validating proofs it is not sufficient to make certain that each statement in the argument is true, but that one should check that there is good reason to believe that each statement follows from the preceding statements or from accepted knowledge.

At our university, the students' first experience with a proof-based mathematics course is a class in introductory real analysis. The class is a typical first course in real analysis, covering topics such as limits for functions of one variable and sequences, differentiation, the Riemann integral, and Taylor's theorem. Students usually follow the course in their third or fourth year at university. The course constitutes one third of the students' workload for the semester. The formal statement of definitions, theorems, and proofs of theorems plays a central role in analysis courses. To succeed in these types of courses it is therefore important that students are able to understand proofs and prove simple statements themselves. Prior to this course the students have very little experience with mathematical proofs. The course also differs from previous courses the students have done in that the students are expected to solve more problems rather than exercises, as well as arguing more than performing calculations. Here we make the traditional distinction between problems and exercises, where, by a problem, we mean a task for which the solution method is not known in advance (see e.g. Lithner, 2008). An exercise is a task which is not a problem. Whether a task is a problem or exercise may vary from student to student.

In order to develop an in-depth understanding of proofs and proving, it is important to foster *creative mathematically founded reasoning* (Lithner, 2008), meaning that an argument should be novel (to the reasoner) and plausible (some support for strategy and solution), and it has a mathematical foundation (tied to mathematical properties of the objects involved). Since the students taking the real analysis course have in their previous courses worked with tasks that primarily require applying learned techniques, they are accustomed to utilizing *imitative reasoning* (Lithner, 2008) to a larger extent than creative reasoning. Imitative reasoning entails memorizing a solution or using an algorithm so that solving a task simply consists of writing down the solution from memory. A typical example is proving a central theorem from a course, where students in some cases simply memorize the proof from the textbook or from a lecture, without necessarily understanding it (Lithner, 2008). In real analysis, the students need to improve their creative reasoning to prove unseen (to the students) theorems as well as argue for their solutions to problems and exercises. Creative reasoning is closely interrelated with problem solving skills, since solving problems in real analysis will to a lesser extent require the use of learned strategies and imitative reasoning than what the students have experienced in earlier courses. Problem solving skills involve regulatory aspects such as monitoring and evaluating progress, and evaluating the arguments justifying the solution of a problem (Schoenfeld, 1985). Monitoring and evaluating progress can also be seen as metacognitive processes (Garofalo & Lester, 1985; Schoenfeld, 1987). In fact, metacognition has been linked to problem solving skills in mathematics research since the 1980s (Schneider & Artelt, 2010). Several researchers have, for example, seen a link between Polya's (1945) problem solving heuristics and metacognition (Garofalo & Lester, 1985; Schneider & Artelt, 2010; Schoenfeld, 1987). Students' metacognitive abilities, in particular the regulation of metacognitive processes, is therefore connected to their problem solving abilities. According to Schneider and Artelt (2010), there is a need for more knowledge of mathematics students' metacognitive monitoring and regulation: "There is a need to focus on behavior relevant to strategy selection, cognitive monitoring, and evaluation of cognitive processes" (p. 154).

This paper describes a seminar model designed to enhance students' understanding of proofs in real analysis. The model is based on having weekly small-group seminars with an assistant teacher in which students take turn presenting solutions to the weekly problem set. The other students are encouraged to ask questions, and the assistant teacher will also ask questions, aiming to guide the students into a productive discourse in which they can learn real analysis. Although some work in this area has been done (e.g. Alcock & Wilkinson, 2011; Hodds, Alcock & Inglis, 2014; Weber & Mejia-Ramos, 2014), there is still need for more studies regarding specific instructional interventions that mathematics professors might use for structured proofs to be effective (Fuller et al., 2014; Weber, 2004). Weber and Mejia-Ramos (2014), for example, argue the benefits of giving students the opportunity to use proofs as a communicative tool in meaningful mathematical activity and for both the prover and the larger community to participate in the construction and validation of the argument: "In our perspective, the act of proving a theorem can be understood as an interactional accomplishment between the prover and his or her audience" (Weber & Mejia-Ramos, 2014, p. 90). Even though metacognitive skills improve as the students get older, studies point at inadequacies also among university students' metacognitive abilities (Zan, 2000).

