The teaching of mathematical knowledge for teaching – a learning study of primary school teacher education

JORRYT VAN BOMMEL

A group of Swedish teacher educators conducted a learning study in order to identify critical features concerning the teaching and learning of *Mathematical knowledge for teaching* (MKT). Three seminars and 300 tests were analysed using variation theory revealing four critical features to take into account in teaching student teachers in mathematics education: namely their need to i) formulate proper goals for a lesson, ii) outline the lesson plan in detail, iii) shift perspective from the role of being a teacher to being a mathematics teacher, and iv) understand the underlying mathematics of the lesson topic at hand. Thus, these are the four features of importance to the learning and teaching of MKT.

The teaching of mathematics has been the object of many research studies and several theories have attempted to describe what is involved in such a task. *Mathematical knowledge for teaching* (MKT) is one of the theoretical frameworks developed to describe the knowledge necessary for teaching mathematics (Ball, Hill & Bass, 2005; Ball, Phelps & Thames, 2008). Researchers in several countries have tried to identify teachers' MKT via multiple-choice tests (Ball & Hill, 2008; Delaney, et al., 2008; Fauskanger, Jakobsen, Mosvold & Bjuland, 2012). At this point research on MKT and related issues focussed on defining, describing and testing MKT, and it has been suggested that we now need to continue by implementing MKT (Askew, 2008; Elliott et al., 2009). Ball et al. (2009) argue for the need of developing suitable approaches to the teaching of MKT. Grevholm and Anthony (2010) state in more general terms that the content and form of pre-service mathematics education should be the focus of future research. There is a need, in other words, to develop knowledge on how to teach MKT. Some attempts have been made to describe the teaching of MKT, for example, a problem solving cycle developed for teaching MKT

Jorryt van Bommel,
Karlstad University

Bommel, J. van (2014). The teaching of mathematical knowledge for teaching – a learning study of primary school teacher education. *Nordic Studies in Mathematics Education*, 19 (3-4), 185–201.

(Koellner et al., 2007) and tasks to promote MKT development (Suzuka et al., 2009). This study also contributes specific knowledge of the teaching of MKT by focussing on the instructional interaction during a learning study conducted by mathematics educators in teacher education.

Background

A team of teacher educators wanted to improve their teaching because they were dissatisfied with the students' results at the end of previously completed courses. The teacher educators described, for example, that the student teachers could indicate several common mistakes in an addition algorithm, and that they could describe suitable strategies for adding two numbers. However, this seemed to be two isolated facts (elements) to the student teachers, as they could not relate suitable strategies to common mistakes. To enhance coherent understanding, the educators decided to conduct a learning study. Among all the components important to teaching mathematics they tried to find crucial aspects to focus on. They considered, for example, representation forms, teaching aids, real life adjustments, applications in other subjects, pupils' prerequisites, suitable exercises, and finally decided upon five elements: *Pupils' prerequisites*, *Models of explanation*, *Related (hands on) materials*, *Suitable exercises* and C*urricular documents*. It is important to note that the elements per se were not in focus, but the coherence between them. Having defined the five elements, MKT seemed a suitable framework to describe the overall focus of the study and the object of learning. In order to find the critical features, a learning study was conducted. The research question was to identify and describe these critical features concerning the understanding of the coherence between the elements.

Five elements and MKT

After deciding upon the five elements, the elements were related to the domains of MKT (figure 1). As for the element *curricular documents*, the teacher educators wanted to include both the (national) syllabus, but also other kinds of curricular documents such as local course plans and student teachers' decisions in selecting and identifying goals. This element is directly linked to the MKT domain "knowledge of content and curriculum", where knowledge of such documents is related to the teaching and learning of mathematics.

