What's there in an *n*?

Investigating contextual resources in small group discussions concerning an algebraic expression

ELISABETH RYSTEDT, CECILIA KILHAMN AND OLA HELENIUS

This small-case study combines a content related and a dialogical approach, in an in-depth analysis of how three 12-year-old pupils in a video recorded small group discussion construe the meaning of the letter *n* in an algebraic expression. The findings indicate that the pupils used a rich variety of contextual resources in their sense-making attempt. They also tried out a wide range of interpretations of the letter indicating that their conception of an algebraic letter was rich but unstable and that the dialogue was instrumental in helping them move from primitive to more advanced interpretations. In addition to previously known difficulties of understanding letters as variables, we found that the meaning of the communicative convention "expressed in *n*" proved an obstacle, and conclude that learning mathematics is as much about learning a specific communicative genre as learning about mathematical objects and relationships.

For at least two decades, communication has been a major focus in mathematics teaching reforms. It is argued that giving pupils opportunities to engage in speaking, writing, reading and listening has dual benefits: they communicate to learn mathematics and they learn to communicate mathematics (NCTM, 2000). This has been accompanied by an interest in the role of communication and collaborative learning (Dillenbourg, 1999), initiating a shift from individual to the collective in mathematics education research, dubbed "the social turn" (Lerman, 2000). Although a significant focus has been on social issues (e.g. Walkerdine, 1988), and in the development and use of mathematics as such (e.g. Bishop, 1988), the role and effect of communication and collaboration continue to generate research interest. In the case of small

Elisabeth Rystedt, *University of Gothenburg*
Cecilia Kilhamn, *University of Gothenburg*
Ola Helenius, *University of Gothenburg*

Rystedt, E., Kilhamn, C. & Helenius, O. (2016). What's there in an *n*? Investigating contextual resources in small group discussions concerning an algebraic expression. *Nordic Studies in Mathematics Education,* 21 (1), 5-26.

group collaboration, several factors seem to influence children's learning. Literature reviews as well as randomized controlled studies indicate that collaborative learning is more effective for conceptual learning than for rote learning (Cohen, 1994; Phelps & Damon, 1989). Other studies indicate that individuals who are active in the group benefit more (van Boxtel, van der Linden & Kanselaar, 2000; Gillies, 2008; Webb & Mastergeorge, 2003). However, even when the content is conceptual in nature, a dialogue can be ineffective. Sfard and Kieran (2001) studied a case of two 13-year old boys' communication and concluded that, "interaction with others, with its numerous demands on one's attention, can often be counterproductive. Indeed, it is very difficult to keep a well-focused conversation going when also trying to solve problems and be creative about them" (p. 70).

In this article we analyse a situation where three 12-year-old pupils engage collaboratively in an algebra task. The situation was chosen from a larger video material because the dialogue as such seemed very productive while the resulting written answer was simple and incorrect. This tension between the apparent quality of the dialogue contrasting the resulting answer placed this situation squarely between the visionary high expectations on collaborative learning in documents like NCTM standards (2000) and the cautionary conclusions drawn from empirical research by Sfard and Kieran (2001). Was the quality of the dialogue only a surface feature or was the answer a poor representative of the learning that took place?

The formulation of the task put the interlocutors in our situation in an unfamiliar algebraic context involving the use of letters, connecting the study to a large field of research about pupils' (mis)understanding of algebraic letters (Bush & Karp, 2013). However, what we want to see is not only *in what ways* pupils understand the subject matter but also *how they reason* and if, and if so why, their interpretations change through the interaction. We will combine an analytical perspective concerning content using categories developed by MacGregor and Stacey (1997), with a dialogical approach (Linell, 1998) to analyse communication. Linell's work on communication sheds light on how shared understanding between speakers is built up by means of contextual resources, such as the environment, prior utterances and background knowledge. Whilst Linell's work mainly concerns everyday interaction, Ryve (2008) has elaborated the dialogical approach further into a framework suitable for mathematics education research. Our aim is to investigate how understanding of the algebraic letter *n* can be generated and elaborated by pupils in a small group discussion. We pose the questions: what interpretations of the algebraic letter *n* emerge in the group, and how do these interpretations relate to the contextual resources made use of in the discussion?

Before the empirical study is reported we will take a closer look at the theoretical framework of *dialogical approach* and the concept of *contextual resources*, and present a review of previous research concerning the learning of variables and algebraic expressions.

Dialogical approach

Although an increasing number of Nordic studies in mathematics education use a dialogical approach, the concept itself and its theoretical origin, dialogism cannot be seen as coherent and well defined (Ryve, 2008). Dialogism can be seen as the opposite of monologism, which accounts communication as transfer of information, cognition as individual processes and contexts as fixed environments (Linell, 2011). In a dialogical approach three principles are fundamental: *sequentiality*, *joint construction* and *act-activity interdependence* (Linell, 1998). *Sequentiality* indicates that an utterance acquires meaning from its position in a sequence and cannot be understood if it is "taken out of its sequence which provide its context" (p. 86). As a *joint construction*, the dialogue is something the actors possess, experience and do together, so that no part is completely one single individual's product or experience. It is a collective construction of "mutually coordinated actions and interactions" (p. 86). The *act-activity interdependence* implies that acts and activities are intrinsically related and are implicated by each other. All acts and utterances are situated within an embedding activity, which can be seen as representing a specific *communicative genre*. Depending on type of activity and genre, the participants act in different ways, which also have an impact on the activity itself. Communicative genres are ways of interacting, involving implicit historically established and cultural routines, norms and interactional patterns.

