A textbook in linear algebra – the use and views of engineering students

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This paper has a twofold aim. One is to analyse parts of a linear algebra textbook to seek for its relevance to engineering students. Another is to present an analysis of engineering students' views about this textbook. Results from the textbook analysis indicate that characteristics like motivating examples and visual design of text and pictures may appeal to engineering students. This is confirmed by analysis of students' views, showing that the textbook is appreciated, with examples as the most valued part. The textbook's design with theory presented in small portions, often in examples using specific values to illustrate theoretical arguments, seems to be a success factor.

In mathematics courses at tertiary level, it is expected that, to a large extent, students work individually with the content (Wood, 2001). They study theory and examples, and they solve tasks. In such work, the mathematics textbook is an important tool and plays a main role in students' learning of mathematics. Then it becomes important to investigate what learning opportunities are offered by the textbook, and, perhaps even more importantly, to know if such opportunities are recognised by the actual students. Thus, the present paper will deal with both these aspects; the textbook content and the students' perception and opinions of it. The textbook is in linear algebra; *Linear algebra with applications* by Lay (2014). In this book we will look closer into the chapter about *Vector spaces*. The students in focus are engineering students in their fourth year of studies to obtain a master's degree in engineering, and they take a course on linear algebra that is partly based on this textbook. Due to the limited amount of research results on textbooks on tertiary level, and even less when it comes to engineering students' use of textbooks, we ask the following research questions:

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- How is the theme *Vector spaces* presented in the textbook? What aspects of the textbook's content fit with readers being engineering students?
- How do the engineering students explain their use of and learning from the textbook in linear algebra?

Seeking answers to questions like these – adjusted for other mathematics themes and other groups of students – are important to any university teacher, knowing the importance of the textbook to the students. Both authors have experienced teaching linear algebra to prospective engineers and listened to the students' questions and comments about the course and the literature. Thus we see the need for more substantial knowledge about the students' use of and learning from the textbook in linear algebra.

The next section presents the theoretical background for the study, followed by a presentation of the methods we used and methodology. In section 4 the results with analyses are given, divided between textbook analyses and interpretations of input from a group of engineering students. This is then followed by a discussion in order to seek answers to our research questions.

Background and literature review

Not much research has been done with respect to university students' use of their mathematics textbooks, and even less about engineering students' use. Therefore, those actually having been published become important starting points. Also, a short survey of learning theories in linear algebra will be included, justifying why the chapter about vector spaces is selected for analysis. Finally, since the textbook is used in a particular educational setting, a short presentation of engineering students' preferences is given.

A recent review of research on textbooks in mathematics education claim that the bulk of textbook research has been on textbook analysis and comparison (Fan, Zhu & Miao, 2013). Fan and colleagues point to the study of Randahl (2012) on the role of the textbook at tertiary level, and emphasise that this level has received very limited exposure in previous research. According to the authors, textbook analysis is a broad term including analysis of a single textbook or a series of textbooks, which often focus on how a topic or topics are treated or how a particular idea or aspect of interest is reflected in the textbooks. This definition is relevant to what we are doing in this study. We are studying a textbook in linear algebra and especially the chapter on vector spaces in this book. Textbook content may be analysed in different ways.

Randahl (2010, 2012) analysed the text offered in the calculus book by Adams concerning a part of the chapter on derivatives in calculus and she also studied the students' use of the book. She found the text in the introduction formal and with few links to possible applications. Most of the exercises demanded procedural treatment and few needed to be solved with conceptual insights and justification. With many exercises and the more demanding ones at the end of that list most of the students never arrived at those tasks. The engineering students preferred to study lecture notes instead of the book and used the book as a source of exercises.

Lithner (2003) studied students solving exercises in textbooks and the focus was on their mathematical reasoning. The author found that students often imitated what had been shown in a solved task in the book and that this process was possible for most of the tasks. He also tried to characterise students' different kinds of reasoning.

Sierpinska (1997, p. 5) reported a study where the researchers try to "understand the complicated mechanisms through which mathematical meanings were being established and stabilized in the interactions between the different tutors, their students, and linear algebra texts". Two layers are discerned in the text, the didactical and the mathematical. Contrary to the study by Sierpinska we have not investigated the discourse between students, tutors and text and thus we cannot build on her use of the notion format of interaction.

For a more general discussion on different methods for analysis of textbooks, we refer to Rezat and Strässer (2012, 2013). They mainly refer to what they call the socio-didactical tetrahedron with student, teacher, mathematics and textbook at the vertices. In the present paper, we use a framework that was developed and used by Grevholm (2012) as part of the work in the Nordic textbook network (Grevholm, 2011), see figure 1. In our textbook analysis, mainly the visible and invisible properties in the middle part of the framework are utilized. Visible properties are those that can be observed by just looking at the pages of the book, while invisible properties demand also some sort of analysis, exploration or reflection before the reader can draw some conclusions. Some features of the book can have both visible and invisible properties, as for example the tasks or exercises. The number of tasks and their position in the text can be seen directly but to know how cognitively demanding they are one needs to investigate them more deeply. The theoretical foundation of the book refers to the author's standpoint concerning how the book is rooted in theory and how the mathematical theory is treated in the text both from mathematical and didactical point of view.