The ultimate goal with the research project is to foster the students' understanding and their reasoning abilities in real analysis such that it also forms a basis for further studies in mathematics. To this end, it is important to help the students develop thought processes that are productive for learning real analysis. Related to this are the students' ability to think about and regulate their own thinking. This article aims at studying how the oral presentations affect such aspects of students' metacognitive awareness. The following research question guided the research:

> How can oral presentations of real analysis problems for peers contribute to students' metacognitive regulation?

Metacognition

In its simplest form, metacognition can be defined as ”thinking about thinking” (Flavell, 1979; Lai, 2011), or somewhat more precise: ”the monitoring and control of thought” (Martinez, 2006). Beyond this broad idea of what constitutes metacognition, different definitions flourish in the educational literature (Lai, 2011). In spite of different definitions, there seems to be an agreement that metacognition consist of two main components: *cognitive knowledge* and *cognitive regulation* (Lai, 2011; Schneider & Artelt, 2010; Schraw, 1998; Schraw, Crippen & Hartley, 2006). Based on the work of several researchers (e.g. Cross & Paris, 1988; Schraw, 1998; Schraw et al., 2006), Lai (2011) summarized cognitive knowledge to entail (1) knowledge about oneself as a learner and factors affecting cognition, (2) awareness and management of cognition, including knowledge about heuristics and strategies, and (3) knowledge about why and when to use a given strategy. Cognitive regulation is defined through the three sub components *planning*, *monitoring*, and *evaluation* (Cross & Paris, 1988; Schraw, 1998; Schraw et al., 2006). According to these researchers, planning involves identification and selection of appropriate strategies and allocation of resources that affect performance. Planning includes goal setting, activating relevant background knowledge, and budgeting time. Monitoring includes the self-testing skills necessary to control learning, and means attending to, and being aware of, comprehension and task performance. Evaluation refers to the ability of assessing the processes and products of one's learning, and includes revisiting and reevaluating one's goals and conclusions (Lai, 2011; Schraw, 1998; Schraw et al., 2006). Thus, the regulatory aspect of metacognition means making use of metacognitive knowledge, by actively monitoring and evaluating the strategies chosen for solving problems, and the arguments chosen to support the reasoning.

Garofalo & Lester (1985) highlight the importance of regulatory metacognitive behaviors in mathematical performance, and exemplify regulatory metacognitive behavior to include ”[...] planning courses of actions, selecting appropriate strategies to carry out plans, monitoring execution activities while implementing strategies, evaluating the outcomes of strategies and plans, and, when necessary, revising or abandoning nonproductive strategies and plans" (p. 166). According to Schraw (1998), metacognitive regulation improves performance in a number of ways, "including better use of attentional resources, better use of existing strategies, and a greater awareness of comprehension breakdowns" (p. 114). He claims that by improving one aspect of regulation it is likely that other aspects may improve as well. There appears to be a close connection between such metacognitive skills and well-established skills involved in successful problem solving (Polya, 1945; Schoenfeld, 1987) and creative mathematically founded reasoning (Lithner, 2008). It is therefore of interest to investigate how students express metacognitive regulation while working with the seminar model. This can give insight into how students think about real analysis and give information that can help later interventions. The focus in this paper will be on the students' metacognitive regulation, defined through the three components presented, while working with the problems they were to present for their peers.