Two analytical concepts in dialogical approach are central in this study, *communicative project* and *contextual resources* (Linell, 1998). The most important characteristic of a communicative project is the involvement of a goal, which the actors are oriented toward in their interactions even when it is not articulated. Communicative projects tend to be hierarchically organized; a large project could be divided into smaller projects, which in turn consists of smaller ones. The two concepts *context* and *resource* are used together in the term contextual resources, because context is nothing in and by itself, it cannot be seen "objectively" or in singular (Linell, 1998). There are various traditions to explain contexts: One is to see "contexts as embedded within and emergent from" the activities themselves (Linell, 2011, p. 93). Utterances actualize contexts and actualized contexts initiate new utterances in a mutual process between utterances and contexts. Another tradition is to see context as more or less ”stable outside environments” (ibid.), which implies that contexts already exist, before the sense making activity. An attempt to bridge the two aspects is to treat contexts both as resources for and as products of the interlocutors' activity. Linell (1998) defines contextual resources as potential contexts made relevant through the dialogue. The status of a contextual resource is thus not inherent in things or processes. A resource is a resource for somebody, for a purpose, in a situation. In this study socio-mathematical norms (Yackel & Cobb, 1996) are seen as one of several potential contextual resources.

Prior research about algebraic letters and expressions

The gatekeeper status of school algebra (Caj & Knuth, 2011) has generated ample research interest. Quinlan (2001) presents a strong correlation of pupils' achievement on algebraic tasks and their level of symbolic understanding. Hence pupils' interpretations of algebraic letters may be particularly critical to their learning in algebra. In the literature about learning variables Küchemann (1978), and MacGregor and Stacy (1997) are frequently referred to, and for this reason we want to take a closer look at their results and what type of data they are based upon. Küchemann (1978; 1981) investigated how pupils, age 13–15, gave algebraic letters meaning, and found that different tasks invoked different interpretations of the letter, hierarchically ordered from less to more challenging.

Küchemann investigated how well students were able to solve tasks that were qualitatively different concerning how the letter could be interpreted and treated. The two least challenging types of tasks were those that could be correctly solved when the letters were ignored or evaluated by trail and error. Achievement decreased in tasks where the letter could be interpreted as a name or a label for a specific object, or as an unknown quantity or measure. Even more challenging were tasks where the letter represented a generalized number, such as ”what can you say about c if $c+d=10$ and c is less than d?”. The greatest challenge was in tasks including a dynamic relation between two expressions including the same variable. Notable is that Küchemann only used the term variable in the last level, in a rather narrow explanation of what characterizes a variable. A contemporary definition of variable used in Sweden defines it as a quantity that can assume any value in a given set (Kiselman & Mouwitz, 2008, p. 21). Such a definition would incorporate several of Küchemann's categories as variables. This indicates that conceptions of variable change over time and that different approaches to algebra need different definitions of variable (Usiskin, 1988). Letters used in algebra can take on many roles, as described by Philipp (1992), for example the role of labels, constants, unknowns, generalized numbers, varying quantities, parameters and abstracts symbols. Since all these uses of letter cannot have the same static interpretation of meaning, pupils need to experience a wide range of situations in which letters appear (Kilhamn, 2014).

MacGregor and Stacy (1997) set out to analyse how algebraic letters were interpreted by pupils, 11–12 years old, who had not yet been taught algebra in school. Interpretations were analysed in relation to how they correspond to Küchemann's hierarchical order, but instead of testing the pupils on tasks with ready-made algebraic symbols and expressions, the pupils were asked to construct an expression including an algebraic letter. One of the two items was: "Sue weighs 1 kg less than Chris. Chris weighs y kg. What can you write for Sue's weight?" (ibid, p. 5). Since this investigation was based on written answers the researchers described the reasoning they assumed in relation to each category. Their results revealed the following six interpretations of algebraic letters:

- Letter ignored (no letter at all is used in the answer)
- Numerical value (a value related to the situation)
- Abbreviated word (w = weight)
- Alphabetical value (y = 25 because y is the 25th letter in the alphabet)
- Different letters for each unknown (e. g. Sue's weight is "*o*")
- Unknown quantity ($y - 1$)

MacGregor and Stacey found that the three first interpretations in their own list correspond to Küchemann's lower division of interpretation and the last one to his higher division. The two categories "alphabetical value" and "different letters for each unknown" were not explicitly included in Küchemann's hierarchy. Interesting to note is that MacGregor and Stacey (1997), along with other subsequent researchers such as Izsák, Çağlayan and Olive (2009), have used the category "letter ignored" in a quite different way to what Küchemann (1978, 1981) did. Instead of representing tasks that could be correctly solved ignoring the letter, the category came to represent a category of solutions where students incorrectly ignored the letter, producing an answer where the letter is missing. In this article we will use the categories in line with MacGregor and Stacey (1997) since we analyse different ways in which pupils interpret the same task rather than pupil's interpretation of letters in different tasks.

All six interpretations in the study of MacGregor and Stacey were also identified in a larger sample of 2 000 pupils who had been taught algebra.

Interpretations, they argue, which do not correspond with the general mathematics notation ”are thoughtful attempts to make sense of a new notation or are caused by transfer of meanings from other contexts” (ibid, p. 15). For instance alphabetical interpretation of letters could be connected to pupils’ prior experience of codes or the strategy used to label tasks in textbooks as a, b, c etc. When pupils see letters in such ways, it can reinforce their belief that each letter has a fixed value depending on the order in the alphabet (MacGregor & Stacey, 1997). In a discussion of the described study, Radford (2000) states that the didactic question generated by such elaborate catalogues of pupils’ misunderstandings in algebra, is how those non-algebraic meanings are successfully transformed by the pupils ”up to the point to attain the standards of the complex algebraic meanings of contemporary school mathematics” (p. 240).