The element *Suitable exercises* was chosen, as it is an indicator of lesson coherence. For this study suitable exercises were studied in relation to the chosen content and is therefore placed in the domain "knowledge

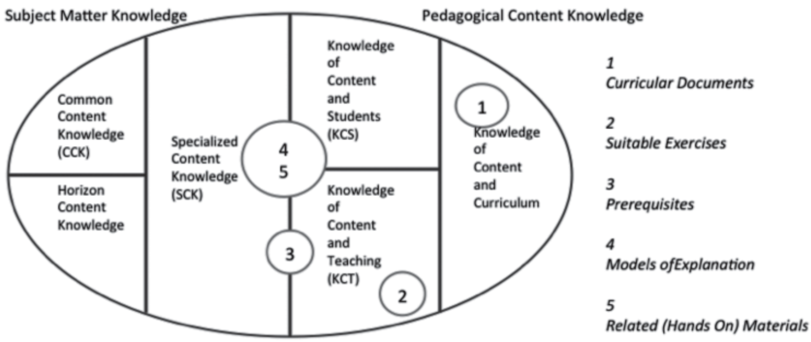

Figure 1 *The five elements of MKT focused on in this study within the six domains of MKT*

on content and teaching". Required *prerequisites* deal with the pupils' prerequisites, e.g., what is necessary to understand upcoming exercises. Common rules and procedures have to be taken into account in describing the prerequisites, which makes specialized content knowledge of importance. Knowledge of content and teaching is also of importance, as a student teacher needs to be able to filter out such prerequisites from instruction materials that can be used in class. The final two elements were placed at the centre of the egg, including three domains: *specialized content knowledge*, *knowledge of content and students* and *knowledge of content and teaching*. Both elements depend on all three domains: *hands on materials* was chosen as an important element, as teaching aids are considered to be a powerful tool to support the learning of pupils if the teaching aids are chosen properly and adequately. Student teachers need to use knowledge of all three domains in order to be able to choose suitable hands-on materials (teaching aids). The same goes for *models of explanation* as different models of explanation serve different purposes. Models of explanations are related to common mistakes, but also to knowledge of content and students (e.g. early grades addition models, differ from the models used in higher grades, where negative numbers appear).

Methodology: Learning study and variation theory

In the following section, the design of a learning study will be explained and the specifics of this study will be clarified along with variation theory used for analysis.

First, a learning study aims to build innovative learning environments and to conduct research into theoretically grounded

> innovations. Second, it aims to pool teachers' valuable experiences into one, or a series of, research lessons to improve teaching and learning. (Pang, Linder & Fraser, 2006, p. 31)

The two aims in the citation above relate to the origin of the learning study. Learning study has been described as a form of design research (Marton & Lo, 2007) relating to the first aim: *build theoretical innovative learning environments*. Learning study also has its origin in lesson study, where teachers *develop and evaluate lessons collaborative*. What distinguishes learning study from design research is the fact that it is the teachers, not the researchers, who own the process (Marton & Lo, 2007); teachers choose to conduct a learning study, they decide on the object of learning and they design the lessons (ibid.). The contribution of the researcher is less controlling than in design research. The role of the researcher in a learning study can be described as a participating expert. As for the difference between lesson and learning study, the explicit use of a theory of learning, often variation theory, is specific to a learning study (Runesson, 2005). The following design is often adopted in a learning study (figure 2):

Firstly, an object of learning is decided upon and literature regarding the object of learning is collected and considered. In the second stage, a pre-test is developed and conducted after which the results are analysed. A lesson plan is finalized and implemented in class. A post-test, often identical or similar to the pre-test, is conducted and analysed again. By putting the analysis of the post-test in relation to the conducted lesson, the lesson plan can be revised and implemented in a new class. Such a cycle can be repeated several times.

The analysis, as well as the planning of the test and lessons are guided by variation theory (Runesson, 2006). In variation theory, learning is seen as a change in understanding a phenomenon (Marton & Pang, 2004). Such a change in understanding results in a more powerful way of seeing the phenomenon (Marton & Tsui, 2004). In a learning study such a

