Finding Erik and Alva: uncovering students who reason additively when multiplying

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This article presents a study in which grade 5 students' responses to multiplicative comparison problems, a well-known method for distinguishing additive reasoning from multiplicative, are compared to their reasoning when calculating uncontextualised multiplicative tasks. Despite recognising the multiplicative structure of multiplicative comparison problems a significant proportion of students calculated multiplicative problems additively. Therefore, multiplicative comparison problems are insufficient on their own as indicators of multiplicative reasoning.

Multiplicative reasoning, which is described below, involves recognising structures in multiplicative situations, handling transformations of quantities and coordinating composite units (Schwartz, 1988; Sowder et al., 1998; Van Dooren, De Bock & Verschaffel, 2010; Vergnaud, 1983, 1994). It is distinctly different from additive reasoning, develops slowly (Clark & Kamii, 1996; Thompson & Saldanha, 2003) and underpins not only our enumeration system (Chandler & Kamii, 2009; Nunes et al., 2009; Steffe, 1994) but key topics such as proportionality, functions and fractions (Empson, Junk, Dominguez & Turner, 2006; Sowder et al., 1998; Vergnaud, 1994). A well-known method distinguishes additive reasoning students from multiplicative reasoning students by their ability to solve multiplicative comparison problems. Since this method does not take calculations into consideration there is a risk that it may fail to identify students who approach multiplicative calculations by means of additive reasoning. This article describes a study where this issue was investigated. In the following, before presenting the study, the nature of multiplicative reasoning is described, followed by a brief review of what the literature says with respect to the development of students' multiplicative reasoning and its evaluation.

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The nature of multiplicative reasoning

Repeated addition is typically construed as "the primitive model associated with multiplication" and resistant to change (Fischbein, Deir, Nello & Marino, 1985, p. 6). Even though multiplication is typically introduced by repeated addition, and there are connections between additive and multiplicative reasoning, multiplication and addition belong to different conceptual fields (Bakker, van den Heuvel-Panhuizen & Robitzsch, 2014; Vergnaud, 1983, 1994, 2009). Multiplicative reasoning – in contrast to additive reasoning – does not develop without instruction (Sowder et al., 1998) and requires a conceptual leap on the part of the learner (Chandler & Kamii, 2009; Simon, 2006; Steffe, 1992, 1994; Tzur et al., 2013). A conception of multiplication as repeated addition is insufficient when multiplication is applied to fractions; to repeatedly add a fraction a fractional number of times, as in $1/3 \cdot 2/5$, is hard to conceptualize (Fischbein et al., 1985; Greer, 1992; Simon, 2006; Sowder et al., 1998; Thompson & Saldanha, 2003).

One way to understand the conceptual differences between addition and multiplication is to consider the quantities that are involved; there is more than one quantity involved in multiplication but not in addition (Vergnaud, 1983). In the example *Sofia bought 3 apples, each apple costs 5 kronor, she paid 15 kronor*, there are three different quantities, apples, kronor/apple and kronor. In an area problem there are two quantities, units of length and units of area. In contrast, addition invokes just one quantity (Barmby, Harries, Higgins & Suggate, 2009), as in, *Sofia had 3 apples and bought 5 more apples*. Thus, multiplication is referent transforming and addition is referent preserving (Schwartz, 1988). Schwartz emphasises the transformation of quantities as the basis for understanding why multiplication is more than repeated addition.

Another way to understand the difference between multiplicative and additive reasoning is that while addition deals with single, nested units at the same level of abstraction (Clark & Kamii, 1996; Steffe, 1992), multiplication deals with composite units on several levels of abstraction, also called higher order numbers (Chandler & Kamii, 2009). Consider an example where Martin puts his marbles in six boxes with twelve marbles in each box. The ability to simultaneously look at a box representing both one box and twelve marbles is a key aspect of a child's construct of multiplicative thinking (Simon, 2006; Steffe, 1992). Indeed, an understanding that an increase of one in the multiplier implies an increase in the product equal to the magnitude of the multiplicand – $3 \cdot 6$ increased to $4 \cdot 6$ is *six* more not one more – is the foundation for multiplicative operations and the conceptual basis for distributivity (Tzur et al., 2013).