Linchevski and Herscovics (1996) claim that it seems easier for pupils to interpret letter symbols in equations with one variable than letters in expressions. The reason could be, they state, that a letter in an equation can be perceived as a placeholder and the equation will intuitively be interpreted arithmetically. Early in school mathematics pupils have become accustomed to solving mathematical tasks by producing a numerical answer. This leads to an expectation that the same is true even for an algebraic expression (Kieran, 1981). An arithmetic expression such as $3 + 2$ can be calculated to give the answer 5, but the algebraic expression $3 + 2a$ cannot be calculated until the value of a is known (Tall & Thomas, 1991). Pupils are often reluctant to accept an algebraic expression as a final solution, which is referred to as acceptance of *lack of closure* (Collis, 1975). A background to this dilemma is that an expression can be treated both as a conceptual object in its own right and as a process to be carried out when the variable is known (Hewitt, 2012; Gray & Tall, 1994).

In brief, the studies we report (Küchemann, 1978; MacGregor & Stacey, 1997) are based on written tests and pupils’ individual comprehensions. In both studies the pupils’ interpretations have been classified in different categories related to a hierarchical order. Hence, there is an individualistic and a relatively static view, which permeate this often-cited research. In this study we build on the study of MacGregor and Stacey (1997), but instead of looking at pupils’ individual written work, making assumptions about their reasoning, we investigate how the understanding of an algebraic letter may be generated and elaborated by pupils in a group discussion. Our aim is not simply to identify these interpretations among our pupils, but rather, in line with Radford’s (2000) challenge, to investigate in what way these interpretations emerge in a group discussion and how such interpretations are consolidated or transformed as a result of the contextual resources invoked.

Method

This study is a part of an international video study entitled VIDEOMAT about introduction to variables in algebra (Kilhamn & Röj-Lindberg, 2013). Within the study five consecutive lessons were recorded in five different 6th and 7th grade classes in four schools in Sweden. The teachers were informed that they would be video taped during their four first lessons on introductory algebra, following the normal curriculum. The fifth lesson was designed by the project, consisting of small group discussions around three tasks adapted from the 8th grade TIMSS 2007 survey (Foy & Olson, 2009) given to the teachers after the fourth lesson. The aim of the fifth lesson was to study pupils' communication when working with algebra tasks they had not been specifically prepared for. All the pupils in the class were informed that the tasks were intended for 8th grade. Informed consent had been obtained in writing from the parents and orally from the pupils. Data for the small-case study presented here come from a 15-minute dialogue in a group of three 12-year-old pupils:

> Hasse has 3 jackets more than Anna. If n is the number of jackets Hasse has, how many jackets does Anna have expressed in n? Write an expression to describe how many jackets Anna has expressed in n.

It is the second task they work with during the lesson, following a task concerning an equation based on the formula $y = 4x + 30$. The pupils' written solutions were collected and the dialogue was video recorded and transcribed verbatim. Excerpts[1] in this article have been adjusted slightly to increase readability and translations have been made to capture the essence of what was said.

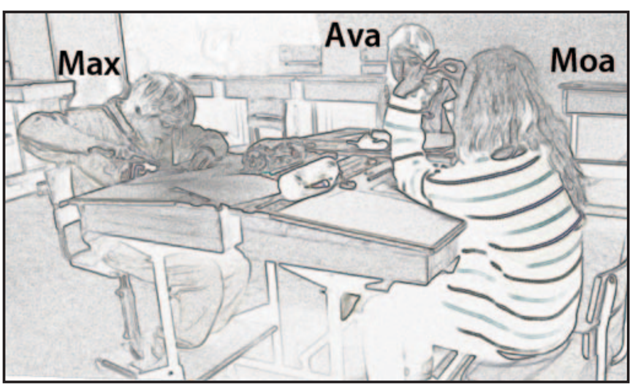

Figure 1. *The pupils in the group*

Data analysis

The data analysis is guided by our two research questions approaching the dialogue from a resource perspective and from the perspective of interpretations of the variable *n*. The process of turning raw video data into researchable and presentable units followed an analytical model introduced by Powell, Francisco and Maher (2003). The model consists of seven interacting, non-linear phases: 1) Viewing attentively the video data, 2) Describing the video data, 3) Identifying critical events, 4) Transcribing, 5) Coding, 6) Constructing a storyline, and 7) Composing a narrative. In our case, critical events are defined by instances in the dialogue that highlight either a special contextual resource (Linell, 1998) or a particular way of understanding the variable *n* (MacGregor & Stacey, 1997). In line with the dialogical approach, the storyline is presented in terms of *topical episodes*, which means relatively bounded sequences in terms of the content and how the actors organize their interaction (Linell, 1998). Topical episodes consist of discursive events connected by content, sometimes spread out over a period of time and intertwined with other topical episodes. For the coding and analyses of what contextual resources the pupils make use of, we build on Linell (1998). Cate-gories suggested by Linell have been modified to fit the empirical data in an iterative process. The list of contextual resources presented below is therefore to some extent the result of an analytical process.

Contextual resources

1. Surrounding physical situation in which the interaction takes place, the "here-and-now" environment with its persons, objects and artefacts
2. Co-text: what has been said on the same topic before the utterance or episode in focus
3. Background knowledge
   - 3.1. Prior knowledge in relation to mathematics
   - 3.2. Prior knowledge in relation to situations outside mathematics
   - 3.3. Knowledge and assumptions about the real world
   - 3.4. Assumptions about the communicative project
   - 3.5. Assumptions about the actual situation in terms of socio-mathematical norms
   - 3.6. Knowledge and assumptions about each other

Some contextual resources can be observed directly by listening to the discussion and/or looking at the pupils' actions, whereas others are mediated resources, not directly manifest in the situation. We will henceforth use the term resources to mean contextual resources. In contrast to the coding of resources, the coding and analysis of different interpretations of *n* makes use of a priori categories as identified by MacGregor and Stacey (1997).