Collaborative teaching settings and metacognitive regulation

There is a strong correlation between higher-order thinking, metacognition, and in-depth learning (Lithner, 2008; Schoenfeld, 1987). Based on Vygotsky's theory of learning, many researchers advocate the use of language and social discourse to promote higher-order thinking. They further highlight the importance of student interaction and collaborative teaching settings for spurring deeper thought, thus encouraging the deve-lopment of metacognitive skills (e.g. Hogan, 1999; Kuhn & Dean, 2004; Martinez, 2006; Schneider & Artelt, 2010; Schoenfeld, 1987; Schraw et al., 2006). According to Schraw et al. (2006, p. 120), "[...] collaboration can be viewed as a tool to support approaches that encourage an inquiry orientation, the utilization of strategies, the development and sharing of mental models, and the making explicit of personal beliefs". Thus, teaching settings where students are encouraged to collaborate – exchange their ideas and make their thought processes visible to others – therefore provide great potential for in-depth learning through activating metacognitive thinking. "Social interaction among students should be used to cultivate their metacognitive capacity. If students are encouraged and guided to think critically together, then their spoken reasoning will ideally make their cognitive tools available to one another" (Martinez, 2006, p. 699). The role of the teacher is important for facilitating this process, and several researchers argue teacher questioning as being significant for promoting student reasoning and reflection (Davis, 2003; Mueller, Yankelewitz & Maher, 2014; Sahin & Kulm, 2008). Metacognitive regulation also plays an important role in oral communication of information (Davis, 2003; Martinez, 2006; Schneider & Artelt, 2010).

Similar seminar models as presented in this paper have been studied before. Reisel (1982) for example, describes a model where students were given tasks to solve and present to other students. In this model students were in particular allowed to ask a tutor for help in preparing their presentations, as well as give and receive feedback on their presentations. Another example is the Warwick Analysis project (Alcock & Simpson, 2002). In this project, a first year analysis course was modified so that the students were given a minimal text and were expected to produce most of the results themselves by solving problems together in groups of 3–6 students with staff support. One major difference between the two studies mentioned and our study is the guidance given to the students. In our study the students were given no special guidance about the presentations except for what they got during the presentation itself. This was done because we wanted to try an intervention that did not require allocating extra resources, and an intervention that would be simple for anyone to try without much special qualifications.

Setting and methods

Case description

The Real Analysis course at our university consists of lectures and seminars from August to December every year. In August 2015, we introduced a seminar model as part of the research project described in the introduction. Two hours a week, the students met to discuss a set of tasks they had worked on prior to the seminar. Each student had prepared one task in particular that they presented for the group, and each student had the particular responsibility of providing feedback to one other student's presentation. The aim of the model was to offer the students an opportunity to articulate their thoughts and make their reasoning explicit through arguing, justifying, and discussing mathematical content. After the students had presented, commented upon, asked questions, and responded to the comments and questions, the seminar tutor summarized the important concepts inherent in the specific task and complemented the discussion and the mathematics of the task. The tutor, in this article given the pseudonym Christopher, was not one of the researchers, nor was he involved in the examination of the students at the end of the semester.

Prior to the start of the seminar the researchers had a conversation with Christopher to discuss how the tutoring should be done. He was instructed that the students should be the first to give feedback. After the students have given their feedback, the tutor could give feedback and ask questions. The tutor was instructed to follow tutoring principles similar to those given in Mason (2002). In particular, the tutor should try not to tell the students what to do, but try to guide the students through open questions and constructive criticism whilst being supportive. If appropriate the tutor would also briefly put the problem into a context in real analysis to help the students understand why this problem is important and where it fits into the course material.

All students following the Real Analysis course had the opportunity to participate in the seminar. Written and oral information regarding the seminar model and the corresponding research project was given at the start of the semester. Sixteen students signed up for the exam in December. However, only nine students followed the lectures and the seminars on a regular basis, and all nine students agreed to participate in the research project. One student chose to withdraw from the study early in the semester. With nine students, we decided that two groups were necessary to have appropriate group sizes. Because of timetable conflicts, we ended up with one group with six students and one group with three students. Participation in the study was optional, and students were allowed to participate in the seminars independently of whether they wanted to be a part of the study or not. The students were informed about the study at the beginning of the semester, and were given the option of withdrawing from the study at any time without having to give any reasons. One student did not want to be filmed. This was solved by seating this student separately from the rest of the students when we were filming, and turning off the cameras when this student was presenting.

