Developing practice through research into university mathematics education

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The paper provides a very brief outline review of research into some key issues that affect students' performance in mathematics in higher education. Community of practice theory is used to frame and focus the discussion. Policies regarding the recruitment of students, institutional practices for grouping students and the cultures of teaching and learning mathematics are considered. The research reviewed provides a context for examining the contributions of the research reports included within this thematic issue of NOMAD. The reports address three themes: regular approaches adopted in teaching mathematics in higher education, innovative approaches to teaching and learning, with emphasis on student participation in the educational process, and the characteristics of mathematical knowledge students appropriate. The paper endorses calls for large scale studies, especially those which relate teaching approaches, both regular and innovative, to the qualities and characteristics of students' learning. The absence of a single overarching theoretical framework that embraces all the studies is also perceived as an obstacle that interferes with scientific developments in the field of researching university mathematics education. However, the value of teachers researching their own practice and their students' learning is argued to be crucial for developing knowledge "in practice" and this underscores the value of the papers included in this issue of NOMAD, both for the authors and the inspiration of other higher education mathematics teachers who, it is hoped, will be inspired to engage in similar studies.

Over three decades ago a UK Government inquiry into mathematics education within schools observed "mathematics is a difficult subject both to teach and to learn" (D.E.S., 1982, § 243). It could be assumed that at higher education it is not so difficult because students choose to attend higher education and their programme of study, so they should be more

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motivated. Moreover, higher education courses are populated by students who have demonstrated competence within the subject and students are more mature and generally more tolerant of demands on their patience and effort. We do not want to suggest that teaching and learning mathematics in higher education is more or less difficult than at school level, but it certainly is challenging and it is different, as the papers within this thematic issue of Nomad demonstrate.

In this paper we review some of the published research and claims about teaching and learning mathematics in higher education and explain how the papers in this Nomad TI contribute to the international field.

In her plenary lecture at the first conference of the International Network for Didactics Research in University Mathematics (INDRUM), Michèle Artigue (2016) reflected on the complexity that had emerged over four decades since the field began to be established. Artigue spoke about the variety of questions addressed, theoretical approaches taken, changing context of higher education and multiplicity of discourses within the field. The papers in this thematic issue of Nomad reflect Artigue's perspective in their variety as they focus on the nature of mathematics taught and learned, the way mathematics is presented, approaches to motivate students to work on mathematics and mathematical meaning, and different viewpoints or approaches on the nature of teaching. The papers also report studies set in different theoretical frameworks – constructivist, cultural-historical activity theory, local theories of teacher knowledge and perspectives, modelling perspectives, critical mathematics education, cognitive science and psychometric analysis. It is notable that apart from this paper, theoretically there is no overlap with the theories set out in a special issue of the journal *Research in Mathematics Education* (Nardi et al., 2014) that also focused on research in university mathematics education. It does seem important however that the literature is taking theory seriously at last, as being central to the ways in which teaching and learning are conceptualised; much of the earlier literature presented narratives of good practice in which theory is somewhat implicit in what is described (Treffert-Thomas & Jaworski, 2016).

Background

The proposal for this thematic issue of Nomad was an outcome of on-going collaboration between the Mathematics Education Centre at Loughborough University (MEC-LU)[1] and the Norwegian Centre for Research, Innovation and Coordination of Mathematics Teaching (MatRIC)[2]. One of the intentions for this thematic issue, stated in the proposal, was to "provide a showcase for Nordic research in university mathematics education and demonstrate how this is connected to the international network of researchers in this area." There are widespread attempts to motivate the development of teaching and learning mathematics and research on university level mathematics education throughout Europe. For example, the **sigma** network in the UK[3], KHDM in Germany[4], and the European Researchers in Mathematics Education (ERME) Topic Conference, INDRUM[5]. A significant element of this effort is to connect HE mathematics teachers across institutional and national boundaries, and to share good practice, innovation and research evidence. Contributions in this issue of Nomad reflect the pan-European nature of research and collaboration in the field.

To demonstrate the coherence and focus of what, superficially, may seem a rather eclectic set of papers we adopt community of practice theory (CPT) as a meta-level theoretical perspective. In CPT the operative terms that we apply are: *enterprise*, *engagement*, *repertoire*, *participation*, *belonging and identity* (Wenger, 1998). We define the enterprise as doing mathematics, this may be as a research mathematician, or as a necessary part of some other professional role – such as economist, engineer, teacher, health worker or scientist. Students join the enterprise as novices and the educational process is intended to support their inward trajectory to become full participants in their chosen profession. The repertoire includes the established mathematical discourse associated with the enterprise and the discourse that emerges through the educational process.