Results

The results build on a 15-minute dialogue. From the pupils' point of view the whole dialogue is one communicative project aimed at solving the task. In the process of analysis we have identified five different topical episodes and used these as our units of analysis, investigating interpretations of *n* and use of contextual resources. The results will be presented first as a description of the five topical episodes, each followed by an analysis of resources and interpretations of *n*. At the end of this section there are two tables, summarising resources and interpretations of the letter *n* identified in the group discussion as a whole. Before the pupils start, they are instructed to work in groups, read the task very carefully, think about and discuss what it is about and what is asked for. While the pupils work in groups, the teacher circulates the room, interacting with different groups.

Topical episode 1: The alphabet track

In this first sequence Moa has just read the task aloud. After 25 seconds Ava suggests the answer "2" phrased as a question. Moa starts to count on her fingers to see which number the letter *n* has in the alphabet.

Excerpt 1, 00:03:30

6	Ava	2?
7		(4.0)
8	Moa	but
9		(1.0)
10		a, b, c, d, e, f, g, h, i, j, k, l, m, n (Moa counts on her fingers)
11		(1.0)
12		14, 14 jackets
13	Max	how did you figure that out?
14		(1.5)
15		that you should count the alphabet
16		(5.0)
17	Moa	but Anna also has expressed *n*, then she also has 14 jackets
18	Ava	though, he has more jackets than Anna
19	Moa	yeah
20	Max	<u>3 jackets more</u>

Ava's answer "2" (line 6) could indicate a plausible number of jackets for Anna, if Hasse has 3 more. If so, Ava simply solved the task without considering the letter *n*, making it an example of *letter ignored*. We see that Moa utilised a well-known situation, using *the alphabet* (line 10), as a sense making resource by giving *n* the value 14. The utterance "14, 14 jackets" (line 12), indicates an attempt to give a number as an answer to the question. This could indicate the application of the norm that in a mathematical task *the answer is a number*. Max' *meta-question "how did you figure that out?"* (line 13) can also be taken into account as a resource indicating a focus not only on finding the answer but also on understanding each other. His question suggests that in this group it was not uncommon to use *each other's competence* as a resource.

The group continues to examine the alphabet track leading them to new results and statements. In Excerpt 2 it is essential to know that the phrase "expressed in *n*", in Swedish is "uttryckt i *n*" – the preposition "in" is a one-letter word, "i".

Excerpt 2, 00:04:16

23	Moa	14 jackets
24	Max	in::: n:::
25	Moa	there are not, there are not many persons, who have 14 jackets
26	Ava	nohhh::: not even me (Ava giggles)
27	Max	in n
28	Ava	nohhh::: I am joking
29		(2 .0)
30	Moa	then it must be that she has 11 jackets

During the girls' dialogue Max is looking at the text in the task and loudly saying "in *n*" twice (in Swedish "i *n*"), with emphasis on one or both of the two letters "i" and *n* (lines 24 and 27). He then asks Moa to count to the letter "i" in the alphabet, justifying his suggestion by referring to what they can read in the task: "expressed in *n*" (in Swedish: "uttryckt i *n*"):

Excerpt 3, 0:04:48

36	Max	eh, double check it, count to i in the alphabet.
37	Moa	a, b, c, d, e, f, g, h, i. 9 (Moa counts on her fingers)
38	Max	because it says Anna expressed in *n* (Max looks at the paper)
39	Moa	in *n*. how many jackets has Anna expressed in *n*
40		(Moa leans forward making a gesture with her hands)

41 she has expressed so many jackets
42 <u>in</u> the letter ***n***, <u>in</u> ***n***, in <u>the *n*</u> (Moa nods her head)

In Excerpt 2 the pupils alluded to knowledge about the real world, *"there are not many persons who have 14 jackets"* (line 25), as a resource although they accepted the unrealistic circumstance when suggesting 11 jackets for Anna, assuming $n = 14$. The dialogue is an example of the norm that tasks and answers do not need to be realistic in math.

In Excerpt 3 the discussion took a new turn when the letter "i" (meaning the preposition "in") is regarded as a possible algebraic letter. It is possible to trace the origin of this idea in the earlier part of the dialogue. In Excerpt 1 Moa used the alphabet to find a value for ***n***, which appeared to be a new idea to Max, in Excerpt 2 he showed that the two letters "i" and "n" were confusing to him. In Excerpt 3 he grasped Moa's idea and wanted to test the alphabet to find a numerical value for "i". He employed the co-text as a resource, schematically illustrated as: *The alphabet* $\Rightarrow$ *if* $n=14 \Rightarrow$ *maybe* $i=9$. In itself it is a logic argument. Max deployed the text in the task as a resource (line 38). Moa tried to clarify the difference between the letter "i" as a preposition and the variable ***n*** through gestures (line 39–42).

The alphabet track is finally abandoned after a suggestion from Ava to try the Spanish alphabet instead "to see if it will work better", at which they all laugh. Although they do not investigate the alphabet track further, the letter "i" comes up again in the dialogue between the teacher and Moa in Excerpt 5, and Max utters an isolated "i" twice more without comments. The episode lasts 3 minutes, after which the pupils seem to take a minute off task, Max fiddles with the rubber, Ava yawns and Moa raises her hand to call the teachers attention.