Procedure and instrument

The data collected in the course of the semester includes observations and video recordings of five group sessions, a written questionnaire at the beginning of the semester, written solutions to two sets of tasks, and two semi-structured group-based interviews at the end of the semester. This paper reports on parts of the data material: one question from the questionnaire and on interview findings concerning student' metacognitive regulation. An interview study is a reputable way of investigating students' ideas and thoughts (Kvale &Brinkmann, 2009; Sharma, 2013), and there are many advantages to researching peers sharing such experiences in small groups (e.g. Krueger & Casey, 2000).

Three students from the main group, here given the names Anders, Tom, and Anna, and two students from the second group, named Sara and John, chose to participate in a group-based interview after the exam.

These students are about the same age, early to mid-twenties. They have the same background in mathematics, and they all share a fondness for mathematics and a motivation for learning. They knew each other well after working quite closely together throughout the semester. Anders, Tom, and Anna from group 1 were much more talkative in the interview setting than Sara and John from group 2, and they willingly shared their thoughts and reflections. Thus, the members of group 1 in particular contributed to an engaged conversation. We have therefore included more data from the group 1 interview.

Both interviews were conducted in the beginning of February in 2016. Two researchers were present at interview 1 (33 minutes) and three researchers were present at interview 2 (56 minutes). The interviewers were not involved in the teaching of the course. The interviews started with an introduction of the purpose of the study, an appreciation and acknowledgement of the students' participation in the research project, and a brief explanation of the structure of the interview. The interviews were recorded and transcribed by one of the researchers.

The interview guide included four themes, each consisting of four to five questions. The main themes concerned the students' reflection on 1) Their experiences from taking the real analysis course and their development throughout the semester; 2) The real analysis course and their mathematical reasoning and argumentation; 3) The seminar model tied to aspects of learning; and 4) how the model and the course had affected their thoughts about reasoning, understanding, and learning in mathematics. Although the interview guide emphasized numbered and specific questions within the themes, the implementation was semi-structured in the sense that the conversation, to a great extent, followed the students' lead (Kvale & Brinkmann, 2009). Because of the students' engagement in the conversation, themes and ideas could be revisited multiple times during the interview. After asking a question we let the students discuss freely, and did not ask a new question before either the students had finished discussing amongst themselves or the discussion changed towards the next question on our guide. The questions were formulated such that the students were given the opportunity to reflect on both positive and negative aspects of the seminar model, and how this way of working affected their learning.

Analysis

Mainly, the interview analyses rested on "a general reading of the interview text with theoretically founded interpretations" (Kvale & Brinkmann, 2009, p. 233). Both authors listened through the interviews and read the interview transcripts multiple times. All utterances concerning reflections involving aspects of metacognitive regulation – planning, monitoring, or evaluating – in connection with the seminar model and the oral presentations, were then coded individually by both researchers. Based on the individual coding and following discussions, we decided on ten utterances altogether that met the criteria. Meaning was generated by interpreting the students' utterances and conversation in light of the theoretical framework and background (sections 2 and 3). In the analysis process, we found that the students' utterances could be classified within three themes relevant for addressing the research question: Metacognitive regulation while preparing for the oral presentations, metacognitive regulation during presentation of problem solutions, and the role of the tutor for spurring aspects of metacognitive regulation. We decided to focus on these three themes in this article, and chose to use all but a few utterances as excerpts in the analysis. The remaining utterances were either very short answers to one of our questions or they overlapped with excerpts chosen to illustrate a particular point in the paper. In the latter cases, the most illuminating utterances were chosen as excerpts.

Students' responses to one of the questions in the questionnaire is included as well: *How would you describe the feedback from the tutor on the presentations?* The analysis process of the questionnaire followed the same procedure as the interview transcripts and responses to this one question were coded and thus relevant for answering the research question.