To accommodate goals of teaching and learning development, CPT is expanded to include individual's agency to change practice, and here the operant terms are *critical alignment* and *systematic inquiry* (research) (Biza, Jaworski & Hemmi, 2014). At HE, teaching, learning and research in university mathematics education is an arena of inquiry composed of three intersecting practices: students' practice, teachers' practice and mathematics education researchers' (MERs) practice. Mathematics teaching developmental projects, such as MatRIC, KHDM, and **sigma** act as brokers between the three practices. Explicitly or implicitly they share a vision that entails supporting students in the educational process to move along a trajectory from being peripheral participants (as receivers) to full active contributing participants and agents in their HE. Further the projects are agents of change and transformation within the educational process, uniting the practices of education and doing mathematics by promoting students as co-creators of knowledge and co-producers of education, and innovative R&D based education.

The intended outcomes expected to arise from developmental actions include teachers' participation in professional development events and activity meetings, events, small scale action research, teaching development trials, innovative teaching, and teachers changing their practice because of engaging within the networks. The production, dissemination and influence of reports arising from research and development activity promoted and facilitated by the developmental networks and meetings such as INDRUM and the CERME Thematic Working Group "University Mathematics Education"[6], in CPT terms, constitute a reification of developmental projects as brokers and agents of change.

The institutional context

Understandably, funders want to see results, meaning that the actions of developmental projects have an impact on resolving the problems that are endemic in higher education mathematics – poor performance, high failure, poor progression and high drop-out. However, the few larger scale studies that are available indicate that the major sources generating these problems are at structural and systemic levels rather than at the level of teaching and learning practices. For instance, it is widely accepted that one of the most significant factors that determines a student's progress is her/his prior knowledge. A study by Opstad, Bonesrønning and Fallan (2017) at Norway's largest university revealed that students in Norway who had chosen a more theoretical mathematics course at their upper secondary school had a systematic advantage over their peers with a weaker mathematics background. In the UK, the acclaimed Advanced Level General Certificate of Education courses in mathematics and associated examinations have been shown to promote a highly proceduralised education in mathematics which creates problems for students when they face the abstraction and formalism of university mathematics (Hawkes & Savage, 2000; Minards, 2012; Nardi, 2008). Students gaining entry to high profile mathematics departments in the UK are expected to attain grades A or A* in these examinations, although lower grades can be acceptable elsewhere. It is nevertheless true that students, widely, struggle with the mathematics they encounter in their university education. However, in Norway, at least for some study programmes containing mathematics the entry demand relates to overall performance from upper secondary school and does not make specific demands about the nature or level of attainment in mathematics.[7] For most programmes where mathematics is a central subject (e.g. Bachelor of mathematics, mathematics teacher education for upper secondary school, or engineering) there are certain requirements on the type of courses taken at upper secondary school although only very few require the most advanced level. Also, in Norway, the Mathematics Council surveys basic mathematical competencies of incoming undergraduate students who normally have at least 60 ETCS points of mathematics in their programme (Nortvedt & Bulien, 2016).[8] The survey has been carried out biennially for over three decades. The most recent results available, that is from the test completed in the autumn 2015, revealed that a little less than half of the nearly 5500 students from 19 Norwegian HE institutions could not score over 50 % on items reflecting the grades 1 to 10 school curriculum. The test does show some small gradual improvement over recent years, especially amongst the younger students, but the results over the first decade and a half of this century are persistently around this outcome: 50 % of incoming students cannot manage 50 % of the grade 10 mathematics curriculum. If substantial improvement in the national profile of students' performance in mathematics is to occur, one approach could be to impose higher demands on students' mathematical competence on entry. An alternative approach could be to adapt HE entry mathematics courses so that there is not such a wide gap between students' starting knowledge and that assumed by the course; if the same level of competence at graduation is expected, the course would need to be longer, or there would need to be more time given to mathematics. Often decisions about such changes lie outside even the scope of institutions, as they are set by government ministries of education and research or professional councils that regulate the education of entrants to the profession.

