Teachers' mathematical knowledge for teaching in relation to the inclusion of history of mathematics in teaching

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This article discusses how the inclusion of history of mathematics in mathematics education draws heavily on a teacher's mathematical knowledge for teaching, in particular horizon content knowledge, in the context of curricular changes. We discuss the role of history of mathematics in school curricula, its inclusion in textbooks and its consequences for the mathematical knowledge needed for teaching. We address the matter from three national settings (Denmark, Norway and the United States). These settings exemplify how, in particular, teachers' horizon content knowledge needs to be broader than what is necessary for only the current curriculum.

When talking about mathematical knowledge for teaching (MKT) we often find ourselves returning to the question of "what is mathematics?" From a historical and epistemological point of view, what mathematics is has changed and developed through time and space. Hence, also what is to be considered mathematical knowledge for teaching varies from country to country, over time periods, and in between reforms according to what the content and goals of the school subject "mathematics" is at a particular time and place.

One example of this concerns whether elements of the history of mathematics have a role to play in the school curriculum. In some countries, history of mathematics is part of the school curriculum while in others it has never been; some places it is part of mathematics teacher education whereas in other places it has never been so; in some particular schools the mathematics program is structured around history and philosophy of mathematics (see e.g. Fried, 2001) while in most schools these topics are rarely or never touched upon. This affects the kind of mathematical knowledge that is needed for teaching.

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Smestad, B., Jankvist, U. T. & Clark, K. (2014). Teachers' mathematical knowledge for teaching in relation to the inclusion of history of mathematics in teaching. *Nordic Studies in Mathematics Education*, 19 (3-4), 169–183.

It was recently argued by Mosvold, Jakobsen and Jankvist (2014) that development of practically every domain of MKT may benefit from the study of history of mathematics. In our opinion, teachers' horizon content knowledge (HCK) is the domain of MKT, which has the most to benefit from the study of history of mathematics. In this particular article, however, we address the relationship between horizon content knowledge and history of mathematics from the "opposite direction", i.e. we discuss how the inclusion of history of mathematics in mathematics education draws heavily on a teacher's horizon content knowledge in the situation of curricular changes. More precisely, we look at the introduction of (elements of) history of mathematics in Danish upper secondary school; in Norwegian primary and secondary school; and in US high school (since the US is the origin country of the MKT framework). We discuss the role of history of mathematics in school curricula, its inclusion in textbooks and its consequences for the mathematical knowledge needed for teaching. Although we address the matter from different national settings, the study of this article is not as such a comparative one. Instead, we use these examples to discuss the role of HCK in times of curricular change. First, we turn our attention to the theoretical aspects of MKT and HCK.

MKT and HCK

Many mathematics education scholars have lamented the fact that there is too little research on the "relationship between teachers' mathematical knowledge and the quality of teaching and learning in secondary school mathematics" (Goos, 2013, p. 973). The quality and quantity of the mathematical knowledge that will improve teaching and learning has yet to be determined. Ball and her colleagues have examined the question of, "what do teachers need to know and be able to do in order to teach [mathematics] effectively?" (Ball et al., 2008, p. 394) extensively. Their work has led them to first identify four domains of mathematical knowledge for teaching within Shulman's (1986) original categories: *common content knowledge* (CCK) and *specialized content knowledge* (SCK) within the category of subject matter knowledge, and *knowledge of content and students* (KCS) and *knowledge of content and teaching* (KCT) within the category of pedagogical content knowledge. Furthermore, Ball and her colleagues also identified an additional category within each of the initial ones: *knowledge of content and curriculum* (KCC) (within pedagogical content knowledge) and *horizon content knowledge* (HCK) (within subject matter knowledge).

Horizon content knowledge (HCK) is defined as that which makes teaching more skillful "when teachers have mathematical perspective on what lies in all directions, behind as well as ahead, for their pupils, that can serve to orient their navigation of the territory" (Ball & Bass, 2009, p. 11). According to Mosvold et al. (2014), most attention has been given to teachers' perspective on mathematical content that lies ahead, where "ahead" in MKT context is usually interpreted as "later grades." But it should be noticed that Ball and Bass (2009) argue that horizon content knowledge also includes knowledge of what lies behind, and that "behind" here may be taken to mean also orienting instruction to the discipline and making judgments about what is mathematically important. That is to say, horizon content knowledge is in fact connected to "what mathematics is", both as a taught and as a practiced discipline.

