When does a variable vary? Identifying mathematical content knowledge for teaching variables

CECILIA KILHAMN

In what sense is x in the expression $x + 2$ a variable? What do teachers need to know about variables in order to create optimal learning conditions for students? The aim of this study is to understand the mathematical issues and demands of teaching the concept of variables, to outline a body of *Specialized content knowledge* for teaching (SCK). Data from two lessons in two Swedish grade 6 classrooms, with complimentary focus group interviews, were analysed using the *Mathematical knowledge for teaching* framework. Findings suggest some aspects of SCK to be an awareness of the different roles of the algebraic letter x in the expression $x + 3$, the equation $x + 3 = 8$ and the formula $x + 3 = y$, an appropriate use of the terms unknown and variable, and the importance of mathematical contexts for expressions.

What does the letter x in the expression $x + 2$ represent? Is it a variable? Does it vary? This study deals with issues raised in the algebra group at the Congress of European Research in Mathematics Education 2013[1] around a paper comparing two lessons on introducing variables held by different teachers revealing differences concerning the approach to algebra and the meaning of a variable (Kilhamn, 3013). The algebra group questioned if one of the teachers was teaching variables at all since she only chose examples with unknown numbers. Both teachers claimed they were teaching variables, building on the same curricula documents. The present study addresses questions about what knowledge of variables these teachers showed, and what knowledge of variables is needed for teaching. A re-analysis of the two lessons and complimentary data was carried out using the framework of MKT; *Mathematical knowledge for teaching* (Ball, Thames & Phelps, 2008). Instead of comparing the two lessons they are here seen as complementary examples, supplying a contrast that make issues of mathematical knowledge for teaching visible.

Cecilia Kilhamn,
University of Gothenburg

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Since research shows that students' problems with algebra are more related to learning conditions than to cognitive limitations (Cai & Knuth, 2011; Kaput, Carraher & Blanton, 2008), it is essential that teachers have the mathematical knowledge necessary to create optimal learning conditions. An assumption of the present study is that mathematical knowledge for teaching is important and makes a difference in the classroom, as outlined in theory by Ball et al. (2008) and generally shown in many previous studies of subject matter knowledge for teaching (for a thorough review see Hill et al., 2008). The study does not set out to prove a correlation (as for example done by Hill, Rowan & Ball, 2005), but to identify a distinct body of identifiable content knowledge that matter for teaching through analysis of classroom activities, as called for by Ball et al. (2008). The aim of this study is to *understand the mathematical issues and demands* arising in the teaching work as featured in video observations and to *outline a body of specialized content knowledge for teaching variables*.

Mathematical knowledge for teaching as a theoretical frame

Building on Schulman's (1987) categories of knowledge described as *Content knowledge* (CK) and *Pedagogical content knowledge* (PCK) several frameworks have been developed to describe and measure knowledge for teaching mathematics (Kaarstein, 2014). One of these is *Mathematical knowledge for teaching* (MKT), defined as "the mathematical knowledge needed to carry out the work of teaching mathematics" (Ball et al., 2008, p. 395). In the MKT framework the PCK domain includes *Knowledge of content and students* (KCS), *Knowledge of content and teaching* (KCT) and *Knowledge of content and curriculum* (KCC). The CK domain includes *Common content knowledge* (CCK), *Specialized content knowledge* (SCK) and *Horizon content knowledge*. Since the MKT framework is still young the domains of knowledge may not be sufficiently elaborated or exclusive. Ball et al. (2008, p. 401) writes: "That we are able to work empirically as well as conceptually helps us to refine our categories; still, we recognize the problems of definition and precision exhibited by our current formulation". While acknowledging some of the problems related to the definition of categories and how these are operationalized, as for example reported in Kaarstein (2014), this article will focus attention to the category Specialized content knowledge (SCK), in this case specific knowledge about the concept of variable that is necessary for teaching.

Ball et al. (2008) describe SCK as; reasoning, insights, understanding and skills related to mathematics but not needed in other settings than teaching. Such knowledge include for example the ability to understand the source of a mathematical error (not simply spot it) and to choose numbers, examples and representations that strategically highlight mathematical ideas students need to distinguish. In an example given by Ball et al. (2008) SCK is described as the knowledge needed to analyse a student error from a mathematical point of view (when and why did it occur, what steps were taken, what assumptions made?), whereas KSC is valuable in order to be prepared for the occurrence of that particular error among the students. When choosing appropriate representations a teacher uses SCK to judge in what way different examples and representations will make mathematical ideas explicit, whereas knowing how and when they are to be deployed effectively is part of teachers KCT (ibid).