From the left side of the model – factors influencing the textbook – we use the curriculum, the theory of learning and teaching, and research on mathematics and mathematics education. From the right side of the model we mainly refer to students, the learning, students' use of the textbook, how students express themselves and connections to other subjects (or applications).

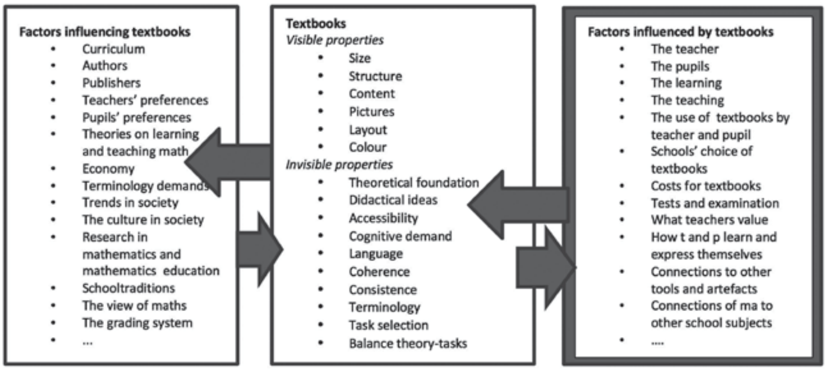

Figure 1. *Theoretical model of factors related to issues on textbooks in mathematics and their influences (Grevholm, 2012)*

Linear algebra is a domain within mathematics that is perceived as difficult for most students. According to Dorier and Sierpinska (2001) this stems from three sources of difficulties. First, there is the pedagogical approach. Algorithms are utilized but students have difficulties when it comes to proofs (Rogalski, 1990). With respect to textbooks, Sierpinska concludes that

> Modern academic linear algebra textbooks often offer structural arguments to justify certain basic statements. While usually short and elegant, they represent a level of theoretical sophistication that leaves a beginning student with a feeling that either nothing has been proved, or that the proved fact is of little significance.
>
> (Sierpinska, 2000, p. 236).

Second, there is a problem with languages used in linear algebra. The formal language with new definitions, concepts and theorems, appear without connection to previous mathematical knowledge. Students' difficulty with grasping the theoretical concepts and language is termed the "obstacle of formalism" (Dorier, 1997; Dorier, Robert, Robinet & Rogalski, 2000). There are, however, also a geometric language, an algebraic language, and an abstract language. In linear algebra there is a frequent move between these languages, also reflected in textbook dealings with concepts. Conversions that may seem obvious to textbook authors and teachers, may appear as difficult to the students and hinder their understanding (Dorier & Sierpinska, 2001). Third, linear algebra requires a "cognitive flexibility" in terms of moving between different languages. This appears as difficult to students, and Sierpinska (2000) postulates certain features to be responsible for this, as "students tend to think in practical rather than theoretical ways" (p. 209). Practical thinking is necessary for students to get a grip on abstract linear algebra objects, but the ability to connect to theoretical, structural thinking is vital for the understanding (Dorier & Sierpinska, 2001).

For engineering students we see a need to be offered both conceptual and procedural learning by the textbook. These concepts were introduced by Hiebert and Lefevre (1986) who define conceptual knowledge as connected networks of knowledge, thus being rich in relationships. Procedural knowledge is defined in two parts, both step-by-step procedures for how to solve mathematical tasks and familiarity with the symbolic representations that are used when solving such tasks. It is natural to note the connections of these concepts to the earlier similar concepts of relational and instrumental understanding (Mellin-Olsen, 1981; Skemp, 1976). We return to them in the discussion. Conceptual and procedural knowledge are, however, complex terms. Star (2005), for instance, points to different levels of such knowledge, both having deep and superficial dimensions. Relationships between conceptual and procedural knowledge has been of particular interest, and Byrnes and Wasik (1991) offer a more detailed discussion of different approaches to such relationships. In the context of engineering students and textbooks Randahl (2012) uses and discusses conceptual and procedural learning.