Results

Metacognitive regulation while preparing for the presentations

All the students emphasized that their preparations for the oral presentations contributed to improving their understanding, illustrated by Anders' reflection on his cognitive processes:

Anders: I think there was at least one presentation where I ... I think I did it in a very cumbersome manner. I had a really long thing one day ... It was about ... Riemann integrals and so on. But I remember when I was presenting it, I had to sit down and think about ... ok ... but why does this go from here to there? Why do we use this here ...? And then I had to think about what is this, really, when you present it graphically and look at ... ok, when you have a point here, why does it work and why does it not work to use it if you ... And then I remember I learned more by simply going through my own arguments so that I could say anything at all.

Here, Anders explained how he made an extra effort of making sure the argument he used, making his reasoning visible, was plausible and mathematically founded (Lithner, 2008). He exemplified how this extra effort involved asking himself specific *monitoring* questions (Cross & Paris, 1988; Schraw et al., 2006) in the process of preparing for the presentations ("why does this go from here to there? Why do we use this here ...?, ... why does it work and why does it not work"). By asking himself these types of questions, he was modifying his reasoning (Garofalo & Lester, 1985) in order to evaluate the plausibility of his arguments (Lithner, 2008). Searching for logical consistency in the argumentation is important for understanding proofs (Alcock & Weber, 2005). In the particular example he used, he had solved the task in a more complicated manner than was necessary. During the process of preparing for his presentation, Anders felt forced to go through his own argument again and more thoroughly, and expressed that he learned more by doing this. He further explained that he used more time solving the problems he was assigned to present than he did solving any of the other problems given in the course, and that he pushed his reasoning further in order to understand the problem. It is thus reasonable to assume that this extra time and cognitive effort involved similar metacognitive aspects as those illustrated by the example. This could be positive, as research has pointed out that some students spend a lot less time studying proofs than their teachers expect (Weber & Mejia-Ramos, 2014). On the other hand, this is not necessarily the optimal use of time for all students, as extra time spent reviewing arguments may result in the students spending less time on other courses and tasks. Anders' thoughts on how preparing for the presentations influenced his cognition was supported by Anna and Tom:

Anna: [...] You want to do it properly, and ... I don't think I would have had that thought about it [the problems] in the same way, that you have to be careful and do it correctly, and in a way that it is easy to understand. If I were not going to present it for anyone, I don't think I would have thought through it as carefully, and I would probably not be as concerned with how I worded things and my solution if I weren't going to present it.

Tom: Absolutely

According to Weber (2012), university students do not spend enough time studying proofs, and they may learn little from studying proofs following the traditional "definition-theorem-proof" teaching format (Weber & Mejia-Ramos, 2014). Encouraging the students to present their reasoning to their peers may involve more, and more effective, study-time, as the preparations spur deep metacognitive reflections at the core of the mathematical content.

Metacognitive regulation during presentation of solutions

On a question regarding the nature of the students' thought processes during the presentations, and whether or not their reasoning changed course sometimes while presenting, Tom and Anders said:

Tom: I think it happened a couple of times. What happened was that I realized that something was maybe not as correct as I thought it was, but I just continued writing what was in my notes and let the other students comment afterwards.

[...]

Anders: I thought I had a solution, I don't remember which one, but I realized that what I did was not correct, and thought "why have I done this here?" And then I thought no, no, I can't continue. I just have to say that I need to pause.

During their presentations both Tom and Anders described what can be considered monitoring of their task performance (Cross & Paris, 1988; Schraw et al., 2006). When Anders asked himself "Why have I done this here?" he attempted to make his thought process explicit to himself so that he could evaluate the work he was presenting. The excerpt illustrates well that it was *during* the presentation Anders discovered that the argument he used did not measure up. Asking oneself such evaluating questions is important for reflecting upon the reasons for one's choices and argumentation (Garofalo & Lester, 1985). They both acknowledged that presenting a problem solution to their peers influenced their on-line cognitive processes. A change in their reasoning happened during the presentation of their thoughts and ideas, a point supported also by John from the second group:

John: I think it was very good to do it orally. Because if you do it on the blackboard in front of 3 other people ... Or if we had 3 other people then it was ... It had to be correct. And often I made a mistake that I didn't think about myself. And that was useful.