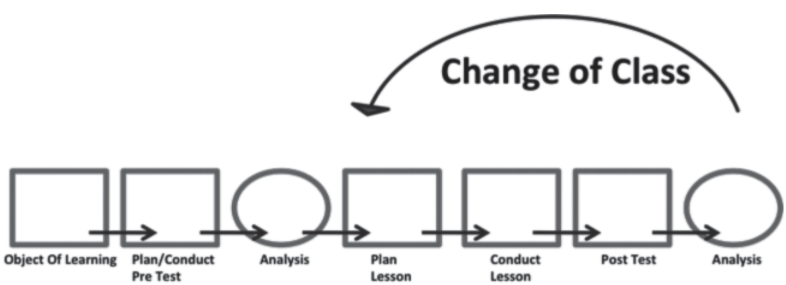

Figure 2. *Common design of a learning study*

phenomenon is called the object of learning and in the course of a learning study critical features of the object of learning are looked for. When critical features are dealt with, a change in understanding the object of learning can be established. "Dealing" with the critical features means that pupils have to be able to *discern* the critical features *simultaneously*. Such simultaneous discernment is supported by the use of *patterns of variation* (Runesson, 2006): 1) *contrast*, in which the understanding of the object of learning is enhanced by means of giving relevant contrasting cases, 2) *separation*, where the critical features are dealt with one at a time, 3) *fusion*, in which more then one critical feature is visible and, finally 4) *generalisation*, where the obtained insights into the object of learning can be generalized, thus providing a more complete understanding of the object of learning. When a critical feature has been identified as a possible critical feature, it is enacted in the lesson using the patterns of variation. Then it is validated through an analysis of the pupils' understanding of the object of learning. The data used in such analysis include the enacted lessons, and the pupils' pre- and post-tests.

Design of the present study

In this study a similar design was adopted but some changes were made in the original design. The cyclic process was kept but conducted in one group of student teachers (van Bommel & Liljekvist, 2008). MKT as the object of learning made it possible to keep the object of learning constant but to vary the mathematical content to which the object of learning was applied. Instead of changing classes as in figure 2, the topic was changed while the aims of the learning study, as well as the cyclic aspect of the design, were maintained, as shown in figure 3. The term *seminar* was used instead of the term lesson (van Bommel, 2012).

In line with regular learning studies, the seminars and the pre- and post-tests were planned and analysed in terms of variation theory. The

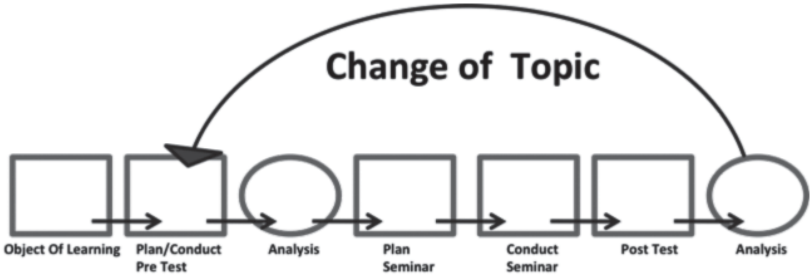

Figure 3. *Adjusted design of the learning study*

teacher educators chose the object of learning, as a crucial aspect of a learning study is the ownership of the teachers (teacher educators). This included, for instance, the choice of elements. After the choice of the five elements, the object of learning was related to MKT: to be able to give relevant connections between the five chosen elements of MKT. It aimed to develop the student teachers' MKT, concerning the five elements chosen by the teacher educators.

How to test MKT?

The assessment had to capture the interrelatedness of the five elements. The research team decided upon "writing a lesson plan" as a form of test, which was larger in scope than, for instance, the individual items in the LMT[1] (Ball & Hill, 2008). Student teachers were asked to write a lesson plan on a given mathematical topic (number sense, algebra etc.) equivalent to the mathematical topic that had been dealt with during the preceding seminars in the same course week. The five elements were given as suggestions to consider in their lesson plans. The feedback given was the same as previous years and commented the elements and their coherence. Although the improvement was compared to a control group (see van Bommel, 2012), these improved results of the student teachers are not in focus in this article; the identification of the critical features are.

As the tests contained the term *lesson*, the term *seminar* was introduced to describe the "lessons" given by the teacher educators. In this way *planning a lesson* (as in the student teachers' tests) could be distinguished from *planning a seminar* (part of the learning study cycle).