These distinctions between multiplication and addition are typically based on how we, as adults and mathematics educators, view arithmetical situations. Such perspectives, which necessarily differ from how students reason in multiplicative situations, provide tools for helping us infer students' understanding of multiplication from their articulated solutions to multiplicative problems (Tzur et al., 2013). The brief description of multiplicative comparison below is also a construct that we as researchers employ as a tool. It is not to be read as equivalent to how students think when they work on contextually-presented multiplicative tasks (Greer, 1992).

Various classifications of multiplicative situations have been presented in the literature (e.g. Greer, 1992; Mulligan & Mitchelmore, 1997), one of which, multiplicative comparison situations, is of importance to this article. A multiplicative comparison situation would be Max has three times as much money as Mollie. Such situations are perceived differently compared to equal group situations, such as Martin's marbles above, and generally considered harder for learners (Thompson & Saldanha, 2003). Multiplicative comparison is not only the foundation for proportionality but the "the cornerstone of all [mathematics] that is to follow" (Lesh, Post & Behr, 1988, p. 94). Multiplicative reasoning more generally refers to the ability to reason with composite units, simultaneously perceiving the parts and the whole (Clark & Kamii, 1996; Steffe, 1992; Tzur et al., 2013).

Evaluation of students' multiplicative reasoning

From the literature we know that multiplicative comparison problems are effective in distinguishing between those students who reason additively and those who reason multiplicatively (e.g. Clark & Kamii, 1996; Van Dooren et al., 2010). In Clark and Kamii's (1996) study students' reasoning was categorised into four levels:

- Level I, not yet numerical thinking
- Level II, additive thinking where the student adds one or two more irrespective of what numbers are involved
- Level III, additive thinking where the student adds the number that should have been multiplied
- Level IV, multiplicative thinking, multiplying with the correct number

Both the additive levels involve the child adding (or subtracting) a number to the given amount instead of multiplying (or dividing). These studies conclude that it can be inferred from students' answers whether their reasoning is additive or multiplicative, given that problems are formulated so that the answers are discernibly different. For example, with respect to the problem, what number is three times as much as 50, an additive answer of $50 + 3 = 53$ is discernibly different from a multiplicative answer of $3 \cdot 50 = 150$. Multiplicative thinking in Clark and Kamii's study was divided into two sublevels differentiated by immediate or not immediate success. If a child gave an additive answer he or she was shown an answer that another child had correctly given and asked to evaluate it. Children who changed to the correct multiplicative answer were categorised as multiplicative without immediate success. Their study was conducted with numbers within the multiplication table, which meant that they could not distinguish repeated addition in their analysis since students may have used memorised number facts.

Van Dooren and colleagues (e.g. Fernandez et al., 2012; Van Dooren, De Bock, Evers & Verschaffel, 2009; Van Dooren et al., 2010) have conducted a number of studies investigating students' reasoning to proportional and non-proportional problems. These studies show that students tend to reason additively to proportional problems in early grades, as found by Clark and Kamii (1996). However, when they learn about proportional reasoning they tend to apply it ”everywhere”, even to additive problems. Their research shows that students' reasoning depends heavily on numbers and ratios in the problems, where ”easy numbers” (e.g. numbers within the multiplication table) and whole number ratios elicit multiplicative reasoning irrespective of the situation (Van Dooren et al., 2009; Van Dooren et al., 2010).

Clark and Kamii (1996) conducted clinical interviews and Van Dooren and colleagues (2010) gave students written tests. Both studies comprised multiplicative comparison problems and focussed on students' answers independently of how they undertook any calculations. A solution to three times as much as fifty calculated as $50 + 50 + 50$ reflects an understanding of the multiplicative character of the problem even though it

> is clear that this approach for solving missing-value proportionality problems is based on the repeated-addition character of multiplication, and therefore has characteristics of additive reasoning. Nevertheless, we categorize it as multiplicative, as it appropriately handles the multiplicative character of the problem situation.
>
> (Van Dooren et al., 2010, p. 363).

Even though repeated addition can be viewed as a ”primitive model” for multiplication (Fischbein et al., 1985) or as multiplication with ”characteristics of additive reasoning” (Van Dooren et al., 2010) there is a consensus that repeated addition is insufficient when numbers are large or rational. Therefore, this study sets out to investigate the sufficiency of multiplicative comparison problems for uncovering students' multiplicative reasoning.