Many authors (at least since Heppel, 1893) have noted how working on history of mathematics may help people see that mathematics is not "fixed and ready-made". Or as phrased by Siu and Siu (1979, p. 563) that "mathematics-in-the-making" is very different from "mathematics-as-an-end-product." Ball (1993) also noted how Schwab argued that teaching should be informed by what it means to "know" something and how ideas develop, which could also be enriched by studying history of mathematics. Later in the same article, Ball claimed, "I try to focus on significant mathematical content and I seek to fashion fruitful representational contexts for students to explore. To do this productively, I must understand the specific mathematical content and its uses, bases, and history [...]" (Ball 1993, p. 394).

Attempting to extend and clarify the definition of Ball and Bass (2009), Jakobsen, Thames, Ribeiro and Delaney (2012) proposed a more practice-based definition of HCK:

> Horizon Content Knowledge (HCK) is an orientation to and familiarity with the discipline (or disciplines) that contribute to the teaching of the school subject at hand, providing teachers with a sense for how the content being taught is situated in and connected to the broader disciplinary territory. HCK includes explicit knowledge of the ways of and tools for knowing in the discipline, the kinds of knowledge and their warrants, and where ideas come from and how "truth" or validity is established. HCK also includes awareness of core disciplinary orientations and values, and of major structures of the discipline. (p. 4642)

As evident from this quotation, and in particular the remark of "where ideas come from," the knowledge of the history of mathematics may play a key role in developing teachers' horizon content knowledge. Ball et al.

(2008, p. 395) talk about knowledge for teaching beyond the "obvious" knowledge of "topics and procedures that [teachers] teach" as well as "how teachers need to know that content" and how one might seek to "determine *what else* teachers need to know about mathematics and how and where teachers might use such mathematical knowledge in practice" (p. 395, emphasis added). This "what else" fits naturally in the domain of HCK. In her empirical study on pre-service mathematics teachers, Clark (2012) argued that studying topics in the history of mathematics indeed contributes to the "what else" described by Ball et al.

MKT in times of curricular change

Curricular changes – many of which are considered innovations in light of improved standards for all students – are an inevitable aspect of the mathematical education of students. Manouchehri and Goodman (1998) noted that, "research has shown [...] that teachers' content and pedagogical content knowledge influence how they teach and evaluate content" (p. 27). Manouchehri and Goodman investigated middle school teachers' evaluation and implementation of standards-based curriculum materials over a period of two years. Among other interesting results, they found that teachers struggled with buying into the role that the new, standards-based materials would have in their classroom teaching, especially after years of successfully teaching mathematics using familiar methods and materials. The background and experience of the teacher played a prominent role in teacher decision-making with regard to how and what to implement from the new curricular materials. And, not surprisingly, the teacher's orientation to teaching, the strength of their content knowledge, the flexibility of their pedagogical content knowledge, and their beliefs about student ability and needs contributed to the effectiveness of implementation. That is, there is often a great division between what is intended and what is actually implemented as a result of new reforms.

Whereas Manouchehri and Goodman (1998) analyzed results with respect to teacher knowledge divided into content knowledge and pedagogical content knowledge, more recent investigations examine the integration of two instructional resources when teaching mathematics: teacher knowledge and curriculum materials. Charalambous and Hill (2012) utilized the MKT construct to:

> [...] capture the knowledge teachers need to effectively undertake the various tasks involved in teaching this subject, including knowing why an algorithm works, defining mathematical terms appropriately for students at a particular grade level, selecting and

> using representations and examples, analysing student errors, and evaluating student non-conventional ideas. (p. 445)

Charalambous and Hill emphasized that it is the enactment of curricula that matters. And, the strength of a teacher's MKT (although Charalambous and Hill did not delineate the role of any one of the six domains within the construct) is an important force in this enactment.