Data collection

The data considered in this study were collected during the spring of 2009, in a course on didactics of mathematics in primary school teacher education. The course was a twenty-week full time course and was given near the end of the program (year 3). Student teachers who took the course chose mathematics as their minor, which meant that it was the first and last time they encountered mathematics in the teacher education program.

Participants

A group of 48 teacher program students participated in the study. The tests were taken as part of their course assignment but the student teachers could choose if they wanted their tests to be included in the research study or not. The research group consisted of teacher educators and researchers. Two teacher educators conducted the teaching, while in total five teacher educators/researchers were involved in the planning and analysis. In this article, the teacher educators are referred to as teacher educators when reporting on their seminars, but as researchers in reference to situations in which their role was that of a participating researcher at other stages of the learning study. The author of this article was only involved as a researcher (not one of the teacher educators) and had no prior relation to the student teachers. The author and the research group were acquainted prior to the study.

Data

The data collected, consisted of written tests and video recordings of seminars. The video recordings consisted of three seminars (7.5 hours in total) given in the teacher education course. The three mathematical topics covered were number sense, rational numbers and spatial sense – the object of learning of these three seminars stayed the same: to be able to give relevant connections between the five chosen elements of MKT. The course consisted in total of around 20 similar seminars, each with a new mathematical content in focus, but these seminars were not part of the learning study. Prior to, and after, each learning study seminar, the student teachers were tested. The effect of other seminars was thus possible to take into account. In total 3 pre- and 3 post-tests were written by each of the student teachers, in total around 300 tests.

Analysis

The analysis consisted of two parts. The first part contains the analysis of the tests, where the appearance of and coherence between the elements were identified and categorized. A coding system was developed for this categorization in which the elements were a) identified for existence, and b) put in relation to each other. As the research question was to identify and describe the critical features concerning the understanding of the coherence between the elements, it was of importance to look upon the existence and understanding of such coherence.

The results of the analysis of the tests led to insights into what the student teachers understood or not and what the seminar should be focussing on. Such insights were formulated as possible critical features and enacted in the proceeding seminars. Variation theory and its patterns of variation guided the planning of proceeding seminars. The analysis of post-tests was related to the planned and conducted seminar and gave more knowledge concerning the identified critical features.

Coding

For each lesson plan, the description of the elements was categorized (table 1): code 0: *not referred to*, code 1: *indirectly referred to*, code 2: *directly referred to*. As for coherence, four further categories were used: code 3: *irrelevant connection*, code 4: *vague*, code 5: *relevant*. When one element was not referred to, connections with other elements were not possible to make and such connections were noted with code 8: *not possible*.

Table 1. *Codes used in analysis*

Code	Element	Connection
0	Not referred to	
1	Indirectly referred to	
2	Directly referred to	
3		Irrelevant
4		Vague
5		Relevant
8		Not possible

Examples of codes 1, 2, 3, 4, 5 are given below. Code 0 and 8 are not possible to exemplify, as it is only non-existence that is coded in those cases. To make a comparison between codes possible, each of the examples were taken from the first pre test, which was on number sense.

Element directly referred to (2)

St42 T1 Pupils have learned all numbers up to ten. They can write the symbols and we have worked with number sense.

Element indirectly referred to (1)

St3 T1 The lesson is conducted in whole class where all pupils have almost the same potential.

The two examples above show a discrepancy between the way the element prerequisite was referred to. In the first example the student teacher explicitly mentions prerequisite and states in detail what that prerequisite consists of. At this point it is not of importance if the description is sufficient or not. In the second example the student teacher (St3) states that pupils have almost the same potential, which was seen as an indirect reference to the element prerequisite, as the content of the potential is not clarified.

When the descriptions of the elements were coded, an analysis of the connections could be made between the elements. The student teacher in the example below (St42) shows coherence between the described prerequisites, stated at the beginning of the lesson plan, and the exercise proposed. ”Learned all numbers up to ten, writing the symbols” are stated as necessary prerequisites. The exercise described demands such prerequisites, and the connection between these two elements was therefore coded as relevant (5)

Connection relevant (5)

St42 T1 Pupils have learned all numbers up to ten. They can write the symbols and we have worked with number sense. Now we will strengthen their number sense and work practically with partitioning numbers.