Method

In this section I describe the tasks, how they were constructed and administered, frameworks for analysis, as well as rationales for my choices. The first task, a written test with three items, was constructed according to the method for distinguishing multiplicative reasoning from additive by solving multiplicative comparison problems. The second task, two calculation items, was constructed to map students' reasoning when calculating. But first I present the participants and why they were chosen for this study.

Participants

The study was conducted with 22 fifth grade students from two different classes. Fifth grade students were chosen since they will have experienced many of the mathematical topics included in the multiplicative conceptual field, including multiplication, ratio, fractions and simple proportion. Despite these experiences, we know that many students at this age have not yet become multiplicative reasoners (Clark & Kamii, 1996; Fernandez et al., 2012). Therefore, it is reasonable to assume that they will have reached different stages in their development of multiplicative reasoning.

The students came from diverse ethnic, economic, and social backgrounds. Among them were several with Swedish as a second language and a few with diagnoses relating to, for example, dyslexia or ADHD. This diversity should not be construed as representative of the population, although it may enhance the opportunity for uncovering different levels of reasoning. All student names are pseudonyms.

Task 1 – Multiplicative comparison problems

The first task was a written test of ten word problems reflecting different multiplicative situations, given for another purpose. Three of the items were multiplicative comparison problems and the students' answers to them alerted me to the issue of multiplicative comparison and students' ability to reason multiplicatively. In this article I draw on these three problems (see table 1) since they are well connected to the research on students' multiplicative reasoning while the other problems are not. Thus, I refer to these three items as task 1 throughout the following. The rationale for using this task was to distinguish students who reason additively from those who reason multiplicatively by their answers in a similar manner as earlier research has demonstrated fruitful (Clark & Kamii, 1996; Van Dooren et al., 2010).

The problems were formulated in simple language and contained all necessary but no superfluous information. Moreover, according to their teachers, all students had had a rich experience of this type of word problem. All numbers in the problems were chosen to be "easy", typically multiples of 50. This was to help keep students' attention on the situation and not the calculation (Tzur et al., 2013). The use of easy numbers should elicit multiplicative reasoning (Van Dooren et al., 2010).

The task was given in whole class settings during ordinary mathematics lessons. The oral instructions stressed that it was not the answer itself that was of importance, but the way the students reasoned and calculated to get the answer. An example of an additive situation (a comparison as subtraction) was presented and students were invited to offer suggestions as to how to solve it. Their suggestions were written on the board to model how their reasoning could be written.

In order to avoid test fatigue, the ten problems (including the other multiplicative situations) were divided into two sets of five and given to all 22 students during separate lessons a week apart. No time limit was given and no student needed more than 20 minutes to complete each set of five questions. Students who wished could have the texts read aloud to them either by the author of this paper or their regular mathematics teacher.

Table 1. *The multiplicative comparison problems*

Item	Type of situation	The word problems
1	Comparison as multiplication	Sofia has 50 kronor. Martin has 3 times as much money as Sofia. How much money has Martin got?
2	Comparison as division, rate missing	Sofia has 50 kronor. Martin has 150 kronor. How many times as much money has Martin?
3	Comparison as division, rate given	Max has 150 kronor. This is 3 times as much money as Mollie. How much money has Mollie?

Framework for analysing task 1

Drawing on the earlier work of Van Dooren et al. (2010), students' answers to the three word problems were categorised as reflecting additive reasoning, multiplicative reasoning or other/non analysable reasoning. Below are shown examples, all from item 3, to illustrate typical solutions for each category.

Additive reasoning

Here the student (figure 1) showed that the meaning of t*hree times as much* was not understood as multiplicative, but as additive, as *three more*.

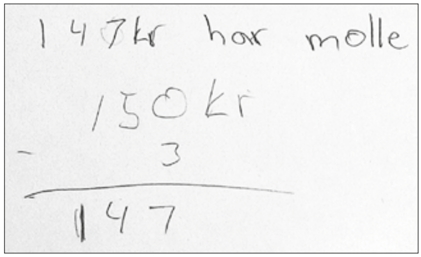


Figure 1. *Additive reasoning*

Multiplicative reasoning

In this category both multiplicative reasoning written by multiplicative operations (left) and multiplicative reasoning written by additive operations (right) was accepted (see figure 2) in line with Van Dooren et al. (2010).