Engineering students study mathematics as a service subject. To these students mathematics may appeal if it is context-based and related to their future profession (Steen, 2001). McGregor and Scott (1995) emphasize the importance of matching the mathematics courses to the students' perceptions of engineering needs. This involves including material that students accept as applicable. Varsavsky (1995) stresses that engineering students' perceived lack of relevance of mathematics to professional disciplines can demotivate their acquisition of skills. Kümmerer (2001) points to that many engineers are not primarily interested in mathematics , and even if engineering students express a good deal of respect towards the subject they believe that for them most of the mathematics is useless. Kirkup, Wood, Mather, and Logan (2003) observed that teaching mathematics as a service subject seemed to be done in an ad hoc way and there were difficulties connecting to other disciplines. The importance of making mathematics relevant – especially for non-specialist students – by including real-world examples has been emphasized in a number of studies (Abdulwahed, Jaworski & Crowford, 2012). In her research studies about engineering students' task solving strategies, Rensaa (in progress) logged episodes where students sought to follow algorithms. Such strategies – when interpreted in an anthropological context – were found to be not only negative. Imitative solution processes may serve as increasing a student's self-confidence and inspire to search for systems to organize solutions of related problems. However, procedural tasks need to be accompanied by more conceptual ones and Rensaa points to the importance of taking the context into account when explaining behaviour and searching to modify the task solving activity.

To our knowledge there is little documentation of algebra courses in engineering programs. Therefore, an interview with an experienced colleague, Hans Thunberg, at KTH Royal Institute of Technology in Stockholm was done to support the research literature. When asked for a comparison of courses for general university students and courses in master's programs for engineering students he claims (authors' translation):

> Concerning the choice of content, the courses are, in principle identical, (precalculus, elementary algebra, basic linear algebra and analysis in one and several variables). The analysis courses are "calculus" courses and not rigorous epsilon-delta courses in real analysis. Courses at university typically have 50–100 % more time (that is more rewarding study points) for the same content, compared to engineering courses. Whether this means that one actually goes deeper into the content or not is not obvious.

He continued:

> There is an ongoing discussion in the technical education programs if courses should more strongly emphasize applications and interpretation, but there is no consensus about this; other people are of the opinion that the important thing with mathematics in engineering education programs is the generality of mathematics and training of logical thinking and problem solving.

The given description of the situation coincides with what both authors have experienced as teachers of engineering students at different universities in different countries and in different times.

Methods and methodology

The studied group

The present investigation was carried out in a group of 59 year–four students, where one of the authors was the responsible teacher. These engineering students had previously completed a bachelor's level engineering program, and had continued to a master's programme, which means two additional years of study. In the university where the research took place, the instruction at the master's level was given in English. Thus, students were used to communicate in English and no translation of responses from students was necessary. This was not the students' first meeting with linear algebra; they had completed a combined course on calculus and linear algebra in their first year of studies in order to know the basic tools needed for the professional disciplines. In the previous course, they had met another linear algebra textbook, a textbook in Norwegian. Thus, for students in the present course, it was their second experience with a linear algebra textbook. In addition, they had read a number of calculus books previously, and for that reason they were somewhat experienced textbook readers. However, in the previous combined mathematics course, the linear algebra part was small. It made just one fifth of a 10 study point course, and students mainly got acquainted with basic concepts and operations on matrices. Thus, a more extensive course was included in their master's level studies. A postponement of linear algebra is in accordance with Carlson's recommendations (1993), who lists, as one of four reasons why many students have difficulties in linear algebra, that the course is taught too early in their studies.

Data collection

Two types of data were collected. One was data given by exploring the textbook of Lay (2014). We searched for visible and invisible properties and what the textbook offers to the reader, but with main focus on the chapter about vector spaces, chapter 4. The reason why this chapter was chosen, was that its content – particularly the generalisation of vector spaces from the concrete and visual R^2/R^3 to R^n for $n \geq 4$ and other types of vector spaces – is cognitively difficult for students (Dorier & Sierpinska, 2001). Then it becomes particularly important to design, explain and formulate the presentation of this content well.

The second type of data was answers to a rather extensive questionnaire giving both quantitative and qualitative data. The quantitative results came from questions asking the students which parts of the first six chapters in the textbook they had read, one chapter in each question. The students were to mark out the parts they had read; main and selected part of theory, applications, examples, exercises or no parts. Multiple choices were possible and every part of a chapter that the students meant that they had read could be marked. Additionally, the questionnaire contained three open questions, in which responses to the following open question was of main interest to the present study: "What is your overall opinion of the textbook and your use of it?" Additionally, the questionnaire contained two other open questions; "What do you mean by learning in linear algebra? And how do you know that you have learned something?" and "Try to describe what you do and how you do it when you learn linear algebra". The answers to all these questions were coded in a joint coding process. Since the focus of the present paper is students' opinions about the textbook, the first open question was given main attention. However, both use and learning from the textbook was an issue, thus selected responses to the questions about learning were included.