Topical episode 2: Using given numbers

About a minute after the Spanish alphabet suggestion Ava overhears another group talking to the teacher, catching the numbers 9 and 12. She whispers "it is 9 and 12". Ava begins to look for connections between the numbers she overheard: "It can in fact be $3 \cdot 4$, look here 3, 6, 9, 12". When Moa objects to Ava's suggestions they drop the idea.

The number 3 appears in the task and 9 has been identified earlier (Excerpt 3). It is possible that Ava connected these numbers, recalling the known number fact $3 \cdot 4 = 12$. In this episode it is clear that Ava used *the pupils in another group* and *the teacher* as resources. The norm that in a mathematical task you *operate with the given numbers* can be taken into account as a resource in this situation.

Topical episode 3: The teacher's input

When the teacher comes to the group, Moa asks about the *n* and the teacher explains:

Excerpt 4, 00:10:23

84	Teacher	[...] it can be like this that n, n can be any number of jackets, it can
85		be that Hasse has 7 jackets, it can be that Hasse has 15 jackets,
86		depending on how many jackets Hasse has, so how many jackets has Anna then?
87	Moa	well, 3 less then
[...]		
103	Moa	but isn't it just to take and say for example, well 7 becomes 4, and
104		then you take minus 3 equals 4
105	Teacher	yes. exactly. write it down and think about how to express it

The pupils used *examples of how the value of n* can vary presented by the teacher as a resource without grasping the generality of the examples ($7-3=4$; $15-3=12$). The dialogue indicates that they interpreted the teachers input as an invitation to choose ”any number”, e g 7, and then decide to *fix the value of n to exactly 7*, rather than as examples of how the value of *n* can vary.

In Excerpt 5 the teacher leaves the particular examples, lifting the discussion to a general level.

Excerpt 5, 00:12:11

125	Teacher	if it was an unlimited* number of jackets
126	Moa	yes
127	Teacher	how could you write a formula then?
128	Moa	n minus i
129	Teacher	yes, but then we don't know what i is, it can't be unlimited or so
130	Moa	n minus 3 ... or ...
131	Teacher	I think that sounds good

* Probably she means ”unkown”.

It is not possible from the data to understand why Moa answered ”*n*–i” (line 128) when the teacher encouraged her to write a formula. One explanation could be that the term formula signalled the use of letters, and she applied the co-text as a resource influenced by their earlier discussion involving the letter ”i”. She then delivered an accurate answer ($n-3$) and the teacher supported it implicitly. However, the pupils did not discern it as the final answer for the task. We interpret the teacher's *suggestion of* *writing a "formula"* as an attempt to support the pupils with a potential resource, which the pupils made use of in their sense-making process. It is a treacherous suggestion since a formula usually involves at least two variables and an equal sign, whereas here the pupils are asked to write an expression using one variable.

Topical episode 4: Generalising "3 jackets more"

After Moa's initial suggestion that Anna has 14 jackets (topical episode 1), there is no evidence that they have any problems understanding that Anna has three jackets less than Hasse, when Hasse has three more than Anna. Three times in the dialogue they express: "Anna has three less", "She had three jackets less" and "well, three less then". They also articulate several times that they understand the arithmetic relation. $14-3=11$ in Excerpt 2; $15-3=12$ and $7-3=4$ in topical episode 3 and $5-3=2$ when the teacher asks how many jackets Anna has if Hasse has 5. When using this *arithmetical relation* as a resource, it appears easy for them to give correct examples of the mathematical generalisation: the number of Hasse's jackets = the number of Anna's jackets plus 3 $\Rightarrow$ the number of Anna's jackets = the number of Hasse's jackets minus 3.

Topical episode 5: Negotiating an answer

At the end of their discussion Max explicitly asks Moa how to think:

Excerpt 6, 00:12:41

158	Max	I still don't understand. do you understand Moa? how are you supposed to think?
159	Moa	that n is something, you shall take away 3. so, it is like x and y, that n
160	Max	you can write [...] 7 minus 3 is 4
161	Moa	no. yea, but then n minus 3 is equal to 4, only because some thing minus 3 shall be 4
162	Max	n, so <u>the n</u>
163	Moa	yes, minus 3 is equal to 4, only because, it could be [...] well
164		it could be x, only that minus 3, it <u>should</u> be 4, then, then
165		you need a number which when you take away 3 becomes 4.

Max initial question was not about the correct answer to this specific task. Instead, he used a meta-question as a resource in an attempt to get an understanding of *"how are you supposed to think"* (line 158). This indicates that he is engaging in a slightly different communicative project of wanting to know something about the way of thinking. In this sequence the pupils clearly availed *each other's competence* as a resource.

Moa utilised their shared prior mathematical knowledge, ”*n* is like *x* or *y*”, as a resource when she tried to answer Max' meta-question (line 159). She also explained to Ava that ”*n* is like *x* and *y*, that we have worked with, and they can be any number”. We cannot be sure about how Moa interpreted the connection between these letters, but it did help her to make sense of the *n* and she used it in an attempt to communicate that sense to the others. Again, the socio-mathematical norm that the answer is a number, in this case ”equal to 4”, is present. Max suggested the written answer $7-3=4$ (line 160) and Moa consequently insisted that something minus 3 should be 4 (line 161; 163–165). In the final negotiation they end up giving the particular example when $n=7$ as their answer.

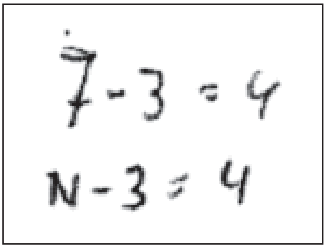

Figure 2. *Ava's written answer*

Throughout the five topical episodes we find two mediated resources permeating the interaction. One is related to socio-mathematical norms, when the excerpts show that the pupils were convinced that *the task had a meaning* and *was possible to solve*. The other one is related to pupils knowledge about each other, since there were *few misunderstandings* and it was palpable that they *understood each other's humour*, for example when joking about the Spanish alphabet.