For many, if not all students, their first experience of mathematics in higher education can be something of a shock. Accustomed to school classes of 25 to 30, they often participate in lectures in auditoriums with 200–600 other students. We acknowledge many efforts to address the quality of mathematics education and to engage students actively within their mathematics education, at national levels such as the aforementioned **sigma** network in the UK, KHDM in Germany and MatRIC in Norway, also at institutional levels such as projects at the Norwegian University of Science and Technology, Norway's largest university, and the Mathematics Education Centre at Loughborough University in the UK. Also, we note international efforts to improve higher education in general, such as the European University Association project "European Forum for Enhanced Collaboration in Teaching"[9]. When mathematics is taught as a service subject for other programmes such as engineering or sciences, the mathematics is taught often as a generic course with few if any applications to the students' chosen programme of study. Further, there is an embedded widespread culture of teaching mathematics in higher education. The London Mathematical Society (LMS, 2010), while recognising the advances in technology that influence teaching and learning mathematics, sets out an argument for (chalk) board based teaching:

> The lecturer must be able to create, to write out, during the lecture itself, a large body of argument [...] To deliver this style of lecturing the lecturer will require several boards, with large total area, on which writing is clearly visible to all of the students. In large lecture theatres especially, this may require chalkboards. (p. 3)

The LMS report concludes:

> [...] when used in conjunction with the technological and other developments associated with learning mathematics and for reasons outlined above, lecture boards remain an important technology for teaching mathematics in an exciting and interactive way, leading to a good understanding of the subject. (p. 4)

Through the lens of community of practice theory it is reasonable to suggest that although mathematics teachers seek to demonstrate the way mathematics is developed in practice, the mathematics demonstrated is that which, in Brousseau's terms (Brousseau, 1997), is the product after an institutionalization process. It does not demonstrate the way that mathematicians, or mathematical modellers encounter, engage with and resolve problems. It only shows the way their solutions, once found are presented in a rather refined manner to their community. The practice of the mathematics lecture is far removed from the mathematics as variously practiced by research mathematicians, engineers or scientists.

In addition to the large number of students attending a lecture, a remote lecturer at the board at the front, and an understandable intimidation against asking questions, the large group of students creates other educational disadvantages. Assessment and feedback, widely recognised as having a significant effect on students' performance (e.g. Black & Wiliam, 1998; Hattie, 2009) becomes more difficult, and often checking assignments and small group work (these may still be as large as 50 students) is often devolved to student or graduate teaching assistants who have very little pedagogical or didactical education. This is hardly a conducive context in which to learn and the results of a meta-analysis of 225 studies by Freeman et al. (2014) that pointed to

> Students in [STEM] classes with traditional lecturing were 1.5 times more likely to fail than students with active learning. [...] results hold across the STEM disciplines, that active learning increases scores on concept inventories more than on course examinations, and that active learning appears effective across all class sizes-although the greatest effects are in small ($n \leq 50$) classes. (p. 8410)

However, literature in mathematics education describing "active learning", in terms of innovations in courses and inquiry-based teaching-learning activity, has often consisted of non-research-based reports from enthusiastic practitioners. So such studies, while suggesting the improvement of learning and students' responses to the teaching experienced, have not provided research evidence for their claims. Treffert-Thomas and Jaworski (2016) describe these reports as *professional* or *pedagogic*, rather than research reports.

The culture of educational provision also appears to have an effect on students' attitudes towards learning. In Norway, and many other countries, class attendance is not mandatory. It is argued that in higher education students are adults, they should be allowed their independence to choose for themselves the most educationally effective use of their time. The advent of streamed lectures also opens the possibility for students to study when most convenient for them, often working around a paid job. A large national survey of mathematics students on STEM programmes in Norway was carried out by the Norwegian Association of Higher Education Institutions in 2013 (UHR, 2014). Nearly 3000 students from 19 HE institutions responded, about 40 % of the student respondents attended one of the two largest most prestigious universities in Norway. About 13 % of the respondents admitted that they attended lectures for 50 % of the time or less, and around 40 % responded that they did not attend other types of class (problem solving, exercises). It could be perceived that if attendance at lectures and other classes is not mandatory, then attendance is optional, and maybe unimportant, rather than a recognition of students' responsibility for their own learning and their right to choose how they engage. It may also be suggested that *some* students are not well prepared to be peripheral participants (Lave & Wenger, 1991) in the practice, and show little motivation to engage in the processes that will ensure their learning towards full participation.