Recapitulating the problématique

To recapitulate, as already mentioned the study of history of mathematics may contribute to a teacher's mathematical content knowledge, including HCK, which has been argued and illustrated elsewhere (e.g. Clark, 2012; Mosvold et al., 2014). On the other hand, concrete inclusions of history of mathematics in mathematics education calls for an already well-developed horizon content knowledge of a teacher, which we shall refer to as *a priori HCK* (including also HCK in relation to history of mathematics, but not only). Since mathematics teachers are rarely very acquainted with the history of mathematics (in Scandinavia and the US, at least), what may we expect to happen when new textbooks and curricula make them teach elements of it? Clearly, common content knowledge (CCK) and knowledge of content and curriculum (KCC) are not sufficient in this situation, since these are the elements under revision. It is the "what else" of horizon content knowledge that seems to be called for here – the a priori HCK. (Our reason for referring to this as "a priori" is that an inclusion of history of mathematics in a teacher's practice may itself contribute to this teacher's HCK – which might then be referred to as "a posteriori" HCK.)

Let us turn to the three cases of selected curricula and textbooks from Denmark, Norway and the United States where the inclusion of history of mathematics provides us with illustrative examples of such inclusion, at least in the transition phase from one curriculum to another, and how it draws upon a teacher's MKT, and in particular HCK.

History of mathematics in Danish upper secondary school

History of mathematics has been in and out of Danish upper secondary school curriculum for decades; beginning with a brief mentioning of it back in 1953, the introduction of it as one of three so-called aspects in 1987, and most recently with the reform initiated in 2005, where it to a much larger degree became an integral part (for a detailed discussion, see Jankvist, 2008). The rhetoric of the latest inclusion of history rely on that from the Danish KOM-project (KOM is a Danish abbreviation for "Competencies and mathematics learning"). This mentions three second-order mathematical competencies, or co-called types of overview and judgment, one of these concerning "the historical development of mathematics, both internally and from a social point of view", it reads:

> The object of this form of overview and judgement is the fact that mathematics has developed in time and space, in culture and society. This form of overview and judgement should not be confused with a knowledge of "the history of mathematics" viewed as an independent topic. The focus is on the actual fact that mathematics has developed in culturally and socially determined environments, and subject to the motivations and mechanisms which are responsible for this development. On the other hand it is obvious that if overview and judgement regarding this development is to have any weight, it must rest on concrete examples from the history of mathematics. (Niss & Højgaard, 2011, p. 74)

Having to choose such "concrete examples", teachers must have some knowledge of the history of mathematics – that is, they must have some actual subject matter knowledge about the development and history of mathematics as part of their a priori HCK (or CCK). In the actual upper secondary school curriculum, students are expected to "demonstrate knowledge about the evolvement of mathematics and its interaction with the historical, scientific, and cultural evolution" (Undervisningsministeriet, 2007, second author's translation), a demonstration drawing upon the curriculum's description of the "identity" of mathematics stating among other things that: "Mathematics has accompanied the evolution of cultures since the earliest civilizations and human beings' first considerations about number and form. Mathematics as a scientific discipline has evolved in a continual interrelationship between application and construction of theory" (Undervisningsministeriet, 2007).

This inclusion of history described above is one that relates to a use of "history as a goal" in mathematics education (Jankvist, 2009). Having to use history as a goal, teachers must not only be able to use concrete examples or cases, as argued above, they should also to some extent be able to select *exemplary* cases from the history of mathematics. As it is not the purpose to cover all of the history of mathematics as part of the upper secondary mathematics program, the cases chosen should ideally illustrate, on the one hand, the more general features of the "identity" of mathematics, as called for in the regulations, and on the other hand, assist in the development of students' overview and judgment about "the historical development of mathematics, both internally and from a social point of view", as called for in the KOM-report. Selecting such good, concrete, exemplary cases or examples certainly puts a teachers' a priori HCK in focus – as does deselecting textbooks' poor uses of history.