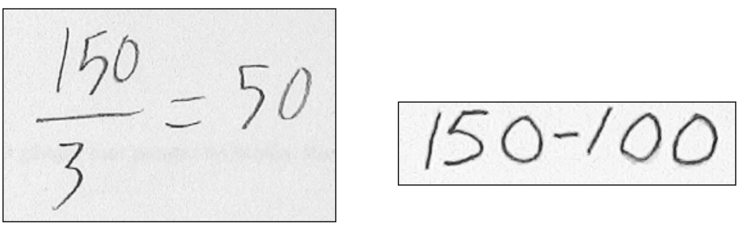


Figure 2. *Multiplicative reasoning written by multiplicative and additive operations respectively.*

Other/no answer

Here were found, see figure 3, solutions that could not be categorised as reflecting either additive or multiplicative reasoning. Items left without an answer were included in this category.

The next step, in order to investigate whether solving multiplicative comparison problems is sufficient to uncover students' reasoning as additive or multiplicative, was to investigate how they undertook multiplicative calculations. Students' use of the distributive property can reflect that the students can handle the numbers as higher order numbers while additive reasoning can be reflected by repeated addition.

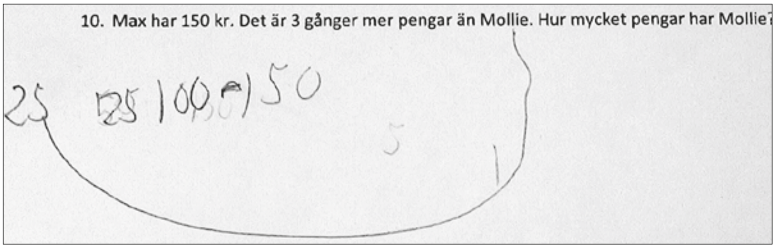


Figure 3. O*ther written solution.*

Task 2 – Calculations

Students were asked, during individual task-based interviews, to calculate $5 \cdot 19$ and $16 \cdot 25$, both written horizontally. The first, as multiplication of a single digit by a multi-digit number had been part of their instruction, was chosen for its potential for eliciting different strategies, since 19 is almost 20 and 5 is half of ten, in the manner of the study by Heirdsfield, Cooper, Mulligan and Irons (1999). The second, since students had not been instructed in the multiplication of two multi-digit numbers, was chosen to explore what strategies they would exploit in solving it. Importantly, 16 and 25 can be partitioned in several ways; for instance, using distributivity to underpin the calculation $10 \cdot 25 + 6 \cdot 25$ or associativity to transform the calculation to $8 \cdot 50$ or $4 \cdot 100$, in the manner reported by Foxman and Beishuizen (2002).

Interviews made it possible to probe students about how and why their strategies worked, as well as questions about the generality of those strategies and their properties. The students wrote with a *smartpen*, a device that captures both writing and audio and has the additional facility of being able to replay earlier writing exactly as it occurred.

Framework for analysing task 2

Students' strategies were categorised according to whether or not they reflected characteristics of additive reasoning. Strategies building on repeated addition or handling numbers as in addition, operating on tens and ones separately, were categorised as displaying characteristics of additive reasoning. Strategies employing implicit use of distributive or associative properties and handling numbers as whole entities were categorised as multiplicative, since they demonstrated an ability to handle composite numbers on a higher level of abstraction. Examples of different types of strategies are given for both categories by use of different students' work, see figure 4–7. Some calculations remained undefined in respect to the students' reasoning, and examples are given in figure 8.

Strategies with characteristics of additive reasoning

Repeated addition, in which one of the numbers was repeatedly added, was employed for both items, in figure 4 is an example for $16 \cdot 25$. By repeatedly adding a number the student did not demonstrate multiplicative reasoning.

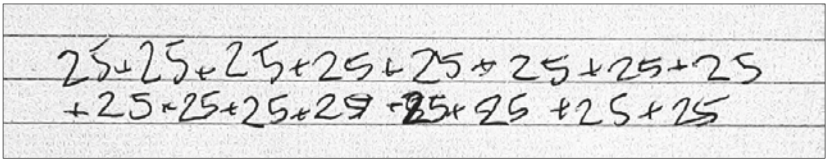


Figure 4. *Repeated addition*

A number of students performed addition-influenced algorithms where the numbers were put in a vertical algorithm and handled as in addition, operating only within columns, see figure 5. Both these strategies reflect that the students multiplied five and six, correctly got 30, multiplied 1 and 2 and correctly got 2. In the left example the student took the 3 tens from 30 and put on top in the tens column and finally added 3 to 2. In the right-hand example the student wrote both 30 and 2 under the line without transferring the 3 tens. Both strategies demonstrated knowledge of multiplication but also reflected influence from the standard algorithm for addition by the work within columns. In the interviews students justified their addition-influenced strategies as correct by statements such as "you take this times this [ones by ones] and add to this times this [tens by tens]" and "you do not need to take all of the numbers right away, you can take it in parts" demonstrating that they treated the numbers as in addition.