[...]

Every pupil has a small bag with chestnuts in it [...] They work in pairs. First they decide how many chestnuts they will have in their bag [...] Now one pupil takes out a number of chestnuts and the other pupils have to ”guess” how many chestnuts are left in the bag.

In the example below, the connection between the proposed prerequisite (pupils have calculated with plus and minus) and the exercise given is relevant but not sufficient. Multiplication is just as important here in order to be able to take part in the exercise.

Connection vague (4)

St22 T1 I trust that pupils earlier have calculated with plus and minus ...

[...]

I write a number on the board, for example 351 ... I tell the pupils that the 3 stands for three hundreds, the 5 for five tens and the 1 for one unit. So it actually says: $3 \cdot 100 + 5 \cdot 10 + 1 \cdot 1$ ...

Finally, the student teacher in the example below (St27) describes clearly that no prerequisites are needed. However, if pupils are supposed to *inscribe something* on a piece of the clay (see below) that makes sense; they need to do more then just copying a symbol. Translations from the pupils' own number system into the Maya system is required for the exercise, which in turn requires certain prerequisites.

Connection irrelevant (3)

St27 T1 Prerequisites: Pupils do not need any direct prerequisites

[...]

> I will start my lesson by telling a story about the number system ... Where do our numbers come from...India – Babylon – Egypt – China – Maya.
>
> [...]
>
> Pupils will get a piece of clay to sculpt and they have to inscribe something (date of birth, a math problem) on it using the twenty number symbols of the Maya Indians. I have clearly marked the different symbols on the board ...

Results: four critical features

As stated before, the analysis of the tests identified critical features that were enacted in the following seminar in order to validate the critical feature. Future teachers must discern these critical features to be able to give relevant connections, as a sign of understanding MKT. Below, the four critical features will be illustrated using test results, and transcriptions of the seminars and of the analysis.

Formulating goals

The first critical feature was identified after the first pre-test and concerned the student teachers' attitude towards curricular documents. When stating their goals for a lesson, a majority of the student teachers either did not state any goal at all for the lesson, or they did the opposite, they copied the whole curriculum and stated that as their goal for their lesson. The ability to refer and relate to the curricular documents in such a way that it supported their lesson plan was seen as a critical feature here. The student teachers had to understand how to specify goals for a lesson in order to be able to make proper connections between the five elements. Contrast as a pattern of variation was used to discern the skill of *formulating goals*.

The test results concerning the occurrence of the element "curricular documents" show that the student teachers improved during the learning study. In the first test 21 % of the student teachers did not relate to the curriculum at all (code 0), whereas 43 % related to the curriculum in an indirect way (code 1). The remaining 36 % related directly (code 2). In the last test only one student teacher referred to the goals indirectly (code 1), while the remaining 98 % were classified as code 2 – direct description. However, the coherence between the elements is of more interest. In table 2 below, this increase is notable and the number of relevant connections between curricular documents and other elements increased from only 9 % in test 1, to 71 % in the last test. Here we also have to keep in mind that the possible connections only contained 49 % of all connections in test 1, but contained 77 % of all connections in the last test.

Table 2. *Occurrence of codes 3, 4, 5 concerning the element "Curricular documents" for test 1 and test 6, in percentage of total connections per test*

Codes	Test 1	Test 6
3 Connection *irrelevant*	36	4
4 Connection *vague*	55	25
5 Connection *relevant*	9	71
Total	100	100

Detailed descriptions

The second critical feature identified was called *detailed descriptions* and followed on the first critical feature. The analysis showed that in the cases where student teachers had described the five elements in detail, the connections were much stronger than in cases where the elements were described less clearly (or not at all). In the seminar, the students were acquainted with a form containing a structured mind-map (Rystedt & Trygg, 2005) where only the headings appeared. The student teachers were supposed to fill in detailed information concerning specific exercises. For example, three questions concerning the goal of the lesson were stated: what should be learned, why should this be learned and how should this be learned. These three questions gave the student teachers the opportunity to relate the choice of hands-on materials to the goal (what) and even to models of explanation (how).