In Jankvist (2008) a detailed analysis and discussion is carried out of the inclusion of elements of history of mathematics in three textbook systems for upper secondary school mathematics, which came out in the wake of the reform. With a few exceptions, the inclusion of history in these books can be described as historical information in the form of minor remarks, e.g. in special colored boxes, often completely separated from the other mathematical content of the textbook, which makes it appear more or less "pasted on". Furthermore, when history is brought in it is very often quite anecdotal in nature, occasionally even bordering festive speeches, as in the case of "the first proof in mathematics" by Thales, where one of the textbooks state: "Imagine: the first proof in history. Here human thought really took a leap" (Jensen & Nielsen, 2005, p. 206, second author's translation). Clearly, making such statements and leaving the inclusion of history at that do not fulfill the curricular ambitions of "history as a goal". Fortunately, in the Danish tradition, upper secondary teachers need not necessarily be very bound to the textbooks in use. Often teachers will deviate from the textbook presentation of a given topic or concept, and occasionally they may also use their own or colleagues' modules in relation to a given topic. However, when it comes to including history, if the teacher is not very knowledgeable about this, he or she is likely to choose what the textbook offers in order to fulfill the curricular requirements. And as illustrated, trusting the textbooks in this matter, one may easily end up being ill off. Hence, again it is a matter of HCK, in particular, for the teacher to be able to separate the wheat from the chaff.

History of mathematics in Norwegian school

In Norway, as in Denmark, the place for history of mathematics in curricula has changed. In the 1974 and the 1987 curricula, there were no signs that students should learn about the history of mathematics (Kirke- og undervisningsdepartementet, 1974, 1987). In 1997, however, the new national curriculum explicitly included history of mathematics, and we will look more closely at the transition between the 1987 and the 1997 curriculum.

The very first sentence in the mathematics part of the 1997 curriculum was, "Man has from the earliest times wanted to explore the world around him, in order to sort, systematise and categorise his observations, experiences and impressions in attempts to solve the riddles of existence and explain natural relationships" (Veiteberg & Hagness, 1999, p. 165).

One of the six main goals for the subject was "for pupils to develop insight into the history of mathematics and into its role in culture and science" (Veiteberg & Hagness, p. 170). Furthermore, two of the subgoals were for students to "have some knowledge of the main features of some other cultures' numeral systems" (Veiteberg & Hagness, p. 178) and "experience the aesthetic aspects of geometry in practical examples taken from architecture, art and crafts and see this is [sic] cultural and historical contexts" (Veiteberg & Hagness, p. 178). There were also seven subgoals containing minor aspects of history of mathematics, for instance that students should "seek historical information on the sexagesimal system and see how it relates to time – days, hours, minutes and seconds, and to the division of the circle and the globe into degrees" (7th grade, Veiteberg & Hagness, p. 177). Thus, at that time in Norway, it could be argued that parts of history of mathematics were – or should be – part of the CCK domain, that is, mathematics that everybody was supposed to learn. This would also influence what would be the knowledge a mathematics teacher had to possess to be able to teach the subject as intended. Prior to the curricular change, however, the knowledge of history of mathematics must be considered part of a teacher's a priori HCK. To get an idea of how the curriculum was implemented, a look at the textbooks is helpful.

The historical content of the textbooks written for the 1997 curriculum was analyzed in Smestad (2002). The analysis showed that a few topics, in particular numeral systems and historical measurement units, were treated in many textbooks. It therefore seems that there was an implicit consensus that these topics should be known by all students (thus forming part of CCK). Most topics, however, are included in only one of many competing textbooks. Thus, the concrete examples are not prevalent enough to be considered common content knowledge, but authors of some textbooks considered them sufficiently relevant to the overarching goals of mathematics teaching to include them in the textbooks. Examples are:

- The story of Florence Nightingale, showing how mathematics was developed to save lives. Knowing that mathematics is connected to a struggle to improve human conditions is an important part of HCK.

- Information on sundials, which helps show how mathematics is connected to other sciences. Knowing that mathematics is connected to other sciences, such as astronomy, is an important part of HCK, as pointed out by Jakobsen et al. (2012).