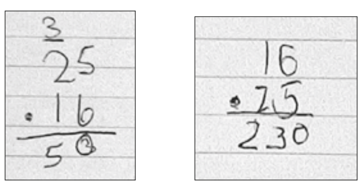


Figure 5. *Addition-influenced algorithm*

The last example of a strategy with characteristics of additive reasoning was the following: "I split 16 into 10 and 6. First I take $10 \cdot 25$, then I have the 6 left, [...] which I add". This demonstrated a partial understanding of the distributive law by splitting 16 and multiplying a part, but when the student decided to add 6 to 250 additive reasoning was demonstrated. The adding of a part of the multiplier revealed that the student did not fully understand the roles of the multiplier and the multiplicand.

Multiplicative reasoning strategies

A common strategy among the students who reasoned multiplicatively was to use the distributive law, see figure 6. When $5 \cdot 19$ was calculated correctly by use of the distributive law, it was either as $5 \cdot (10 + 9)$ or as $5 \cdot (20 - 1)$. In calculations of $16 \cdot 25$ students split either 16 or 25, not both of the numbers, and sometimes a number was split in more parts than two, such as $16 \cdot (10 + 10 + 5)$.

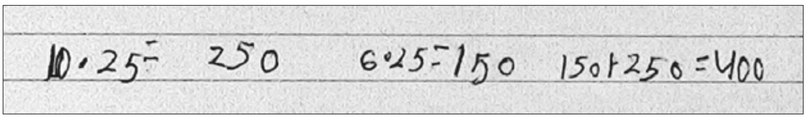


Figure 6. *Distributive law*

Another example of multiplicative reasoning strategies was to perform a successive doubling of the multiplicand where implicit use of the associative law was used, see figure 7. This strategy can also be perceived as using the fact that $16 = 2^4$. However, judging from student explanations of this strategy it seems, as it was an iterative doubling procedure while keeping track of how many times the multiplicand was used. This strategy was only used to $16 \cdot 25$.

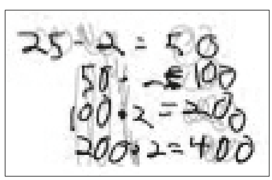


Figure 7. *Successive doubling*

Undefined reasoning in respect to additive or multiplicative reasoning

In this case a correctly executed vertical algorithm could not be categorised with respect to either additive or multiplicative reasoning. Here, as in figure 8, students were able to explain the algorithm procedurally, which is possible without ”any understanding of what actually is happening with the ones, tens, and hundreds” (Fuson, 2003, p. 85), but leaving it impossible to infer whether they used multiplicative reasoning or not while performing the algorithms. (The student who wrote the left algorithm could not erase the miswriting, but was clear about the answer being 95). No answer was also put in this category of undefined reasoning.

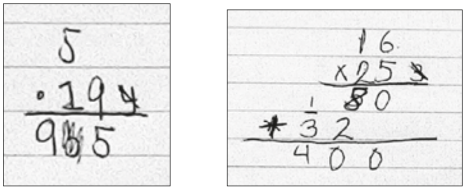


Figure 8. *Correct algorithms*

Results

In this section I present students' profiles from the two tasks, the word problems and the calculations. First I present how students were categorised by each of the tasks, then by a combination of the tasks. Finally two students' profiles are presented in detail as representatives of those students who were not identified as additive reasoners by their solutions to multiplicative comparison problems, but displayed characteristics of additive reasoning when calculating.

With respect to task 1, the eleven students who gave answers categorised as multiplicative reasoning to all three multiplicative comparison problems were considered to demonstrate multiplicative reasoning. There was no student who gave additive answers to all three items, but three students gave answers demonstrating additive reasoning to two of the three items. They were considered to reason mainly additively. Eight of the 22 students had given one answer reflecting additive and two reflecting multiplicative reasoning or one additive, one multiplicative and the third categorised as other, thus interchangeably demonstrating additive and multiplicative reasoning. This mixed reasoning was expected since the students were expected to be in a transitional stage.