This critical feature might seem to be a more general version of the first critical feature, which was only applied to one element, but the team decided to see these as two critical features. Their reason was that the first critical feature was so specifically related to the documents student teachers used that the second feature involved more of a change of attitude towards the curricular document and how to use it in class. As for the other elements, the alternative to state "the whole" was not an option. There is no document describing "the whole hands-on materials". The student teachers had to make up their own descriptions, and when they were able to specify in detail and describe accurately, the connections between other elements fell into place. For example, it seemed that the choices they made for hands-on materials after deciding upon an exercise became more relevant.

Shift in perspective

The teacher educators believed that there would a relation between the student teachers' involvement in an exercise and their ability to look upon the exercise from a mathematics teachers' perspective. Two shifts in perspective were required here. In the first shift, student teachers had to shift from being a mathematics learner to being a mathematics teacher. The second shift had to do with their views on their own role as a teacher. They did not seem to look upon themselves as mathematics teachers but as (general) teachers. The analysis of the tests showed that the student teachers did not focus on the mathematics in their lessons plans. Considerations were often of a general teaching methodological nature. These considerations might be useful, but they were unrelated to the learning of mathematics. Similar instances can be found in the video recordings. In the transcript below, three student teachers are discussing the role of the teacher, while the pupils play a game.

StA You role as a teacher? Well, to explain the game.

StB In a better way than the written rules.

StA Yes, and making up your own rules is perhaps not a good idea for small children.

StB Exactly.

StC Then you walk around and listen so there isn't a pupil doing calculations for all the others ...

StA Mm.

StB Exactly.

StC To speed things up. Everybody should get the chance to think.

StA Mm.

StB Making sure that everyone is taking part [in the game].

(from transcript second seminar)

Their arguments are pedagogical and relevant. However, as the focus of the seminar is on the teaching and learning of mathematics in education, the teacher educators who initiated the discussion expected another focus. In the analysis of the seminar this conversation was related to the results on the tests and on this basis the team started to look for instances during the seminar session where the students had been given the opportunity to distinguish between their role as a teacher (pedagogue) and their role as a mathematics teacher. The team realised that they had been addressing this issue implicitly and that they needed to create opportunities for students to be able to distinguish between the two roles. The teacher educators had not made the shift from teacher to mathematics teacher explicit, and therefore the student teachers did not recognise the shift. For the teacher educators, this shift was implicitly embedded in the question, but the analysis showed that an explicit clarification of this critical feature was necessary in order for student teachers to be able to understand the object of learning. Instead of asking about the role of a teacher in a game, the question asked should concern the shift: what is the role of a *mathematics* teacher in this *mathematical* game?

Mathematical knowledge

The last critical feature identified concerned mathematical knowledge although it was the focus of the study and was not tested. However, the analysis of the tests indicated that the student teachers' mathematical knowledge influenced their ability to understand the object of learning.

In the second seminar, the student teachers were asked to discuss the learning opportunities in an exercise on the different number sets. Three student teachers are having a discussion on mathematics instead. Before they can start discussing the learning opportunities of the exercise, they have to sort out what each number set stands for.

StA	A Rational number is like the square root of ... fractions right?
StB	No.
StC	No, they are irrational.
StB	Irrational ... rational ones are like fractions, kind of.
StA	Rational ones are fractions?
StB	Yes.
StA	Yeah, right and then irrational, the square root of...
StC	Hmm.
StA	That's maybe what we have, the square root of and powers, in some way.

(from transcript second seminar)

The research question aimed at identifying critical features concerning the understanding of the coherence between the elements. Four critical features were identified: curricular documents, detailed descriptions, shift in perspectives, and mathematical knowledge. In the following, each of these critical features will be linked to the MKT framework and implications for teaching MKT will be stated.

Implications

The critical features found imply necessary conditions for student teachers to distinguish between in order to understand the object of learning. These critical features add to recent research (Suzuka et al., 2009) and are a step in defining the relatively undefined area of knowledge in the teaching of MKT (Ball et al., 2009).