- How history of mathematics can show connection to different cultures, e.g. the history of measuring units, numeral systems, algorithms and geometrical patterns – certainly also a part of HCK.
- Some examples from the history of mathematics point at both what lies "ahead" and what lies "behind" (Ball & Bass, 2009). In one textbook, the early history of probability is discussed, giving an insight into how one part of mathematics was developed: by collaboration, trial-and-error, or trying to understand each other's solutions. At the same time, the solutions and formulas they reached in quite a short time also forms part of the distant future for the pupils.

As in the Danish context the emphasis of including history is closely related to a use of "history as a goal" (Jankvist, 2009), since the above examples from textbooks exemplify how history of mathematics illustrates mathematics' connections to human attempts to improve human conditions, mathematics' connections to other sciences, mathematics' connections to different cultures and the connections between the strategies of mathematicians of the past and present strategies. If a teacher is to maneuver within the above meta-issues of mathematics, neither a teacher's CCK nor her KCC is sufficient. If there is no real consensus about which historical examples to use and what to use them for, then the teacher will have to rely on her a priori HCK.

History of mathematics in US high school

For many middle and high school mathematics teachers, the course textbook is the curriculum (Tarr et al., 2008). And, as part of the state governance of educational programs and initiatives, the majority of states in the US employ some form of a textbook adoption process, which then allows districts within a state to select from "approved" materials. In the very near future, most states' textbook editions will be replaced with a "Common core state standards" edition, since 45 states and the district of Columbia have adopted the Common core mathematics standards (National Governors Association & Council of Chief State School Officers, 2010). Thus, natural questions concern what the presence of history of mathematics in current mainstream textbooks, particularly in a standards era in which history of mathematics is given no explicit role, is and what HCK is needed for a teacher to teach them?

For the purpose of this article, two two-volume textbooks from the Prentice Hall "Honors gold series" were examined. The first, *Prentice Hall algebra* (Charles et al., 2011a), contains 817 pages. However, of the 722 pages of content, only four pages contain a history of mathematics reference and each of these are merely superficial references to a historical context within an exercise, as in the exercise found on page 43:

> The Rhind Papyrus is one of the best-known examples of Egyptian mathematics. One problem solved on the Rhind Papyrus is 100 divided by 7 7/8. What is the solution of this problem?

In the second two-volume textbook (comprising 946 pages, 836 of which are content), *Prentice Hall geometry* (Charles et al., 2011b), "history" in the index refers to just nine pages. Again, each of these uses some sort of historical context within an exercise.

Although a brief glimpse into two of the textbooks produced by the largest school textbook publisher in the US only gives a majority view, it is telling of the exposure to history of mathematics high schools students receive in mathematics curricular materials. Such a superficial exposure falls short of the call for reform by the National Council of Teachers of Mathematics (NCTM) in the *Principles and standards for school mathematics*:

> *Mathematics as a part of cultural heritage*. Mathematics is one of the greatest cultural and intellectual achievements of human-kind, and citizens should develop an appreciation and understanding of that achievement, including its aesthetic and even recreational aspects.
>
> (NCTM, 2000, p. 4)

Clark (2014) argued that policy and mathematics education reform documents, such as the *Principles and standards* and the *Common core state standards*, provide rhetoric toward the potential for historical and cultural perspectives in teaching mathematics. These intentions, however, are abandoned before they get to the teacher level as a result of their omission in curricular materials as the textbooks discussed above and, sadly also, high-stakes assessments. However, with strong MKT – primarily a priori HCK, as well as the intersection of KCS and KCC – teachers are empowered if having to prepare historical elements on their own or when selecting already prepared materials and adapting these to fit their own teaching context. For example, a teacher's expertise in knowing his or her students, as well as curricular demands, strengthens the teacher's ability to design instruction to engage students in discussions about their understanding of historical methods and how they equate to what students understand as traditional algorithms. This aspect of a teacher's work also depends heavily on content background as well as "confidence in their knowledge of mathematics" (Manouchehri & Goodman, 1998, p. 38).