With respect to task 2 and those calculation strategies thought to show either additive reasoning or multiplicative reasoning, students showed the same kind of reasoning in both calculations or a mix of additive, multiplicative and undefined reasoning. Eight students calculated both items by multiplicative strategies and seven students calculated both items by strategies with characteristics of addition. The remaining seven students employed a mix of strategies over the two items.

Each student was now categorised as reasoning multiplicatively, additively or with mixed reasoning by two different types of tasks. When these categorisations from the multiplicative comparison problems and the calculations were combined the students' profiles formed two groups: students who showed the same kind of reasoning and students who showed different kind of reasoning to the two tasks.

Students with the same kind of reasoning were identified as additive, multiplicative or mixed reasoners both by the multiplicative comparison problems, and by their calculations. These ten students (in cells A1, B2 and C3 in table 2) are not further discussed in this article since their reasoning was consistent over both types of task.

Table 2. *Number of students categorised as additive, multiplicative or mixed reasoners by the two tasks*

Multiplicative comparison problems	Calculations		
	1. Strategies with characteristics of addition	2. Mixed strategies	3. Multiplicative reasoning strategies
A. Additive reasoning	2	1	0
B. Mixed reasoning	4	2	2
C. Multiplicative reasoning	1	4	6

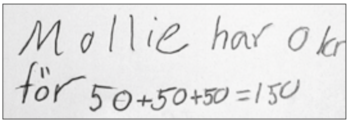
The student in cell A2, was identified as reasoning additively on task 1, but used mixed strategies on task 2 by performing a learned algorithm for $5 \cdot 19$ and repeated addition for $16 \cdot 25$, and is not further discussed here, since the learned algorithm does not let us know more about her reasoning with respect to her calculations. The two students in cell B3 were categorised as mixed reasoners by the word problems and as multiplicative reasoners by the calculation task. In task 1 they solved one item by additive reasoning and the other two correctly by multiplicative reasoning and in task 2 they used the distributive property to both items. These two students were considered as mainly multiplicative reasoners in spite of their additive reasoning to one item in task 1, since they mastered calculations by making use of the distributive property. To master distributivity with two two-digit numbers indicated that they could coordinate composite numbers and understand the role of the multiplier, hence demonstrating that they could reason multiplicatively. They are not further discussed.

The four students in cell C2 were categorised as multiplicative reasoners by the word problems and as mixed reasoners by the calculations. They solved all multiplicative comparison problems correctly and calculated 5 · 19 by the distributive law, but failed to do so for 16 · 25. Since they displayed multiplicative reasoning by a strategy that requires coordination of composite numbers they were considered to be well on their way to develop their reasoning to also comprise two two-digit numbers, even if they could not do that yet, hence they are not discussed further.

The four students in cell B1 were categorised as mixed reasoners by their answers to task 1 but employed calculation strategies with characteristics of additive reasoning to both items in task 2. These students and the student in cell C1, categorised as multiplicative reasoner by task 1 and who showed solely additive calculations to task 2, will be discussed in detail. In the following I present Erik as an arbitrary chosen representative for the four students in cell B1 and Alva who is the sole student in cell C1.

Erik

Erik represents the four students who were categorised as mixed reasoners on the word problems. Here Erik reasoned additively to one item, multiplicatively to one and multiplicatively with an incorrect additive conclusion to the third. Erik wrote his solutions to all items in task 1 by additive operations and in item 3 this led him to draw an incorrect conclusion, see figure 9. Item 3 was *Max has 150 kronor. This is 3 times as much money as Mollie. How much money has Mollie?* When Erik had used fifty three times to get 150 kronor he concluded that Mollie had zero kronor.



[Mollie has 0 kronor since 50 + 50 + 50 = 150]

Figure 9. *Erik's erroneous conclusion to item 3*

Both his calculations in task 2 were performed as repeated addition, see figure 10. He was clear about the possibility of adding five nineteen times or nineteen five times, hence showing awareness of the commutative property. Initially he decided to add five nineteens since "it is higher, then you don't need to take it as many times", see the left hand image of figure 10. First he took the tens from the nineteens and wrote 50. Then he added two nines and got eighteen, which he said quickly indicating that he knew the answer. When he added two eighteens, he took the tens first and wrote 20. Then the two eights was added and he said 16, which he also seemed to know by heart. Then he wrote 36 and was not sure what to add next. After a little while he decided to add nine to thirty-six. He took four from the nine to get forty, which he wrote, and was unsure how much was left of the nine after taking four. He thought it might be three left and checked that by counting three plus four while he knocked a finger rhythmically on the table. Then he counted three plus four plus one plus one while knocking and said ”five left”. He wrote 45 and added 50 and 45 by taking the tens first writing 90 and finally 95.