Using the language to making one's reasoning explicit through explaining, arguing, and justifying is emphasized as important for in-depth learning (Maher, 2005; Mueller & Maher, 2009) and for developing metacognitive skills informing the reasoning process (Hogan, 1999; Martinez, 2006; Schoenfeld, 1987; Schraw et al., 2006). Anna's response on a question regarding how they experienced presenting their arguments to the other students, and how it influenced the argumentation and the plausibility of the arguments, is interesting in this regard: "I thought it was difficult to argue very precisely [orally], to use the mathematical concepts to create an argument, it ended up being more informal". Anders uttered similar thoughts when discussing the benefits of the seminar model in preparing for written vs oral examination:

Anders: ... in a written exam you also have to – you have to include words and explanations that would have been there if you were to present it orally. So [...] it would have been an advantage to describe your thoughts, and that sort of automatically happens orally, or in a way, it happened more automatically after a while when we did our preparations and presented the assignments. I will not say that I became an expert, but I did improve. I think it helped me on the [written] exam, and it would have helped me on an oral exam as well.

Both Anna and Anders highlighted the dissonance between using the formal mathematical language in written argumentation and in oral argumentation, respectively. They also highlighted this cognitive challenge as beneficial for learning mathematics and that practice in oral argumentation influenced the performance of arguing in writing, which was the summative assessment form at the end of the semester. Anders described how the differences in presenting orally compared to handing in something written resulted in him preparing differently, thus affecting his planning. In particular, Anders talked about ”describ[ing] your thoughts” as important. The presentations therefore seemed to help Anders making his thoughts more explicit, which affected his monitoring of task performance.

The role of the tutor

One question in the questionnaire was concerning the role of the tutor: *How will you describe the feedback from the tutor on the presentations*? All nine students gave high evaluations of the feedback that they received from the tutor. Aspects repeated by several students were that the tutor created a friendly and safe learning environment, that he gave thorough feedback that promoted their understanding, that he elicited and used the students' thinking in his feedback, and that he posed questions to spur the students' reflection. The latter is well illustrated by Sara when discussing the seminar model in the interview setting:

Sara: Yes, I was going to say that Christopher [the tutor] was very good at ... He didn't say that something was wrong, but he said ”What did you think there?” Or ”Why have you done it this way?”, in that part. And it felt great that you had to think about it yourself, that you weren't told that this was wrong, but had to explain why, and then you often have an ”aha-moment” (?) – ”Why did I use addition there? I shouldn't add there” ...

Creating a friendly learning environment, small-group collaborative settings, and thoughtful teacher guidance and questioning are all evaluated as highly important aspects for promoting mathematical reasoning and in-depth learning (Mason, 2002; Mueller et al., 2014) as well as explicitly being particularly beneficial for developing metacognitive regulation skills (Schoenfeld, 1987; Schraw, 1998; Schraw et al., 2006). Schraw (1998, p. 123) claims that "... one particular [condition] that is especially important for developing metacognitive awareness, is to have access to an expert's reflection on what he or she is doing, and how well it is being done". Through thoughtful questions and guidance, the tutor encouraged the students to monitor their cognition, and evaluate their solutions and cognitive processes. Sara's explanation illustrates how these types of questions would spur metacognitive thinking and a deeper understanding of what she was doing and why. Thoughtful reflections and questions asked by an expert are in particular emphasized as important to activate aspects of metacognitive regulation (Schoenfeld, 1987; Schraw, 1998; Schraw et al., 2006).