Returning to Artigue's overview, she draws attention to a weakness of research in the field, which is the preponderance of small scale studies. The foregoing overview of systemic, structural and cultural issues, appears to point rather convincingly at fundamental problems with mathematical education in HE. Nevertheless, the research reported seems to have little impact upon policy and practice. There certainly is a need to make a greater investment in large scale studies that will have an impact at policy, institution and programme levels. The absence of large scale studies in the field could be the result of funding policies that favour research on school mathematics education, or that larger scale studies would require collaboration between research institutions that are more accustomed to competing for limited funds. The papers included in this Nomad TI fall within the descriptor "small scale", but they are best understood within the institutional framework critiques above and they contribute to understanding better the constraints on learning that students experience.

Contribution of the studies reported in this NOMAD TI

Artigue writes about the "'schizophrenia' which is rather common to those who teach undergraduate courses: the complete disconnection between what is their lived experience as mathematicians and their lived experience as university teachers" (p. 13). A somewhat nuanced interpretation may be necessary in view of the survey responses reported in the paper by Treffert-Thomas, Viirman, Hernandez-Martinez, and Rogovchenko (this issue), in which they study university mathematics teachers' experience of mathematical modelling and the use and attitude towards modelling in their teaching. Treffert-Thomas et al. observe, "One of the few significant differences found between groups of respondents was the fact that respondents who used MM in research were significantly more likely to use it also in their teaching." All of the papers in this TI are authored by academics and researchers who are teaching mathematics, some within pure mathematics programmes, and some to future scientists, engineers or teachers. All of them are concerned to inquire into the teaching of mathematics and to create favourable and better conditions for students' learning. The papers focus on the quality of learning (Breen, Larson, O'Shea & Pettersson), the characteristics of teaching at university (Mali & Petropoulou) and students' perceptions of "good" teaching (Asikainen, Viholainen, Kopomen & Hirvonen). The potential for mathematics teachers "schizophrenia" is explored (Treffert-Thomas et al.) with a view to introducing mathematical modelling, that is authentic mathematics practices, into the classroom. Similarly engaging with authentic mathematics is at stake in the paper that considers school teachers' interpretation of indices in a continuing professional development context (Kacerja, Rangnes, Herheim, Pohl, Lilland & Hansen). The final papers focus on students as active partners in the educational process, in teaching (Naalsund & Skogholt) and assessment (Jones & Sirl)

Artigue sets out four challenges to the research field at the end of her INDRUM paper (pp. 22–23):

1. How can we maintain some connection between the living field of mathematics, so dynamic and diverse, and undergraduate mathematics education, both in terms of content and practice?

2. How can we make our students really experience the subtle and original combination mathematics currently offers of experimental and deductive games, thanks to the evolution of technology?

3. How can we address the dramatic changes that the technological evolution more generally induces in the ways we and our students access information and resources, learn, communicate, interact, work and produce with others?

4. And, finally, how can we make our students consider mathematics as a resource for thinking about this fast moving world, questioning it, and trying to make it a bit better?

The papers in this issue address the first and fourth of the challenges identified by Artigue and demonstrate how the vision and goals of mathematics teaching developmental activity may be attained. However, none of them address the use of technology, which is surprising given the emphasis on the innovative use of technology that lies at the heart of much of the innovative action in university level mathematics teaching, such as use of video and so-called flipped classroom approaches, computer aided assessment, and digital visualisation and simulation. Moreover, the papers do not reflect significant changes in mathematics practice such as the advent of computational mathematics.

As noted above, the papers in this issue of Nomad may appear more eclectic than ”thematic” as intended. Nevertheless, it is possible to draw attention to two broad and related themes – teaching approaches and learning of specific mathematical content. Teaching approaches can be seen as ”regular practices” or as ”innovative practices”. We consider these in the light of the foregoing discussion.

Teaching approaches – regular practice

Three of the papers consider how mathematics is taught in regular practice, that is without attempts to introduce innovations. Mali and Petropoulou (this issue) report from a study of university mathematics teachers’ practices in Greece and the UK, and in lectures and tutorials. Analyses of two short episodes, one from each setting (a lecture in Greece and a tutorial in the UK), are used to develop an argument for a common ground of university mathematics teaching across settings and approaches. Teachers’ actions within the lecture or tutorial room are characterised as selecting appropriate tasks and challenges, explaining, extending ideas to make connections with other mathematics and concepts, and evaluating students’ responses. The paper draws attention to the complexity of teaching higher mathematics concepts, the need for deep subject knowledge, and a sense of how that knowledge might be presented to enable learning. The complexity at higher education also includes teaching and learning mathematics in several types of setting, the research reported indicates that teachers' actions across settings and across countries are fairly uniform.