Summing up the results

The resources described in the five topical episodes and their relations to the pupils' interpretations of the algebraic letter *n* have been categorised and collected in table 1. It is important to stress that this is not a complete record of all imaginable resources, but the ones we found relevant for the main communicative project of solving the mathematical task.

Five of the six categories identified by MacGregor and Stacey (1997) emerged in the group during the dialogue. The only category not found in our study was interpreting the letter as an abbreviated word. This is

Table 1. *The contextual resources and their relations to pupils' interpretations of n*

Contextual resource		Identified in this paper	Interpretation of letter
1. Surrounding physical situation in which the interaction takes place	1.1	The pupils in the group: each other's competence	
	1.2	The text in the task	Alphabetical value
	1.3	The pupils in another group	Numerical value
	1.4	The teacher:	
	1.4.1	examples of how the value of *n* can vary	Numerical value
	1.4.2	suggestion of writing a "formula"	Different letters for each unknown Unknown quantity
2. Co-text: what has been said on the same topic before the utterance or episode in focus	2	The alphabet ⇒ if $n = 14$ ⇒ maybe $i = 9$	Alphabetical value
3. Background knowledge:			
3.1 Prior knowledge in relation to mathematics	3.1.1	Arithmetical relation: $14 – 3 = 11$; $7 – 3 = 4$; $15 – 3 = 12$; $5 – 3 = 2$	Numerical value
	3.1.2	"*n* is like *x* or *y*"	Unknown quantity
3.2 Prior knowledge in relation to situations outside mathematics	3.2	The alphabet	Alphabetical value
3.3. Knowledge and assumptions about the real world	3.3.1	"2" [jackets]	Letter ignored
	3.3.2	"there are not many persons who have 14 jackets"	
3.4. Assumptions about the communicative project	3.4	Meta questions	
	3.4.1	"how did you figure that out?"	
	3.4.2	"how are you supposed to think?"	Unknown quantity
3.5. Assumptions about the actual topic, in terms of socio-mathematical norms	3.5.1	the answer is a number	Numerical value
	3.5.2	tasks and answers do not need to be realistic in math	
	3.5.3	operate with the given numbers	Numerical value
	3.5.4	the task has a meaning; the task is possible to solve	
3.6. Knowledge and assumption about each other	3.6	few misunderstandings understanding each other's humour	

not surprising as there is no suitable word beginning with "n" that could be used in the context of jackets. However, the pupils showed a related interpretation when they extended the idea of an algebraic letter by assigning a number to a non-algebraic letter in their interpretation of the preposition "in", which in Swedish is the single letter "i".

There are two ways in which to look at the relation between interpretations and contextual resources. First, the chronological order in which the categories emerged follows the hierarchical order from simple to more advanced, as shown in table 2. Secondly, we note that the most advanced interpretation of an algebraic letter as an unknown quantity emerges from contextual resources connected to mathematics (1.4.2 and 3.1.2 in table 1) or to meta questions about their own thinking (3.4 in table 1). In contrast to this, the most basic interpretations of letter, as letter ignored or as alphabetical value, emerge from contextual recourses not primarily connected to mathematics, such as the text in the task (1.2 and 2 in table 1) and other knowledge outside mathematics (3.2 and 3.3 in table 1). Interpreting the letter as a numerical value comes up in relation to almost every evoked contextual resource.

Table 2. *Interpretations of n, which emerged in this study*

Interpretations of algebraic letters (MacGregor & Stacey, 1997)	Identified in the dialogue
Letter ignored	Episode 1
Alphabetical value	Episode 1
Different letters for each unknown	Episode 3
Numerical value	All episodes
Unknown quantity	Episode 3 and 5

Discussion

Our analysis of the small group discussion has shown us that these pupils made use of many different contextual resources in their attempt to make sense of the algebraic letter *n* in the task at hand. Although their final written solution was an incorrect answer to the question and did not show a full understanding of a letter as a variable, we claim that the dialogue showed that learning was happening through the pupils' progression from simple to more advanced interpretations. Conjecturing, testing and eventually abandoning an insufficient interpretation, such as the alphabetical value of the letter, is perhaps one answer to Radford's (2000) question about how non-algebraic meanings are successfully transformed into complex algebraic meanings. Giving pupils this opportunity in discussions such as the one analysed here could be a fruitful path to follow.

The results show that the interpretation of the letter *n* as a [specific] numerical value is well established. It appears in every topical episode and is invoked by several different contextual recourses. This implies that the norm that mathematical questions produce numerical answers needs a lot of work to change if it is not questioned already in the early school years, as suggested by many contemporary researchers (Cai & Knuth, 2011). Given how numbers appear not only in mathematics but also in the everyday world, it is not surprising that the interpretation in terms of numerical value emerges from both mathematical recourses and non-mathematical resources. The final written solution includes the correct expression $n-3$, but in a closed formula $n-3=4$, indicating that pupils want a numerical answer and are reluctant to accept a lack of closure, which is in line with previous research (Collis, 1975; Kieran, 1981). The pupils see the specific case, not the general.