Concluding reflection

So, how can oral presentations of real analysis problems for peers contribute to students' metacognitive regulation? First, the students explicitly stated that they spent a lot of time, more than they usually did solving tasks, when preparing their oral presentations. As discussed earlier in the paper, this is not necessarily positive as it can result in a less optimal allocation of studying time. However, this time involved an, according to them, increased cognitive effort in order to better understand their own reasoning and the plausibility of their arguments. This effort involved asking themselves important monitoring and evaluating questions, which activate and practice metacognitive skills. Secondly, the students reported that it was in the moment of presenting their reasoning they discovered that arguments they used did not measure up. Thus, the action of presenting problem solutions can spur regulative metacognitive reflection. The students further experienced a dissonance between their use of formal mathematical language in oral argumentation and written argumentation, respectively. In presenting their problem solutions, they needed to orally present their written argumentation. This experienced tension made them feel the necessity of better planning how to express their thoughts, and monitoring through making their thoughts more explicit. Third, and last, the students highlighted the role of the tutor as a mentor asking open and focusing questions as very important for their reflection and learning.

In addition to spending more time studying proofs (Weber, 2012; Weber & Mejia-Ramos, 2014), the seminar model discussed in this paper may further help the students to engage in a more active interaction with the proof and better validate each step of the argument, thus gaining a deeper understanding of what a proof is. The lack of such competence among undergraduate students is emphasized as one important reason why understanding proofs is such an obstacle for many students (Alcock & Weber, 2005; Selden & Selden, 2003). In preparing and presenting their arguments to different problems for peers, the students became active participants in explaining and justifying, as well as constructing, their argumentation and their reasoning. Researchers have claimed that using the mathematical language this way enhances metacognitive skills (Schoenfeld, 1987; Schraw et al., 2006), for example by contributing to making the reasoning plausible and anchored in the mathematics at play (Lithner, 2008). Promoting reasoning consisting of logically structured and mathematically founded argumentation is important for understanding proofs and proving (Alcock & Weber, 2005).

The students in our study were all about the same age, they had quite similar background regarding experiences with school system as well as previous university mathematics courses. Thus, one can argue that they represent ”typical” real analysis students at our university. One might therefore argue that similar findings could occur if the study was repeated at another time with other real analysis students. We have also grounded the data analysis in international research. Hence, we can assume that our findings bear relevance outside this particular group of students, also in an international context.

Acknowledgement

We thank Hans Erik Lefdal, Norwegian University of Life Sciences, for valuable contributions in designing the research project.

References

Alcock, L. & Simpson, A. (2002). The Warwick analysis project: practice and theory. In D. Holton (Ed.), *The teaching and learning of mathematics at university level. An ICMI study* (pp. 99–111). Dordrecht: Springer.

Alcock, L. & Weber, K. (2005). Proof validation in real analysis: inferring and evaluating warrants. *Journal of Mathematical Behavior*, 24 (2), 125–134.

Alcock, L. & Wilkinson, N. (2011). E-proofs: design of a resource to support proof comprehension in mathematics. *Educational Designer*, 1 (4).

Cowen, C. (1991). Teaching and testing mathematics reading. *American Mathematical Monthly*, 98 (1), 50–53.

Cross, D. R. & Paris, S. G. (1988). Developmental and instructional analyses of children's metacognition and reading comprehension. *Journal of Educational Psychology*, 80 (2), 131–142.

Davis, E. A. (2003). Prompting middle school science students for productive reflection: generic and directed prompts. *The Journal of the Learning Sciences*, 12 (1), 91–142.

Flavell, J. H. (1979). Metacognition and cognitive monitoring: a new area of cognitive-developmental inquiry. *American Psychologist*, 34 (10), 906–911.

Fuller, E., Weber, K., Mejia-Ramos, J. P., Rhoads, K. & Samkoff, A. (2014). Comprehending structured proofs. *International Journal for Studies in Mathematics Education*, 7 (1), 1–32.

Garofalo, J. & Lester, F. K. (1985). Metacognition, cognitive monitoring, and mathematical performance. *Journal of Research in Mathematics Education*, 16, 163–176.