Concluding discussion

As mentioned previously, the three examples above all to some degree deal with curricular transition phases, the change in focus being the inclusion of elements of history of mathematics in new curricula and accompanying textbooks. Once elements of history of mathematics is part of the curriculum, these elements of history of mathematics are no longer to be considered as part of a teacher's horizon content knowledge, but as part of other sub-domains of mathematical knowledge for teaching, e.g. CCK and KCC. However, in the transition phase from one curriculum not including elements of history of mathematics to another which does, in-service teachers often lack the associated CCK, KCC, etc. And at this particular time, in this particular transition period while implementing the new curriculum and training in-service teachers, a priori HCK comes to play a more crucial role.

In all three countries, policy documents have provided rhetoric that can be used to argue for inclusion of history of mathematics in teaching mathematics. This language was more concrete in Norway than in Denmark and the US. The degree to which this rhetoric translated into historical content in mathematics textbooks, varied widely. In the US, the two chosen textbooks, which more or less define the curriculum, show little more than superficial references to a historical context. In Denmark, the textbooks analyzed varied, but the historical content was often anecdotic and not integrated. In other words, in Denmark and the US, the opportunities of the available rhetoric were largely missed in the textbooks. This puts focus on the importance and necessity of teachers having HCK to rely on. This is especially relevant when textbooks include dubious information and claims, as in the case of the Danish textbooks for upper secondary school. In Norway, there were several examples where history of mathematics played a bigger part. Some topics were included in all textbooks and could thus be seen as part of future CCK once the transition from the old curriculum to the new one, including history, is over.

Although Ball and colleagues probably never intended MKT to be regarded as a "static" entity, the impression we are left with upon reading papers relying on the notion of MKT is often the exact opposite, i.e. that several researchers seem to think of MKT as a universal and timeless notion. But as illustrated by the three cases of including history in mathematics curriculum, this is certainly not the case; MKT is highly dependent on time and place. More precisely, the three examples show that CCK differs between countries and times, and show that it is important to include several different countries (as well as a historical perspective), if the goal is to develop a theory of mathematical knowledge for teaching that is not overly tied to a certain time and place. At the same time, it appears to us that HCK is the one domain, out of the six, which is perhaps most resistant to changes in time and place.

Epilogue

In this article we have addressed the "problématique" of how a transition from one curriculum to another, exemplified by the inclusion of history of mathematics, may put an emphasis on teachers' horizon content knowledge during this period. As Charalambous and Hill (2012) observed in the case studies they compiled, an ambitious curriculum often imposes greater demands on teachers, and thus teachers will experience challenges in implementation (p. 461). However, we argue that equipping teachers with knowledge in history of mathematics assists in developing HCK, which in turn has the power to assist teachers in meeting the demands of ambitious, new, non-traditional, or innovative curricula. Further, the content in the history of mathematics is less prone to change than school curricula. As phrased by Mosvold et al. (2014, p. 56): "History of mathematics may provide teachers with a more stable foundation in relation to knowing only about the mathematical content being present in curriculum materials [i.e. CCK] at a particular time and place."

Of course, often the history of a subject also sheds light on connections to other fields that later developments have made us forget. However, one of the most important points is perhaps that history of mathematics provides us with a repertoire of *authentic examples* to use in the learning and teaching of mathematics (e.g. Jankvist, 2014), and that these examples may *motivate* not necessarily the students, but the abstract mathematical concepts and constructs themselves as well as their coming into being (e.g. Pengelley, 2011). That is, history provides authenticity and meaning – aspects that should be part of every teacher's MKT. Hence, by including history of mathematics in both pre-service and in-service mathematics education courses, we may enhance teachers' MKT – and as part of this, in particular HCK – for the current curriculum, while also better preparing them for future curricular changes.

Acknowledgements

We wish to thank our reviewers as well as the guest editors of this thematic issue for very constructive criticism, which assisted us in focusing the article and more clearly convey our ideas.

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Kathleen Clark is an associate professor in the College of Education at Florida State University. Her primary research interests lie in two fields, mathematics education and history of mathematics. In the former, her research investigates ways in which prospective and in-service mathematics teachers use history of mathematics in teaching and the ways in which the study of history of mathematics impacts mathematical knowledge for teaching. In the latter, her historical research is focused on 17th and 18th century mathematics, with a particular emphasis on the early development of logarithms.

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