The most advanced interpretation of the letter as an unknown quantity is invoked twice in the dialogue. First by the teacher suggesting a formula in topical episode 3, and later in topical episode 5 when the pupils make use of their prior experience of letters in mathematics by stating that the n is ”like x and y”. Since the pupils had prior experience of equations with x and y, and since the preceding task had involved the equation $y=4x+30$, these mathematical associations were potential resources possible to invoke. They did not appear at once indicating that the students did not associate the letter n with previous notions of variable. Only after struggling to find meaning of the letter n in several ways and failing to reach closure, did they step back to ask about their own thinking and relate to other mathematical knowledge, which led them to the more advanced interpretation. Perhaps that insight does not come without the struggle.

The dialogue presented here was chosen because it on the surface seemed very productive. Our analysis based on Linell's dialogical approach confirmed this. The pupils are from the outset placed in a situation where it is the interpretation of the letter n and the associated mathematical conventions on how such symbols works in an algebraic expression that present the major obstacles. Early on it becomes clear that the pupils understand the relationship between numbers of jackets. Since the statement *expressed in n* is unknown for them they invoke different contextual resources that might have the potential to make *expressed in n* possible to interpret. When a resource fails to provide them with support, they move on by invoking new contextual resources that lead them to new interpretations of n. This is a creative endeavour. When a given interpretation fails to fuel further dialogue, a new resource is invoked to keep the dialogue going. Some of these do not lead to new interpretations of n but serve as important intermediate steps to sustain ”the dynamic flow of interaction” (Linell, 1998, p. 183). Invoked resources are of different kinds, ranging from the physical text to tacit socio-mathematical norms (Yackel & Cobb, 1996). For example, the pupils seemed to accept the pseudo-realistic character of the problem (Verschaffel, Greer & De Corte, 2000), and expect numbers as answers to mathematical tasks (Kieran, 1981). We conclude that the dialogue was instrumental in sensitizing the pupils to different meanings of *n*, which in the end led them to meanings aligned with contemporary school mathematics. Our results illustrate the process Radford (2000) calls for when he asks how non-algebraic interpretations are transformed by the pupils. We have shown that this can happen as a result of interaction in a problem-solving activity.

As mentioned in the introduction, the pupils in the studied group do not reach a correct answer, so a fundamental question is if the choice of using small group discussion enhanced learning or not. The detailed research of Sfard and Kieran (2001) shows this it not always the case: "The road to mutual understanding is so winding and full of pitfalls that success in communication looks like a miracle. And if effective communication is generally difficult to attain, in mathematics it is a real uphill struggle" (p. 70). In our case, we cannot assess if the pupils reached mutual understanding or not, we only assessed what was actually expressed. Since they are unaware of the convention of using *n* in the sense required in the task, the situation was very difficult for them.

Conclusions

We have shown that in a very short period of time in a group discussion pupils can make as much as five of the six interpretations of an algebraic letter identified by MacGregor and Stacey (1997). This is in contrast to previous research where students have been attributed to single interpretations. We conclude that interpretations of an algebraic letter can be dynamic and may shift quickly depending on contextual resources invoked, indicating that an interpretation is not a static, acquired piece of knowledge, but more like a network of associations possible to invoke. This implies that what pupils know when they are given the opportunity to discuss in a group and what they know when individually answering a written question are quite different things. Likewise, what is made possible to learn will also be different in the two situations. Connecting back to the dual benefits of collaborative communicative activities in mathematics education (NCTM, 2000), we have shown that in the process of trying to communicate mathematics these pupils communicated to learn mathematics.

Our second result is that the more advanced interpretation of the algebraic letter emerges as a result of invoking contextual resources of mathematical nature. Although such recourses are potentially available they might not be invoked unless the more basic interpretations are found invalid. Pupils' collaborative struggle to make sense can be a fruitful way towards more sophisticated interpretations.

It is also noteworthy that what the pupils struggled with was not only a problem that could be solved by means of mathematics or logic, but also the linguistic convention of the term "expressed in *n*". In the process of handling this situation, we claim that the dialogue provided a helpful context that in a fundamental way helped the pupils explore interpretations of the use of algebraic letters. An important conclusion is that learning mathematics is as much about learning a specific communicative genre as learning about mathematical objects and relationships. In this case, the search for the meaning of *n* involves problem solving, reasoning and communication – many of the skills that during the last few decades have been described as fundamental for the learning of mathematics.

Acknowledgements

The study was funded by NOS-HS (The Joint Committee for Nordic Research Councils for the Humanities and the Social Sciences) and Centre for Educational Science and Teacher Research at Gothenburg University.

References

Bishop, A. J. (1988). Mathematics education in its cultural context. *Educational Studies in Mathematics*, 19 (2), 179–91.

Boxtel, C. van., Linden, J. van der. & Kanselaar, G. (2000). Collaborative learning tasks and the elaboration of conceptual knowledge. *Learning and Instruction*, 10, 311–330.

Bush, S. & Karp, K. (2013). Prerequisite algebra skills and associated misconceptions of middle grade students: a review. *Journal of Mathematics Behaviour,* 32, 613–632.

Cai, J. & Knuth, E. (Eds.) (2011). *Early algebraization: a global dialogue from multiple perspectives.* Berlin: Springer.

Cohen, E. G. (1994). Restructuring the classroom: conditions for productive small groups. *Review of Educational Research*, 64 (1), 1–35.

Collis, K. (1975). *The development of formal reasoning.* University of Newcastle.

Dillenbourg, P. (Ed.). (1999). *Collaborative learning: cognitive and computational approaches.* Oxford: Elsevier.