Hodds, M., Alcock, L. & Inglis, M. (2014). Self-explanation training improves proof comprehension. *Journal for Research in Mathematics Education*, 45 (1), 62–101.

Hogan, K. (1999). Thinking aloud together: a test of an intervention to forster students' collaborative scientific reasoning. *Journal of Research in Science Teaching*, 36 (10), 1085–1109.

Kuhn, D. & Dean, D. (2004). A bridge between cognitive psychology and educational practice. *Theory in Practice*, 43 (4), 268–273.

Lai, E. R. (2011). *Metacognition: a literature review*. Pearson research report. Retrieved from http://www.pearsonassessments.com/hai/images/tmrs/Metacognition_Literature_Review_Final.pdf.

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics*, 67 (3), 255–276.

Maher, C. A. (2005). How students structure their investigations and learn mathematics: insights from a long-term study. *Journal of Mathematical Behavior*, 24, 1–14.

Martinez, M. E. (2006). What is metacognition? *Phi Delta Kappan*, 696–699.

Mason, J. H. (2002). *Mathematics teaching practice: a guide for university and college lecturers*. Chichester: Horwood Publishing.

Mueller, M. & Maher, C. A. (2009). Learning to reason in an informal math after-school program. *Mathematics Education Research Journal*, 21 (3), 7–35.

Mueller, M., Yankelewitz, D. & Maher, C. A. (2014). Teachers promoting student mathematical reasoning. *Investigations in Mathematics Learning*, 7 (2), 1–20.

Polya, G. (1945). *How to solve it?* Garden City: Doubleday.

Reisel, R. B. (1982). How to construct and analyze proofs – a seminar course. *The American Mathematical Monthly*, 89 (7), 490–492.

Sahin, A. & Kulm, G. (2008). Sixth grade mathematics teachers' intensions and use of probing, guiding, and factual questions. *Journal of Mathematics Teacher Education*, 11 (3), 221–241.

Schneider, W. & Artelt, C. (2010). Metacognition and mathematics education. *ZDM*, 42, 149–161.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando: Academic Press.

Schoenfeld, A. H. (1987). What's all the fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189–215). Hillsdale: Erlbaum.

Schraw, G. (1998). Promoting general metacognitive awareness. *Instructional Science*, 26, 113–125.

Schraw, G., Crippen, K. J. & Hartley, K. (2006). Promoting self-regulation in science education: metacognition as part of a broader perspective on learning. *Research in Science Education*, 36, 111–139.

Selden, A. & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, 34 (1), 4–36.

Weber, K. (2004). Traditional instruction in advanced mathematics classrooms: a case study of one professor's lectures and proofs in an introductory real analysis course. *Journal of Mathematical Behavior*, 23, 115–133.

Weber, K. (2012). Mathematicians' perspectives on their pedagogical practice with respect to proof. *International Journal of Mathematical Education in Science and technology*, 43 (4), 463–482.

Weber, K. & Mejia-Ramos, J. P. (2014). Mathematics majors' beliefs about proof reading. *International Journal of Mathematical Education in Science and technology*, 45 (1), 89–103.

Zan, R. (2000). A metacognitive intervention in mathematics at university level. *International Journal of Mathematical Education in Science and technology*, 31, 143–150.

Margrethe Naalsund

Margrethe Naalsund is associate professor in Mathematics Education. She works at Faculty of Science and Technology (Section for Learning and Teacher Education) at Norwegian University of Life Sciences (NMBU). Her main research interests are learning and teaching algebra at primary and secondary school, and learning and teaching real analysis at university level.

margrethe.naalsund@nmbu.no

Joakim Skogholt

Joakim Skogholt is PhD-student in Mathematics. He works at Faculty of Science and Technology (Section for Applied Mathematics) at Norwegian University of Life Sciences (NMBU). His main research interests are applied linear algebra, and learning and teaching real analysis at university level.

joakim.skogholt@nmbu.no