School algebra and the concept of variable

School algebra is a large research area with diverse definitions of algebra. Several reports in the 1990's showed that school algebra was at the time predominantly rule based and procedural (Kieran, 1992), and researchers suggested a broader approach including generalisation, modelling, problem-solving and functional perspectives (Bednarz, Kieran & Lee, 1996). While mathematicians already in the 12th century highlighted the relational rather than the representational aspect of algebra (Subramaniam & Banjerjee, 2011), the invention of algebraic notation in the 16th century had a strong impact on the development of mathematics. Today the learning of algebra can be seen as both learning to reason about relationships and structures, and learning the formal symbolic language of algebra. Algebraic reasoning is prominent in literature about early algebra (Cai & Knuth, 2011; Kaput et al., 2008). An example of knowledge within the KCT domain concerns the relationship between algebraic reasoning and algebraic notation in instruction. Rojano (1996, p. 61) advises against "placing symbolic manipulation as an object of learning in advance of situations that can give rise to it", and Russell, Schifter and Bastable (2011, p. 63) argue that "[students] need to spend a good deal of time articulating general claims clearly in words and then connecting these statements to arguments based on representations".

The term variable was introduced by Leibnitz (1646–1716) to represent a varying quantity linked to the notion of function (Philipp, 1992), but was given a new definition after the introduction of set theory. In a modern Swedish dictionary of school mathematics a variable is defined as a "quantity that may assume any value within a given set" [2] (Kiselman & Mouwitz, 2008). According to Usiskin (1988) different approaches to algebra need different definitions of variable, as shown in table 1. In some curricular texts the term variable is used on a meta level incorporating all uses of letters in algebra (Kieran, 1989), while in others it is used on

Table 1. *Different definitions of variable (Usiskin, 1988)*

Approach	Variable
generalized arithmetic	pattern generalisers
problem solving	unknowns, constants
study of relationships	arguments, parameters representing quantities that vary

a more specific level representing only quantities that vary (Cai, Moyer, Wang & Nie, 2011).

Several studies deal with how students interpret letters in algebra (e.g. Asquith, Stephens, Knuth & Alibali, 2007; Küchemann, 1981; MacGregor & Stacey, 1997). In terms of the MKT framework, knowledge of common student conceptions and difficulties fall within the domain of KCS. Some of these categories of meaning are considered misconceptions although they are correct in some contexts, e.g. ignoring the letter (correct in calculations when the letter indicates a unit; 5m meaning 5 meters), assigning a particular value to the letter (correct when a letter denotes a constant such as e and π), interpreting the letter as an abbreviated word (common in geometric formulas, for example r for radius). Three categories of meaning where the letter represents a number or quantity are correctly used in school algebra, these are:

- a specific (unknown) number
- a generalised number representing several (or any) values
- a proper variable representing a range of values used to describe a relationship.

For the sake of clarity in this article the term *variable* will be used for variables representing a range of values and the term *unknown* will refer to specific unknown numbers, described above as a variables within a problem solving approach to algebra. Letters used to represent variables and unknowns are henceforth called *algebraic letters*.

The term ”acceptance of lack of closure” (Collis, 1975) has been used to describe students' reluctance to accept algebraic expression, e.g. $n + 1$, as representing a quantity without being reduced to a single number by carrying out an operation. In this study algebraic letters appear in algebraic expressions, equations and formulas, making these concepts closely linked to the concept of variable. In accordance with how the teachers in the study apply these terms, *equation* is here used for an equation with one unknown, *formula* for an algebraic equality including two or more variables and *expression* for of a string of numbers, operations and algebraic letters without an equal sign. Since we know that students struggle with algebraic expressions and the meaning of variables in expressions (Bush & Karp, 2013), these concepts are central components in mathematical knowledge for teaching.

Method

Data was collected as part of an international video study[3], where four consecutive introductory algebra lessons were video recorded in several classrooms in each of four countries. Data was collected *in situ* using three cameras focussing teacher, whole class and one student group. Informed consent was retrieved from all participating students and their parents. In addition, several teacher interviews were made including a video recorded focus group interview. As preparation for the focus group interview each teacher was given access to videos of his/her classroom and asked to choose one or more episodes, featuring some aspect of algebra teaching, to share and discuss with the other teachers. The Swedish data involved eight teachers in grades 6 and 7, divided into two focus groups representing diversity in experience and workplace. The group discussion evolved from questions each teacher had posed along with his/her episode, moderated by a researcher.

The analysis builds on data from the first lesson in two of the Swedish grade 6 classrooms, students age 12. The two teachers, Ms B and Ms C, were both trained as generalist teachers for grades 1–7 with approximately 10 years of teaching experience. They were both confident in their teacher profession and expressed interest in professional development as mathematics teachers. Like most grade 6 teachers in Sweden they taught almost all school subjects and were not specialised mathematics teachers. There were 30 students in Ms B's class and 20 in Ms C's class. Both teachers referred to the same textbook and the national curriculum as resources in their planning. The two lessons were chosen for this analysis because they were both planned around the same textbook unit on introduction to variables, and yet a previous analysis had shown that the two lessons were quite different concerning the approach taken to algebra and the meaning of variable conveyed by the teacher (Kilhamn, 2013). Contrasting and comparing the two lessons highlighted issues related to the teaching of variables. Verbatim transcripts were made of the teacher camera videos. Transcripts presented in the article have been translated. Names are excluded using T for teacher and S for student. The two teachers took part in different focus groups and both teachers had chosen their introduction to variables as the episode they wanted to discuss. Data from the focus group interviews concerning these two lessons was also transcribed and included in the analysis, bringing up additional information about the teachers' views on content knowledge needed for teaching variables.