Foy, P. & Olson, J. F. (Eds.). (2009). *TIMSS 2007 user guide for the international database. Released items. Mathematics – eighth grade.* Boston: TIMSS & PIRLS International Study Center. Retrieved from http://timss.bc.edu/timss2007/items.html

Gillies, M. R. (2008). The effects of cooperative learning on junior high school students' behaviours, discourse and learning during a science-based learning activity. *School Psychology International*, 29 (3), 328–347.
Gray, E. & Tall, D. (1994). Duality, ambiguity and flexibility: a "proceptual" view of simple arithmetic. *The Journal for Research in Mathematics Education*, 26 (2), 15–141.
Hewitt, D. (2012). Young students learning formal algebraic notation and solving linear equations: Are commonly experienced difficulties avoidable? *Educational Studies in Mathematics*, 81, 139–159.
Izsák, A., Çağlayan, G. & Olive, J. (2009). Meta-representation in an algebra I classroom. *Journal of the Learning Sciences*, 18 (4), 549–587.
Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, 12, 317–326.
Kilhamn, C. (2014). When does a variable vary? Identifying mathematical content knowledge for teaching variables. *Nordic Studies in Mathematics Education*, 19 (3-4), 83–100.
Kilhamn, C. & Röj-Lindberg, A.-S. (2013). Seeking hidden dimensions of algebra teaching through video analysis. In B. Grevholm (Ed.), *Nordic research in didactics of mathematics: past, present and future* (pp. 299–326). Oslo: Cappelen Damm.
Kiselman, C. & Mouwitz, L. (2008). *Matematiktermer för skolan* [*Mathematical terms for school*]. National Centre for Mathematcs Education, University of Gothenburg.
Küchemann, D. (1978). Children's understanding of numerical variables. *Mathematics in school*, 7 (4), 23–26.
Küchemann, D. (1981). Algebra. In K. Hart. (Ed.), *Children's understanding of mathematics:* 11–16 (pp. 102–119). London: John Murray.
Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 19–44). Westport: Ablex.
Linchevski, L. & Herscovics, N. (1996). Crossing the cognitive gap between arithmetic and algebra: operating on the unknown in the context of equations. *Educational Studies in Mathematics*, 30 (1), 39–65.
Linell, P. (1998). *Approaching dialogue: talk, interaction and contexts in dialogical perspectives*. Philadelphia: John Benjamins Publishing.
Linell, P. (2011). *Samtalskulturer: kommunikativa verksamhetstyper i samhället [Conversation cultures: communicative activity types in the society]*. Department of Culture and Communication, Linköping University.
MacGregor, M. & Stacey, K. (1997). Students' understanding of algebraic notation: 11–15. *Educational Studies in Mathematics* 33 (1), 1–19.
National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston: NCTM.

Phelps, E. & Damon, W. (1989). Problem solving with equals: peer collaboration as a context for learning mathematics and spatial concepts. *Journal of Educational Psychology*, 81(4), 639–646.
Philipp, R. A. (1992). The many uses of algebraic variables. *The mathematics teacher*, 85(7), 557–561.
Powell, A., Francisco, J. & Maher, C. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *Journal of Mathematical Behavior*, 22, 405–435.
Quinlan, C. (2001). Importance of view of regarding algebraic symbols. In H. Chick., K. Stacey., J. Vincent. & J. Vincent (Eds.), *Proceedings of the 12th ICMI study conference: the future of the teaching and learning of algebra* (Vol. 2, pp. 507–514). University of Melbourne. Retrieved from http://hdl.handle.net/11343/35000
Radford, L. (2000). Signs and meanings in students emergent algebraic thinking: a semiotic analysis. *Educational Studies in Mathematics*, 42, 237–268.
Ryve, A. (2008). Analyzing mathematical classroom discourse – initiating elaborations on the usefulness of the dialogical approach. *Nordic Studies in Mathematics Education*, 13(3), 7–29.
Sfard, A. & Kieran, C. (2001). Cognition as communication: rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture and Activity*, 8(1), 42–76.
Tall, D. & Thomas, M. (1991). Encouraging versatile thinking in algebra using the computer. *Educational Studies in Mathematics*, 22(2), 125–147.
Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. F. Coxford & A. P. Shulte (Eds.), *The ideas of algebra, K–12* (pp. 8–19). Reston: NCTM.
Verschaffel, L., Greer, B. & De Corte, E. (2000). *Making sense of word problems*. Lisse: Swets & Zeitlinger B. V.
Webb, M. N. & Mastergeorge, A. (2003). Promoting effective helping behaviour in peer-directed groups. *International Journal of Educational Research*, 39, 73–97.
Walkerdine, V. (1988). *The mastery of reason: cognitive development and production of rationality*. London: Routledge.
Yackel E. & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27(4), 458–477.

Notes

1 Transcript key

(4.0)	Pause in seconds
[...]	Inaudible or excluded
:	Elongation
<u>n</u>	Underline indicates emphasis

Elisabeth Rystedt

Elisabeth Rystedt got a licentiate degree in September 2015 and this article is a part of her thesis. As a PhD student she was a member of the The Graduate Research School in Educational Sciences at the University of Gothenburg. Today she is working 50 % at the National Center for Mathematics Education (NCM) and 50 % as a teacher educator at the University of Gothenburg. Her research interest concerns how pupils understand introductory algebra.

elisabeth.rystedt@ncm.gu.se

Cecilia Kilhamn

Cecilia Kilhamn is a Dr of Philosophy in Mathematics Education and works as a lecturer and teacher educator at the University of Gothenburg. She is also a mathematics education researcher with an interest in classroom research in primary and lower secondary school.

cecilia.kilhamn@ped.gu.se

Ola Helenius

Ola Helenius has PhD in mathematics and is a researcher and deputy director at the National Center for Mathematics Education at the University of Gothenburg. Some of his current research interests include neuroscience perspectives on learning mathematics, socio-cultural perspectives on preschool mathematics and mathematics curriculum design research.

ola.helenius@ncm.gu.se