The paper by Asikainen et al. reports research that attempts to see teaching through students' eyes. Students are asked two open questions, about the knowledge teachers need and about characteristics of good teaching. The study also explores whether teacher education students and future STEM subject specialists have different views. The students involved in the study are in their first year at university; the paper thus reflects views that have been formed through years of school education more than their experience at the university. Such students have been studied in the *Transmath* study in the UK, which reports on research that studies the experiences of students in transition from the last years of schooling to first year university studies. Transmath researchers characterise much of the teaching at these levels as "transmissionist", leaving students dependent on the teachers rather than as thinking individuals in their own right (Williams, 2016). Our interest here is particularly in characteristics of teaching that students do not write about. They do not say that good teaching is characterised by challenging questions and tasks given to students, they prefer their teachers to be patient, clear, inspiring, consistent enthusiastic, encouraging and helpful. Of course, the teachers who also expect students to engage with cognitively demanding and challenging tasks that require deep thinking and understanding may also have all of the characteristics that students identify as "good". Further the students' responses do not indicate that good teachers are those who expect students to be partners in the educational process; although it is obvious that if students have no experience of such approaches, they would not be likely to mention them.

Students' expectations are very influential in teaching development. The experienced teacher may be accomplished in the approaches identified by Mali and Petropoulou (this issue), selecting, explaining, extending and evaluating. These same teachers may have the knowledge, skills and competencies that students have learned to expect and appreciate through many years of mathematics education. When the teacher introduces some innovation, she/he is taking a risk. Challenging students with cognitively demanding tasks causes discomfort for the students and they will probably react negatively. Taking students into partnership in teaching and assessing may cause students to feel they are not receiving the expert teachers' attention. The innovation may not be implemented in a proficient manner, and indeed the teacher may not be aware of all the features of the innovation. For example, the introduction of student response systems gives the teacher of large classes the opportunity to elicit feedback, and the first implementation of such a system may stop at this point. However, the response system also opens the opportunity for the teacher to ask open questions and invite students to discuss in pairs or small groups prior to giving their responses. Inept or naïve innovation may discourage a teacher from persisting and perfecting a new approach.

The discussion above has already touched on the disconnection between teachers' lives as mathematician researchers and mathematics teachers. A feature of excellent practice in higher education is claimed to be teaching and learning that includes research and development within the field. Most mathematics researchers would claim that their research is beyond the understanding of most undergraduate students. The application of mathematics to real or authentic problems need not be out of reach, and the mathematics of industry and commerce is about the solution of such problems – through mathematical modelling. The inclusion of mathematical modelling in undergraduate studies can be educationally beneficial, contribute to working life beyond the university and motivate interest. Treffert-Thomas et al. (this issue) report their survey research into mathematicians' experience of mathematical modelling and their inclusion of mathematics modelling within their teaching. They found amongst Norwegian mathematics teachers who responded that significantly more teachers who used mathematical modelling in their research were likely to include mathematical modelling in their teaching. The challenge facing the pure mathematician to introduce learning opportunities that go beyond the reproduction of institutionalised mathematics is addressed in the paper by Kacerja et al. (this issue) that we consider in the next section.

Teaching approaches – innovative practices

Exploration of the present character of higher education mathematics teaching, as outlined in the preceding paragraphs is important because it is too easy to make unfounded and erroneous generalisations on limited evidence. There is a need to survey the characteristics of mathematics education at both national and international levels that relates teaching approaches to learning outcomes in terms of students' understanding, not just performance in examinations that test, predominantly procedural competencies. The replacement of so called traditional approaches to teaching with innovative approaches must only occur because presumed weaknesses in the former are substantiated by trustworthy evidence, and that the innovative approaches are effective as claimed. The development of teaching and learning mathematics, for future mathematicians, mathematics teachers, engineers, economists, etc. must be based on evidence that supports claims of effectiveness. Studies of innovation are necessary to illuminate the potential for development. In this NOMAD TI there are three reports of small scale innovative studies, all deserve attention.