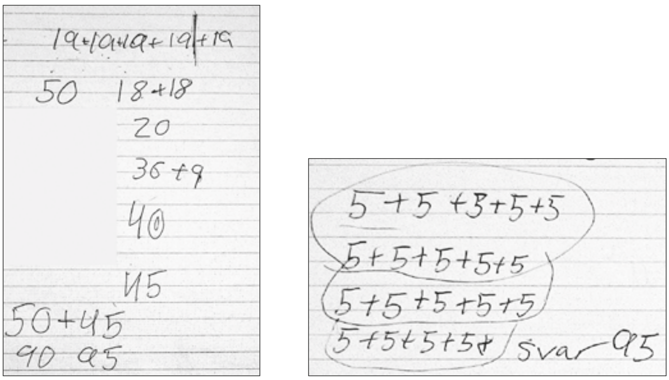


Figure 10. *Erik's calculations of repeated addition for 5 × 19*

After this calculation Erik said it would probably be faster to add the fives instead. When asked how to keep track of adding the correct number of fives Erik wrote nineteen fives, see figure 10 to the right.

Erik: Now one can: five, ten, fifteen, twenty, twenty-five, thirty, thirty-five, forty, forty-five, fifty, fifty-five, sixty, sixty-five, seventy, seventy-five, eighty, eighty-five, ninety, ninety-five.

When Erik was asked to multiply 16 · 25 he said that he needed to add either twenty-five sixteens or sixteen twenty-fives. He preferred to add the twenty-fives since ”it is just to take the twenties first and then add on five, ten, fifteen, twenty and so on”.

Erik was asked but had no alternative strategies than repeated addition for multiplication. He was aware of the commutative property and chose which of the numbers was easiest to add. He knew how to skip count by fives. Even though he correctly reasoned multiplicatively to two of the items in the written test, he must be categorised as a not yet multiplicative reasoner since he had no multiplicative strategies for calculations and related multiplication to repeated addition on both calculation items.

Alva

Alva was categorised as a multiplicative reasoner on task 1 since all her answers reflected the multiplicative structure of the problems. However, she wrote all solutions by additive operations and drew the same erroneous conclusion to item 3 as Erik, see figure 9.

When calculating Alva solved 5 · 19 in a similar manner as Erik. She very quickly said five, ten, fifteen etc. to fifty. Then she paused and continued at a slower pace, taking the two numbers with the same tens together: fifty-five, sixty and sixty-five, seventy and seventy-five until she reached ninety-five. She kept track of the fives on her fingers. She was asked but had no alternative strategy for the calculation, she did not suggest that it would have been possible to add five nineteens, thus she did not demonstrate awareness of the commutative property.

When she worked on 16 · 25 she added sixteen twenty-fives, see figure 4 which is Alva's work, and got the answer 72. She started by adding the tens separately, but treated them as twos, not twenties, thus getting 32 that she added to 40, which was the sum she got for sixteen fives. Alva got 32 by adding 16 + 16 as 10 + 10 + 6 + 6 = 20 + 12.

Alva: Sixteen plus sixteen. Ok, wait. Ten plus ten is twenty and six plus six is twelve, then it has to be thirty-two. Then you take all the fives. You think that is forty, thirty-two plus forty ... seventy-two.

As with 5 · 19, Alva had no alternative strategy for 16 · 25. She knew how to skip count by fives, and did that quickly up to fifty, after which the pace was slower. She was categorised as multiplicative reasoner by the word problems task since all her answers reflected the multiplicative structure of the situation. She proved to reason additively when her only strategy for calculating multiplication was repeated addition.