The importance of curricular knowledge for teachers has been emphasised by Niss (2004), was described as one of the required competencies, by Grossman (1990) and it is also part of the MKT framework. Recently, Land and Drake (2014) emphasized the use of the curriculum in pre-service courses. The way student teachers have to relate to these documents does not become clear from either the competency requirements or the domain in MKT. As this critical feature showed, the student teachers have to be given the opportunity to develop an attitude towards the documents that helps them to identify goals but also to develop the ability to define related lesson goals. During the learning study, this critical feature was made visible by contrasting the old curriculum: LPO94 (still valid at the time of study) with the new curriculum: LGR11. Comparing the differences in formulation and the consequences for teaching enabled the student teachers to discern the importance of the formulation of goals.

Such explicitness was included in the second critical feature as well, but covering other aspects. This critical feature did not involve the choice and reformulation of existing documents, but the student teachers' own wording of their thoughts. The structured mind-map forced the student teachers to write in detail, a tool they then could use and transfer to other exercises and other areas in mathematics.

The lack of shift in perspectives came as a surprise to the teacher educators, as they took for granted that during their 20-week course, the focus was on mathematics. Explicitly stating *mathematics* made the student teachers aware of both perspectives coexisting simultaneously. When teacher educators where aiming at domains described in the MKT framework during their exercises and discussions, the student teachers reflected in terms of pedagogical knowledge. Recently, Palmer (2013) found similar results when asking her students about their work as a teacher. These different identities have to be made explicit to student teachers so they can make decisions from the appropriate perspective.

The importance of mathematical knowledge has been addressed in a recent study by Adler et al. (2014) in which student teachers' notion of mathematical knowledge was studied. The issue of mathematical knowledge requirement was also addressed by Koellner et al. (2007) by including a phase in their problem solving cycle where teachers solve tasks before planning a lesson containing these tasks. The fourth critical feature here showed that common content knowledge could be regarded as a foundation of MKT. Student teachers with poor common content knowledge will not have the same opportunities to develop their MKT as student teachers with rich common content knowledge. This does not mean that common content knowledge automatically indicates MKT, but that there is the chance to develop it in a larger area.

MKT as an object of learning is too big for just one learning study, but the results of this study explain parts of the teaching of MKT as requested (Askew, 2008; Elliott et al., 2009). The teaching of MKT has to be researched as study after study shows that teachers do not possess enough MKT. What ways of teaching MKT are fruitful? What ways are not? Sometimes the change lies in one single word: reflect upon your role as a *mathematics* teacher ...

References

Adler, J., Hossain, S., Stevenson, M. & Clarke, J. (2014). Mathematics for teaching and deep subject knowledge: voices of mathematics enhancement course students in England. *Journal of Mathematics Teacher Education*, 17 (2), 129–148.

Askew, M. (2008). Mathematical discipline knowledge requirements for prospective primary teachers and the structure and teaching approaches of programs designed to develop that knowledge. In P. Sullivan & T. Wood (Eds.), *The international handbook of mathematics teacher education. Volume 1. Knowledge and beliefs in mathematics teaching and teaching development* (pp. 13–36). Rotterdam: Sense Publishers.

Ball, D. L. & Hill, H. (2008). *Mathematical knowledge for teaching (MKT) measures*. Ann Arbor: University of Michigan. Retrieved from http://sitemaker.umich.edu/lmt/files/LMT_sample_items.pdf

Ball, D. L., Hill, H. C. & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 29 (1), 14–17, 20–22, 43–46.

Ball, D. L., Phelps, G. & Thames, M. H. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59 (5), 389–407.

Ball, D. L, Sleep, L., Boerst, T. & Bass, H. (2009). Combining the development of practice and the practice of development in teacher education. *Elementary School Journal*, 109 (5), 458–476.

Bommel, J. van (2012). *Improving teaching, improving learning, improving as a teacher: mathematical knowledge for teaching as an object of learning* (Ph. D. thesis). Department of Mathematics, Karlstad University.