The questionnaire was distributed electronically to all the 59 engineering students in the group and replies were submitted anonymously. An electronic system kept track of which students had submitted answers without revealing their names. Due to the quality assurance policy in higher education programs in Norway, students are used to reply to questionnaires. They do this frequently in different courses for the purpose of evaluation. Thus, responding to the present investigation was just one in a row. Since responses were anonymous, it was not possible to trace which students wrote what, thus the students had no profit from expressing more positive opinions than they actually had. However, the anonymity could raise another obstacle since students may see no profit from taking the time to respond to the questionnaire. To address this problem, submitting an answer was made mandatory in the course on linear algebra. Consequently, of the 59 students, 55 submitted answers; a high response rate particularly as the students were engineering students. Previous non-mandatory data collection had given very low response rate for this type of students (Randahl, 2012). Note that when we later quote students' open responses, they are the exact text without any changes or translations.

The processes of analysis

In the analysis of the chapter on vector spaces in the book we mainly considered the middle sector of the framework in figure 1, which deals with the properties of the book as such. After having read the text back and forth a number of times and used hermeneutic methods for interpreting the text (Ödman, 1979), we explored its visible and invisible aspects.

In this process, it was useful to relate the presentation in the textbook of Lay (2014) to another, similar textbook. For instance, when considering the layout of the book, it was valuable to compare this to what was done in another textbook. Because of content similarity, level of difficulty and organization, the textbook to compare with was chosen to be Anton and Rorres (2014). Still, the analysis was not a stringent comparative analysis since the content of the textbook by Lay was of main interest, while the Anton and Rorres book was used mainly as a reference – not analysed in its own right.

The quantitative data was interpreted according to response rate. In order to analyse the answers to the open questions, grounded theory was used. This theory originated from Glaser and Strauss (1967), but has been further developed in different directions. In Strauss and Corbin's version (1998), which is the one that was utilized in the present analysis, two features are vital. One is the development of theory out of the data where the data is broken down in component parts and coded. The other is the recursive process of reviewing the data and collecting further data to adjust the categories. All three open questions in the questionnaire were analysed together, following a joint coding process. First, one researcher derived the codes and coded the questions according to these. Next, the codes were sent to the other author who made a sampling test. Then both researchers met and made a thorough categorisation of the answers that both researchers had coded separately. The main finding in this joint process was that the codes were proper but selected ones needed some specification. The clarification and the thorough discussion about students' statements made answers to all the open questions more clear and one of the researchers could then make a second, adjusted coding.

Analyses and results

Analysis of the vector space chapter in the textbook

The structure of the chapter *Vector spaces* is simple and clear. It starts with an illustration from real life, explaining how received data from a space flight is given by functions. This imposes the ability to handle function spaces, which is one of several types of vector spaces. The example motivates learning about general vector spaces. The sections in the chapter are built up by examples where the author brings in one theme or concept followed by theory. A couple of "Practice exercises" are presented in each section, and they have solutions at the end of the section. These solutions serve as models for students in how to write their solutions of a task. A great number of exercises, often 30-40, are given in each section.

For simplicity, we refer to the textbook by Lay (2014) as the Lay-book while the textbook by Anton and Rorres (2014) is referred to as the Anton-book. As described previously, we mainly used the middle part of the framework by Grevholm (figure 1) to explore the properties of the textbook. The first visible property is size. The sizes of the textbooks are about the same, covering similar contents in approximately the same amount of pages. This applies when we omit the "Study guide" in the Lay-book, and omit the generalisations and themes in the Anton-book that are not dealt with in the Lay-book. The page sizes in the Lay-book are 2 centimetres longer than in the Anton-book.

When analysing structure and content of the vector space chapters in both books, these are rather similar except that the Anton-book has gathered discussions about the Euclidean vector spaces in a separate chapter and general vector spaces in a chapter called "General vector spaces". The Lay-book has put all types of vectors spaces together in the chapter called "Vector spaces". This is interpreted as a common characteristic when considering all chapters of the two textbooks: While the Anton-book to a large extent gather themes that can be unified by a common heading – like Euclidean vector spaces – the Lay-book more often spread these themes in other chapters where they are related to the content. Themes from Euclidean vector space discussions are in the Lay-book found in many chapters. Other spread outs are applications and transformations. While the Lay-book introduces applications and transformations connected to relevant linear algebra topics, Anton has separate chapters about these themes.

As for pictures, layout and colour, the Lay-book is set out to be more appealing than the Anton-book. The Lay-book uses a variety of colours in writings, figures and pictures, while the Anton-book keeps a grey scale. Graphs and figures are similar in the two books, but the Lay-book includes some photographs – in colours – in the text, like the space shuttle (p. 205) and cities and suburbs (p. 270). Such pictures represent a visible connection to the real world outside the university classroom. Both textbooks have wide margins to the left on each page, and the Lay-book allows figures to be put in these margins. In the Anton-book, both figures and comments are included in these margins, making them more crowded. The main text on each page covers the same area, but since pages in the Lay-book are 2 centimetres longer than in the Anton-book, the latter appears as more packed with content.