Discussion

Students like Erik and Alva, who demonstrated multiplicative answers to all or most of the multiplicative comparison problems, handled all multiplicative calculations as repeated addition or by addition-influenced strategies. Thus, I question whether they can be considered to reason multiplicatively in spite of correct responses to multiplicative comparison problems. To reason multiplicatively one must correctly solve multiplicative comparison problems (Clark & Kamii, 1996; Van Dooren et al., 2010), distinguish multiplicative problems from additive (Van Dooren et al., 2010) and, with respect to composite numbers, show an awareness of the distributive property (Tzur et al., 2013). When numbers are treated as in the addition-influenced algorithm, in which the tens and the ones are separately multiplied and then combined, multiplicative reasoning is not yet developed to include multi-digit operations. To depend on repeated addition, as did Erik and Alva in all their calculations, could be construed as reflecting a primitive multiplicative reasoning (Fischbein et al., 1985) or a multiplicative reasoning with characteristics of addition (Van Dooren et al., 2010). However, it can also be considered as demonstrating additive reasoning (Bakker et al., 2014; Vergnaud, 1983). In the process of repeated addition the multiplicative character of the situation is avoided since the referent unit is preserved and not transformed (Schwartz, 1988). When Erik was adding 50 kronor three times, he did not need to consider the units, he added within the same unit as the answer.

Erik and Alva would not be identified as additive reasoners by reference to their answers to multiplicative comparison problems. Moreover, their use of repeated addition on the calculation tasks would not necessarily be discovered, since their cumbersome calculations typically yielded correct answers. With respect to the fact that repeated addition is an unsustainable strategy when numbers are rational or real, students like Erik and Alva need to be identified if they are to gain targeted instruction to widen their view of multiplication and calculation strategies. Therefore I claim that it is insufficient to assess students' multiplicative reasoning by means of multiplicative comparison problems without testing how they perform calculations.

Both Erik and Alva demonstrated an erroneous conclusion to item 3, in which they added 50 three times (see figure 9) and concluded that Mollie had zero kronor. This seemed to stem from additive reasoning where the comparison is absolute and not relational. The absolute (additive) difference between 50 and 150 is 100 and the relational (multiplicative) difference is three times. Here both Erik and Alva recognised the multiplicative character of "three times as much as" to split 150 in three equal groups of 50. Then by adding the three 50s they got the total of 150, which made them conclude that Mollie had nothing, all the 50s were used to get Max's 150 kronor, which I construe as additive reasoning.

Moreover, it seems that to write a solution as $50 + 50 + 50 = 150$ or as $3 \cdot 50 = 150$ indicated the reasoning students employed when calculating. One could argue that it is possible that a student might have calculated by multiplicative operations even though he or she wrote an additive expression and vice versa. Naturally, that possibility cannot be ruled out for all students' written solutions, but it seems as these 22 students actually expressed their way of calculating when each students' solutions from the multiplicative comparison problems and their calculations were compared. To the multiplicative comparison problems both Erik and Alva wrote all solutions by additive operations. This made sense as they actually calculated all multiplication as repeated addition. The two students in cell B2, who were categorised as mixed reasoners by the first task and multiplicative reasoner by the second task, as well as all other students who used distributivity when calculating, wrote the majority of their solutions by multiplicative operations. To write an additive operation as a solution to a multiplicative problem might be an indication that a student reasons additively when performing calculations, while multiplicative reasoning students seem to reflect their reasoning by writing their solutions by multiplicative operations. This could be further investigated in another study since if it is true that students' written solutions reflect their way of calculation it might constitute a quick overview for evaluation of students' reasoning.

This study has demonstrated that the use of multiplicative comparison problems to distinguish whether students reason additively or multiplicatively is not sufficient to uncover a specific group of students; students who perform multiplication calculations by additive reasoning, for example by repeatedly add one of the numbers. Since repeated addition is not a sustainable method when calculating with rational or real numbers (e.g. Fischbein et al., 1985; Simon, 2006), it is important to identify students who cannot perform multiplicative calculations in any other way. The perception of multiplication as always reducible to repeated addition has proved to be rigidly rooted and causing problems when the factors are not natural numbers (Fischbein et al., 1985; Greer, 1992; Simon, 2006; Sowder et al., 1998; Thompson & Saldanha, 2003). This implies that students who are left with this constrained conception of multiplication need more specific instruction to widen their perception of multiplication, but first they need to be identified. We have knowledge of multiplicative comparison problems as an effective tool to identify additive reasoners, but this study demonstrates that it is not enough. We also need to identify students like Erik and Alva, who can discriminate the multiplicative character of a problem but solves the problem by additive calculation strategies.

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