Bommel, J. van & Liljekvist, Y. (2008, November). T*esting the same group again and again: an alternative design for a learning study*. Paper presented at the World Association of Lesson Studies International Conference 2008 (WALS 08), Hong Kong, China.

Delaney, S., Ball, D. L., Hill, H., Schilling, S. & Zopf, D. A. (2008). Mathematical knowledge for teaching: adapting U.S. measures for use in Ireland. *Journal of Mathematics Teacher Education*, 11 (3), 171–197.

Elliott, R., Kazemi, E., Lesseig, K., Mumme, J., Carroll, C. & Kelley-Petersen, M. (2009). Conceptualizing the work of leading mathematical tasks in professional development. *Journal of Teacher Education*, 60 (4), 364–379.

Fauskanger, J., Jakobsen, A., Mosvold, R. & Bjuland, R. (2012). Analysis of psychometric properties as part of an iterative adaptation process of MKT items for use in other countries. *ZDM*, 44 (2), 387–399.

Grevholm, B. & Anthony, G. (Eds.). (2010). *Teachers of mathematics: recruitment, retention, professional development and identity*. Linköping: SMDF.

Grossman, P. L. (1990). *The making of a teacher: teacher knowledge and teacher education*. New York: Teachers College Press.

Koellner, K., Jacobs, J., Borko, H., Schneider, C., Pittman, M. et al. (2007). The problem-solving cycle: a model to support the development of teachers' professional knowledge. *Mathematical Thinking and Learning*, 9 (3), 273–303.

Land, T. J. & Drake, C. (2014). Understanding preservice teachers' curricular knowledge. In J.-J. Lo, K. R. Leatham & L. R. Van Zoest (Eds.), *Research trends in mathematics teacher education* (pp. 3–22). Cham: Springer.

Marton, F. & Ling, L. M. (2007). Learning from the "learning study". *Journal of Research in Teacher Education*, 14 (1), 31–46.

Marton, F. & Pang, M. F. (2004). On some necessary conditions of learning. *Journal of the Learning Sciences*, 15 (2), 193–220.

Marton, F. & Tsui, A. (2004). Classroom discourse and the space of learning. Mahwah: Lawrence Erlbaum Associates.

Niss, M. (2004). The Danish "KOM" project and possible consequences for teacher education. In R. Strässer, G. Brandell, B. Grevholm & O. Helenius (Eds.), *Educating for the future: proceedings of an international symposium on mathematics teacher education* (pp. 179–190). Gothenburg: National Centre for Mathematics Education.

Palmer, H. (2013). *To become – or not to become – a primary school mathematics teacher: a study of novice teachers' professional identity development* (Ph. D. thesis). Växjö: Linnaeus University Press.

Pang, M. F., Linder, C. & Fraser, D. (2006). Beyond lesson studies and design experiments: using theoretical tools in practice and finding out how they work. *International Review of Economics Education*, 5 (1), 28–45.

Runesson, U. (2005). Beyond discourse and interaction. Variation: a critical aspect for teaching and learning mathematics. *Cambridge Journal of Education*, 35 (1), 69–87.

Runesson, U. (2006). What is it possible to learn? On variation as a necessary condition for learning. *Scandinavian Journal of Educational Research*, 50 (4), 397–410.

Rystedt, E. & Trygg, L. (2005). *Matematikverkstad: en handledning för att bygga, använda och utveckla matematikverkstäder.* Gothenburg: National Centre for Mathematics Education.

Suzuka, K., Sleep, L., Ball, D. L., Bass, H., Lewis, J. & Thames, M. H. (2009). Designing and using tasks to teach mathematical knowledge for teaching. In D. Mewborn & H. S. Lee (Eds.), *In scholarly practices and inquiry in the preparation of mathematics teachers* (pp. 7–23). San Diego: Association of Mathematics Teacher Educators.

Notes

1 LMT: Learning Mathematics for Teaching Project

Jorryt van Bommel

Jorryt van Bommel is a lecturer at Karlstad University and works mainly with in-service and pre-service teacher training. Her research interests concern mathematics teacher training and professional development of mathematics teachers.

jorrbomm@kau.se