Exploring the invisible properties of the textbook, we interpret the theoretical foundations of the two books as similar. Both books try to present mathematical theory to give a foundation to the exposition. A detailed analysis of the section about bases for vector spaces as an example (section 4.3 in the Lay-book) reveals parallel definitions and theorems to what is found in the Anton-book (section 4.4 in the Anton-book), but results are arranged in slightly different ways. Proofs are provided to all theorems in the section by Lay, but a typicality of the Lay-book is revealed when proving the Spanning set theorem (theorem 5, p. 226). The theorem states that if a set H is spanned by a set of vectors $=\{v_1, ..., v_p\}$, a basis for

H is obtained by removing vectors in *S* that are linear combinations of the remaining vectors in *S*. In the Lay-book, this theorem is motivated by an initial example where three specific vectors in $\mathbf{R}^3$ are stated to span a set *H*, but where a calculation shows that only two of the vectors are necessary. Examples that motivate theoretical arguments are recurrently used in the Lay-book. This could be said to be one of the didactical ideas behind the presentation in the Lay-book and, using the notion of Sierpinska (1997), it can be seen as part of the didactical layer. A similar situation is given by two examples prior to the last theorem in section 4.3 in the Lay-book (theorem 6, p. 228), stating that pivot columns of a matrix *A* give the basis for the column space of *A*. The examples illustrate this fact, and the proof of the theorem can do with a descriptive explanation referring to the examples. The Anton-book does not use examples for motivation in the same way. The Anton-book does not use the same didactical idea and is more weakly didactical according to Sierpinska. The Anton-book has omitted a proof of the pivot theorem (theorem 4.7.5, p. 219). The didactical idea of motivating for theory by examples is accomplished throughout the vector space chapter in the Lay-book and provides a relationship between texts, arguments and results. It makes the chapter appear as coherent and consistent in its form.

Another didactical idea that is interpreted from comparing the content of the two linear algebra textbooks in focus is that some headings referring to similar contents are given different names. For instance, the Anton-book has a chapter called ”Inner product spaces” while the Lay-book calls the chapter with similar content ”Orthogonality and least squares”. There is a difference in emphasis given by these two names. Inner product spaces often include vector spaces with rather abstract ways of implementing a product since such new ways of implementation is the main point in introducing general inner product spaces. Orthogonality and Least Squares on the other side, are often associated with the Euclidean inner product and vector space $\boldsymbol{R}^n$. If this process and the method are implemented in a general inner product space, the calculations may not be straightforward. But at least initially in textbooks, Gram-Schmidt and Least Squares are presented in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$, involving rather procedural mathematics. This is easier to grasp than the abstractness that inner product spaces often represent. It appears as if the Lay-book tries to avoid scaring the students by putting a title on the chapter that refers to more computational concepts by students. Also, in the Lay-book, the inner product concept is more superficially dealt with than what is done in the Anton-book.

The discussion of titles of chapters is related to the interpretation of language and terminology in the textbooks. When comparing the two textbooks, there are mostly coincident descriptions as the language and terminology in linear algebra is common. Both textbooks present well-structured texts, written in a straightforward way. There are however some differences in putting names on theorems and results. An example from the vector space chapter is the relationship between dimensions and column space of a $m \times n$ -matrix A; rank $A + \dim \text{Nul} A = n$. The Lay-book names this "The Rank theorem" (2014, p. 249) while the Anton-book uses the name "Dimension theorem for matrices" (2014, p. 227). This may be a source of some confusion for students if they seek to consult both textbooks, since students often find the formalism in linear algebra difficult. One student expressed his frustration about the linear algebra language in the reply to the open question about the textbook as follows: "Hard to work on math in English, as for the book, it uses words that even google translate cant figure out... not liking the book to be blunt" (response 53). The well-structured text in the Lay-book seems to be of little value to this student. He struggles with the formal language of linear algebra in general.

If searching for a balance between theory and tasks in the Lay-book, the section about bases for vector spaces (section 4.3) may again be used as an example. The balance is to a large extent influenced by the types of tasks that are offered. The particular section contains 38 tasks. Out of these, the first 20 may be interpreted as procedural – involving calculations with numerical values. The following 18 tasks are of a more conceptual type, some being true/false statements about a basis for a vector space, some asking for explanations or proofs. The proportion between conceptual and procedural tasks in the vector space chapter in the Lay-book varies somewhat between sections, but all sections in the vector space chapter contain some tasks that involve discussing, verifying or proving a statement or an argument. In the section about basis of vector spaces in the Anton-book (section 4.4), all the 18 tasks given may be classified as procedural. Other parts of this book contain conceptual tasks, but the amount of such tasks in chapter 4 is rather low.