A common theme in the three papers (Kacerja et al.; Naalsund & Skogholt; Jones & Sirl) is the more active involvement of students in the educational process, teaching, learning and assessment. The participants in the study reported by Kacerja et al. (this issue) are primary school teachers following a professional development programme. The mathematical education of teachers is a highly significant issue in the development of teaching and learning mathematics in higher education, as was asserted above, a major factor in the determination of students' success in mathematics in higher education is the knowledge and understanding they bring with them. Kacerja et al. expect their participants to engage critically with numerical data and indices, the outcome of mathematical procedures, and not just passive or submissive receivers of knowledge. Participants are organised into discussion groups in which they are presented with information that challenges the orthodox interpretation of, in this case the "body mass index". We contrast this approach with the conventional teacher at the chalkboard presenting refined mathematical statements and arguments that students are expected to accept, rather than challenge. We want to emphasise that a critical disposition towards mathematics is necessary at all levels.

Naalsund and Skogholt (this issue), from a study of students' engagement with a proof-based course in real analysis that many find challenging, report an innovative approach that aims to stimulate students' critical engagement, and the development of their "metacognitive regulation" through discussion and peer mentoring. Working on mathematics with the aim of satisfying one's own understanding, and working so that one can explain the mathematics to another stimulate different cognitive challenges and levels of engagement. The study reports from interviews with students who are asked about their experiences of learning, in their preparation, through their presentation and from the feedback provided by the tutor. The report points to this being an effective and, it appears, largely positive experience for the students, but there is no inclusion of any data generated by questions relating to students' affective response. From the CPT perspective that frames this paper one can assert that this approach that places the students in the teaching role also is closer to the practices of mathematician, mathematics teacher and mathematics user than experienced in regular lectures and tutorials such as those analysed by Mali and Petropoulou (this Issue).

Students can also be engaged in the assessment process. It is reported that students' self-assessment benefits learning: "The primary purposes of engaging students in careful self-assessment are to boost learning and achievement, and to promote academic self-regulation, or the tendency to monitor and manage one's own learning" (Andrade & Valtcheva, 2009, p. 13).

Another approach to engaging students in the assessment process is peer assessment, as reported by Jones and Sirl (this issue). Their approach requires only that students rank their peers' work by making multiple decisions between pairs of their peers' responses to a task. Pairs of responses are randomly assigned to students in the group and software is used to collate students' judgments on comparative merit (better/worse) of the pairs to produce a combined ranked list. The study included 132 students. In contrast to the claims about the educational value of self-assessment Jones and Sirl acknowledge a "commonly expressed concern [...] that students receive no written feedback." They argue, however, "that the judging process engages students with meaningful comparisons of the quality of answers, and thereby provides a novel and beneficial form of feedback about their own performance." In situations faced by many HE mathematics teachers with classes of several hundred students, approaches to educationally meaningful assessment that can be efficiently implemented, with or without the use of modern technology, are desirable. A problem with regular assessment approaches used with very large groups of students is that procedural mathematics is favoured and in preparation for such assessments students often do not develop the deep understanding of the subject required for problem solving, modelling and further study.

Exploring students' learning specific topics

Returning to Artigue's paper, she identifies that a strength in the field of researching university mathematics education is "its move from investigation focusing on the student to a more balanced interest in both the student and the teacher" (2016, p. 19). This is evident in the foregoing six of the seven research reports in this TI that report teaching, either by describing regular practice, or describing some form of innovate practice that engages students more actively in the research process. Five of the papers also pay varying degrees of attention to the mathematics content within which the study was conducted – real analysis, proof, calculus, indices and modelling. However, they are not large studies and it is unsurprising that they do not also focus on students' learning, in particular the quality and characteristics of the learning gains achieved. The remaining report in this TI by Breen et al., directly addresses the nature of students' learning and it combines evidence from two separate studies, one conducted in Sweden, the other in the Republic of Ireland, that set out to explore the variety in students' concept images of a basic mathematical idea, inverse function. Many of the meanings students express about inverse functions, when they exist and how they might be explained and represented are not valid. A teacher can only assume that there are a variety of meanings held, and that a sizable proportion of students hold erroneous meanings. Given the significance of prior knowledge when learning something new, the teacher has an extremely challenging task. The paper by Breen et al. also draws attention to the variety of approaches that might be used to expose students' meanings. The authors recognise that the different methods might be the underlying reason for some of the differences between the Irish and Swedish students whose meanings of inverse function have been explored.