The comparison between the sections on basis for vector spaces in the two textbooks illuminates a difference in numbers of tasks that prevails in all the chapters: The Lay-book offers far more tasks than the Anton-book. In quantitative terms, Lay offers 327 tasks in 75 pages of the vector space chapter. This gives an average of about 4.4 tasks per page, while the number for the Anton-book is about 2.4 tasks per page in the similar chapter. With such a vast amount of tasks in Lay, a larger variety is given – including more tasks that ask for justifications of solutions or statements.

Analysis of reading frequencies in students' use of the textbook

Based on the responses to which parts of the chapters that the students have read it may be concluded that the most read units are the examples, followed by theory read in main and selected parts. On average for all chapters, 71 % of the students have studied examples while 48 % have read main parts of the theory. There is a noteworthy higher response rate to having read examples than having consulted exercises. The alternative "Studied examples" is the single one with highest response rate in each and all chapters. A third result, however, is that the introductions to each chapter – which include motivating examples of how the mathematics content may be applied in practical situations – are not studied as much as would have been expected. On average, only 20 % of the students have read these units. Engineering students often emphasize applications as important (Hjalmarson, 2007; Rensaa, 2014), and as teachers we have noticed that students do habitually ask during lectures what different concepts may be used for. Still, the introductions that present current problems from reality, which may be solved by using concepts from the chapters, are not frequently consulted.

Analysis of qualitative responses about opinions of the textbook

The coding process previously outlined established a set of categories to interpret answers to the open question asking for opinion and use of the textbook. Without going into details about interpretations of the answers, students mostly express satisfaction with the textbook. Descriptions coded as good, helpful or useful dominate, found in 67 % of the responses to the question. For instance, one student writes: "The only bad thing about the book is that it is hard to find the chapters you are looking for" (response 7). His only objection is layout problems due to the lack of indication on the top of each page, showing the number of the current chapter. Indirectly this expresses a great satisfaction.

The next to highest frequency of responses to the open question is the category coded as "Examples being helpful". Nearly a quarter of the students have referred to examples when describing their use of the textbook. If combining categories in a contingency table, the most frequent combination is explaining the textbook as good/useful/helpful just because it provides good examples. Examples, often related to practical use, are pointed out as valuable like in the following statement: "The concept behind the exercises are well illustrated with example which i find interesting. I highly recommend this book" (response 17). This statement substantiates what was highlighted in the textbook analysis: Explanations of concepts by motivating examples are appreciated.

In a cross-tabulation of codes, another frequent combination is describing the textbook as good/useful/helpful by referring to its contribution to understanding. One such response is the following: "the text and the examples in the textbook are great, they have made an effort to make it easy to understand. It's a great tool. The problem is that its to much to learn in short period of time" (response 5). The textbook provides clear explanations which may lead to better understanding.

A third rather frequent coded combination of categories is that the textbook is good/useful/helpful due to its well written theory. One student points to that this leads to understanding: "The textbook is very useful for me to learn this course. And i think it talks about the theories clearly, which is more helpful for the international students, because the English in the textbook is easy to understand" (response 35). The language and terminology of the book seem to be taken well by most students and that is particularly important in a linear algebra course.

Discussion

About the analysis of the textbook chapter

In the investigation of visible properties of the textbook, two aspects with significance to engineering students are highlighted. One is the organization of the textbook, in particular examples of use of linear algebra. In the Lay-book such applications are integrated in relevant chapters, while the Anton-book has placed them in a separate chapter. Both organizations have advantages. Presenting applications as realistic problems connected to a linear algebra concept may motivate students to learn the mathematics needed in order to solve such practical problems, while gathering applications in a separate chapter shows the variety of realistic problems existing within the domain of linear algebra. Still, building also on our experiences as lecturers we notice, that engineering students seem to be more in need of continuing motivation in their reading of linear algebra than for instance mathematics students. Engineering students seek to be exposed to real world problems (Hjalmarson, 2007; Rensaa, 2014). However, not all students in the present investigation liked the spread around of some of the themes in the Lay-book. One of the responses to the open question about the textbook states: "It is an ok book, but the subjects are scattered all over the chapters, and it is a bit tricky to get an overview of the topics" (response 12). To this particular student, the Anton-book would probably be more appropriate.

Another visible property of the Lay-book is the more "airy look" together with its use of colours and genuine pictures. These properties may be more appreciated by engineering students than pure mathematics students because engineering students are not primarily interested in mathematics (Kümmerer, 2001), and then the need of catching and keeping their interest by an appealing layout and motivating pictures from everyday life may be necessary.

Among the invisible properties of the Lay-book, the use of examples to motivate for theoretical arguments is a vital one. Such motivating examples are a repeatedly used didactical strategy throughout the textbook. It appears as if the author attempts to demystify theory by giving examples with numerical values or concrete vector spaces. In such examples, calculations and arguments are easier to implement, and the examples are often followed by an extension to a general situation. The didactical idea is to make theory more accessible to readers. Students do probably find it easier to get acquainted with a concept through such numerical derivations, and examples of this kind may be considered as a part of the theory presented in the textbook. The examples embrace both theoretical arguments and more procedural tasks. The reading frequencies in the present study show that engineering students appreciate this kind of examples.

Each section of the vector space chapter of the Lay-book ends with a long set of tasks. What cannot be seen just by looking at the tasks, but is revealed by a further inquiry, is that these problems follow the same system as Randahl and Grevholm (2010) describe for the calculus textbook they studied, starting with computational tasks and increase in demand gradually. The textbook analysis of the Lay-book shows that when offering so many tasks, a larger variety is possible. A variety in cognitive demand is also possible. This typically means starting with numerical tasks involving procedural calculations to obtain a correct answer, and ending with tasks that ask for some type of justification or proof. An advantage with such a system is that it supports some students' need to build self-confidence. The work by Rensaa points to this, as some students need to open their task solution activity with problems that they feel they can accomplish (Rensaa, in progress). Starting with manageable tasks may give self-confidence and may motivate for further – more advanced – tasks. Since many textbooks organize the tasks according to a gradual increase in cognitive demand, authors seem to agree on considerations about self-confidence. What Randahl describes as problematic with such an organization, is that the weak students use much time to accomplish the initial tasks, and do not reach the more advanced ones (Randahl, 2012). Their work with mathematics will then mainly be procedural. Another concern when including so many tasks in each section as the Lay-book has done is the risk of providing more tasks that are similar. One student comments on this in the questionnaire:

”examples are good but questions are straight forward and there are too much similar tasks in this book” (response 23). However, from the comparison with the Anton-book it follows that the large number of tasks in the Lay-book offers the opportunity to include more problems of a conceptual type. Thus, there are both procedural and conceptual tasks, which may satisfy different kinds of students with different kinds of wants. This is an advantage.

Which parts of the textbook were read by the students

The reading frequencies of the textbook show that the students had read theory, either the main or selected parts. In particular, all students had read the main or selected parts of the theory in the chapter about vector spaces. This indicates that they have found the content of the chapter challenging and therefore found it important to read the theory about it. This supports the research finding that vector spaces is regarded as a difficult concept by students (Dorier & Sierpinska, 2001). The result does somewhat diverge from what Randahl found in her research about engineering students in their first year of studies (2012). As mentioned, her investigation showed that these students preferred not to read the book but to read the lecture notes and use the calculus textbook mainly to look up tasks. The importance of students’ taking and using lecture notes is reported by other researchers too (e.g. Bergsten, 2011; Rensaa, 2014). The present students have consulted the theory in all relevant chapters. It may be due to the linear algebra textbook being better than the calculus book, the content of the course on linear algebra being more theoretical or the engineering students having experienced throughout their studies that the textbook can be a useful tool. Also, students may be more mature in year four than in year one, knowing how to carry out their studies better. What is somewhat surprising, though, is the average response rate of 20 % to the reading of the introductory sections in each chapter, since these present applications of the linear algebra concepts. A reason for this could be that realistic problems require rather much knowledge about the presented situation in order to be understood. Real-world problems are usually rather complex. Students know that such knowledge will not be tested in the exams, thus give it lower priority.

Students’ opinions about the textbook

Most students find the book rather easy to read and helpful for their studies. The cognitive demand of the text is obviously at an acceptable level for the engineering students as there are few complaints about the text being too difficult to understand. Much used are the examples, which is not surprising as Lay has managed to give large parts of the text in the form of examples. One student expresses it like this: "I like the book. The examples are easy to follow" (response 25). This indicates that some students find it easier to study theory when it comes as examples. Other aspects that students do emphasize when bringing up the value of examples, are that they are well written and are of help in grasping the meaning of concepts. Linear algebra is a domain that students often find difficult, particularly as they have problems with the formal language. They complain about the many new and unfamiliar concepts that are introduced, the obstacle of formalism. As pinpointed by Dorier and Sierpinska (2001), one of the reasons why so many students find linear algebra difficult is that "Linear algebra is an 'explosive compound' of languages and systems of representations" (p. 270). In such a setting a textbook being described as helpful in explaining concepts is valuable. One student explains it like this: "It's very good, there are many examples and applications. it is the combination of theory and practice" (response 28). This student appreciates the examples and applications, which – aligned with the textbook analysis – are used to offer more familiar approaches to, and motivation for, theory. Mathematics becomes relevant to engineering students if being apprehended as important for their engineering specialization (McGregor & Scott, 1995). The above statement indicates that this is a need that is met by the textbook.