Here, we come up against the notion of ”mathematical correctness”, and the mathematical tradition that puts meaning on such correctness. It is (relatively) easy, for example, to learn a procedure and to apply it correctly when the problem in which to apply it fits exactly the conditions within which it was taught. Thus, we set questions in upper secondary school and university exams that require students to be correct in using the procedures and formulae they have been presented with in the mode in which they were presented. Inverse functions form a concept, not a procedure. There may be procedures in the ways in which we work with inverses, and therefore correct answers associated with this. An incorrect answer here, however, is not just wrong, it reveals a potential depth of mis-understanding that cannot be corrected by correcting the procedure. The big issue is how to tackle the depth of this problem to foster an in-depth understanding of inverse functions.

Discussion and conclusion

As Michèle Artigue has shown, exploring teaching and learning of mathematics in higher education is a complex endeavour with multiple facets. The small collection of papers here makes this clear. We could imagine a special issue based around the principle themes of each of these papers. None of these papers reports large scale research. Papers which come from studies conducted by the teachers who design courses and work with students to promote mathematics learning are by their nature small scale. A central aim of these studies is to involve students in ways which foster engagement with and understanding of mathematics. They are intensive in their fostering of relationships within the teaching-learning community and their focus is these special relationships in innovative practice. This draws attention to a major challenge in our research field, there are many ideas, strategies and actions that might be implemented within the educational arena, but exploring the long-term effect of interventions is very difficult. The *Transmath* project, to which we refer above, is an example of a project looking at teaching and learning on a larger scale: it worked with considerable external funding and a large research team. We cannot expect that one or two teacher-researchers, researching a small intervention in practice, will be able to undertake a study of this scale. However, such small scale projects are illustrative of possibilities in educational practice and start to point towards issues and outcomes that can be explored further. We need both kinds of projects.

Another issue here is that of theory. If there were to be an agreed global theoretical framework for learning and teaching at this level, interventions could be argued analytically in terms of the theory. However, innovation studies are often based (not unreasonably) on fairly local theories – that have their roots in quite different global theories. As we mentioned above, the theoretical perspectives employed here do not overlap with those in the earlier special issue of *Research in Mathematics Education* in this area in Nardi et al. (2014). So, perhaps we can suggest that a problem here lies in the diversity of theories and the problem of connecting research findings that have a different theoretical basis.

Finally, we believe there is another important role of the small scale studies. This is related to the teaching-learning practice of which they are a part. Since all research is designed to provide/enhance knowledge in the field, we should acknowledge that knowledge-in-practice is important. As teachers conduct research into the practices in which they engage, they come to know their practice better and have the opportunity to develop the ways they work with students. Where students can become involved in the research, these opportunities can extend also to the students. We are especially interested in seeing more studies that engage students in innovation research.

So, to sum up, the studies reported here, all of which are small-scale offer insights into learning and teaching which can be explored further by others. Such studies also develop important new knowledge-in-practice which can inform the development of teaching to support improved learning. Nevertheless, we recognise that larger scale projects are necessary to gain wider understandings of teaching and learning at this level. In such cases, funding is required to make the larger scale possible. Theory is central to what is reported. As long as we have considerable diversity of theoretical perspectives, it will be hard to draw conclusions across research studies.

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Notes

1 http://www.lboro.ac.uk/departments/mec/

2 MatRIC is based at the University of Agder with collaborating partners: Norwegian University of Science and Technology, Norwegian University of Life Sciences and the Norwegian Centre for Mathematics Education (NSMO). MatRIC focuses on mathematics teaching and learning in higher education and NSMO focuses at school level.

3 http://www.sigma-network.ac.uk/about/the-sigma-network/

4 https://www.khdm.de/en/

5 https://indrum2018.sciencesconf.org/

6 http://cerme10.org/scientific-activities/twg-teams/

7 For some programmes, such as teacher education for compulsory school, a national minimum grade is specified, but no requirement on the type of course (courses vary in the extent of higher abstract mathematics, such as calculus, that is included).

8 One exception to the minimum of 60 ECTS points of mathematics is the inclusion of students preparing to be teachers in elementary school (grades 1 to 7) who may have only 30 ECTS points of mathematics in their study programme. However, as noted above, these students are expected to have achieved a minimum level of mathematics performance to be accepted into the programme.

9 See http://www.eua.be/activities-services/projects/current-projects/higher-education-policy/effect

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