When asked about their opinion and use of the textbook, one student emphasizes how the examples could be used as *tools* when solving tasks: "It is good to follow the text book. It explains the history, application, theorems, examples. Mostly examples problems in text book helped me to solve the task questions and some diff methods to solve them" (response 31). This student appreciated the variety of methods provided by the examples, which again could be used when solving tasks. Such similarities between examples and tasks have been documented by researchers. According to Lithner, 70 % of the 600 calculus textbook exercises he analysed could be solved by using the presented examples as models (Lithner, 2003). Some of the examples in the Lay-book may be used as templates for solving exercises, supporting this research result and the above student's statement. These examples are typically dealing with procedures. The importance of developing some procedural flexibility while working with tasks is emphasised by researchers (Star, 2005). Consulting examples before doing tasks is an often observed approach, and the strategy is useful to get ideas about a solution. However, it must not become a pure search for algorithms to copy in the solving process, without consciousness about why the solution strategy applies. Some engineering students prefer *doing* mathematics rather than *studying* mathematics. Kümmerer (2001) calls this the ”workman approach”, an approach where following a set of rules ”automatically” gives the answer. Then the *why* is missing. Since examples in the Lay-book embrace both theoretical and numerical arguments, students are offered to learn more than pure procedures from the examples.

Seven students' opinions about the textbook were interpreted as highlighting the books' contribution to understanding. This is valuable. It may, however, be questioned what students put into the phrase ”understanding linear algebra”, since understanding is a non-trivial notion (Sierpinska, 1994). For instance, one can distinguish between instrumental and relational understanding (Mellin-Olsen, 1981; Skemp, 1976), the latter being ”knowing both what to do and why” (Skemp, 1976, p. 20). In the open question about learning of linear algebra, one student connects understanding to learning by the following explanation: ”Generally, I mean that learning is to study something until you understand the theory, and is able to use it in both theoretical and practical problems. You know you have learned something when you are able to solve a problem, and fully understand what is going on. It is my opinion that the understanding af a task is more important than how many tasks you have done” (response 6). The statement indicates a rather deep reflection about learning, pointing to the importance of understanding the situation. Another student provides a clarification of how to know that one has learned something: ”To learn does not necessarily mean to remember something, but to understand it in depth and be able to utilize that information for your own goals. When one have truly learned something, one can easily explain it to someone else” (response 9). The recognition of how thorough the mathematics needs to be understood to be able to explain it to someone else is emphasized by researchers, also within engineering educations (Carberry, 2008). A third student connects his understanding to illustrations: ”This is a good textbook and it is easy to be understand. This book has lots of pictures which gives good illustration. I read the book before I start my homework every time” (response 11). In this statement, illustrations in form of pictures are emphasized. Visualizations may play an important role in learning of mathematics, and this student's reflection upon the role of pictures in the textbook is one that is discussed in research settings. Arcavi (2003) argues how visualizations can be a key component in learning and doing mathematics, but also how they may pose limitations and cognitive difficulties for students. One student, though, expresses a rather peculiar apprehension about the concept of understanding in his reference to the textbook: ”it was so useful for me. those parts, which I could not understand them during the lectures, are available in textbook and they are 85 % understandable ... the main part of text book are the examples, they are so helpful" (response 34). Again the examples are appreciated, although with an odd amount of understanding involved. In summary, it follows that the present engineering students frequently highlight examples and reading of the textbook, but their explanations of how this is done is somewhat varying. It is in accordance with Sierpinska's research, showing evidence that students stating that "I read the example" can mean very different things to different students (Sierpinska, 1997).

Conclusion

Two questions have been addressed in the present paper; asking about the textbook properties and the engineering students' use of the book. The analysis of the vector space chapter indicates that both the design and the content may appeal to engineering students. Pictures, examples and tasks do to some extent relate to applications. Theoretical arguments are motivated by examples in order to make general theory easier accessible. Texts are well-structured. This gives a presentation that meets the demands of the engineering students, which is confirmed by the responses to the questionnaire. A majority of the student responses may be labelled as "finding the textbook useful". When explaining such a view, the students refer to a number of features well aligned with the textbook analysis; examples, explanations and applications are appreciated and easy to grasp. The textbook's design with theory presented in small parts, often in examples using specific values to illustrate theoretical arguments, seems to be a success factor. Engineering students seek relations between learning of mathematics and use of the discipline, and a textbook ought to nourish this need. As one of the students states when explaining his learning;

> For me, learning is knowing the practical use of theory and how to execute said theory. As a computer engineer student specializing in games development, linear algebra is central in the programming I preform. I only know I have learned something if I can accotiate theory to a problem I encounter. (response 30)

The quantitative data about which parts of the chapters that have been consulted suggests that the textbook fulfils this wish since students have read most parts. The text is inviting to the students and it appears as if they take up the invitation and read it.

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