Method of analysis

In accordance with Ball et al. (2009) attempts were made to understand the *mathematical issues and demands* arising in the teaching work as featured in the video recorded episodes without evaluating the teacher's work. Issues can be related to what happens in the classroom and how students react to what goes on, but also to teachers' reflections before, during and after the lesson. Teaching work raises many demands on teacher knowledge and action, and in the flow of action captured by video it is possible to pinpoint such demands and reflect on what knowledge teachers need in order deal with them. Often such knowledge is easy to identify when critical situations appear in the classroom. The analytical approach used is adopted from Ball et al. (2009) and unfolds in three steps:

1. Observing and analysing video documented classroom instruction to identify the work of teaching the content ”variables”.
2. Analysing *mathematical* issues and demands arising in the classroom teaching work and the teachers own reflections from focus group interviews.
3. Considering the knowledge implied by those issues and demands.

The analysis focussed on the teachers' choice of examples (a prominent aspects of SCK) and use of mathematical terminology and algebraic notation. Questions asked to the data were: What aspects of variable come to the fore in the classrooms as a result of teachers' choice of examples and use of terms and notation? What demands on teachers' mathematical knowledge is seen as the activities unfold and what do the teachers themselves bring up as difficult when discussing the content in these episodes?

Results

First a detailed description of each teacher's lesson is presented and then mathematical issues and demands raised in these two lessons are analysed and discussed.

Ms B's classroom – "x is nothing it is just what his age is called"

Ms B's lesson is 42 minutes long, of which 20 are spent in whole class interaction. During whole class interaction Ms B uses the term *variable* on five occasions and *expression* twice. In her introduction Ms B approaches algebra as if it were a language of symbols students learn through activities of generating and translating expressions. The lesson starts with references to previous tasks where a variety of symbols were used in place of a number and the term variable is introduced as something that varies (excerpt 1).

Excerpt 1. Ms B's introduction

T: [00:08:20] Today we will get into what we talked about where one uses letters instead of boxes, lines, teddys, bananas, apples ...

S: cars.

T: dolphins, chickens, well whatever we agreed we've written before, and where one actually, practically, gets to use it. And you have been working with this a tiny, tiny, tiny bit. That thing when I told you, create an equality, and, create an equality where there's a number we don't know anything about. For example, if I write on the board here. [T writes: "Some number added to 2"]

T: That can be, it took a while to write that, didn't it?

S: mm.

T: It can be done much, much simpler. Does anyone have a suggestion, how I could do this? To make it convenient for me?

[Two suggestions are written on the board by two students:

S1 writes: "Some number = 2, $x - 2$"

S2 writes: "$x + 2$" [at which S1 realizes his mistake]

T: Do you agree that this is quite practical? And math is very much about actually finding practical ways of doing things. And now, you'll do tasks where we'll, simply use this way ... variables.

T: Over here I don't know yet, what it is ... this x, x can be anything, so far. I could have written like this:

[T writes: "$x + 2 = 5$

$x + 2 = 7$

$x + 2 = 8$"]

T: Are all these x:s the same?

S: No.

T: There's a nice word in Swedish: it varies. And that's the word variable for you. So that, where we've always said squares or x or z. That's why they are called variables.

After this introduction Ms B presents the first task, taken from the textbook (figure 1) with the instructions to solve it together in groups (excerpt 2).

Osmond is 3 years older than Mohammed.
Leyla is 5 years younger than Mohammed.

How old is Osman when Mohammed is
a) 10 years b) 15 years c) 30 years

How old is Leyla when Mohammed is
a) 10 years b) 15 years c) 30 years

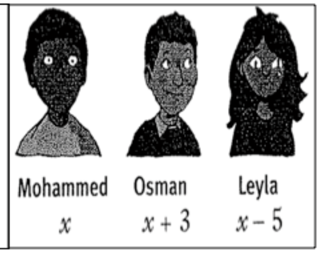

Figure 1. *First task on introduction to variables (Carlsson, Liljegren & Picetti, 2004, author's translation)*

Excerpt 2: Ms B's first task

T: [00:12:58] Now you're supposed to use variables to describe what you are going to calculate, simple as that. You are supposed to arrive at this. Here is something about what I said, what a variable is.

[T points to the task on the board and reads out aloud. Then the students start working. After a short time T interrupts to give additional instructions.]

T: I want you to fill in how you have reasoned using a variable. And keep in mind that I want not just one, I want a twofold result, one could say. I want to know *how old* Osman is, but I also want to find out *how you got hold of* Osman's age.

All students initially solve the task without including x, declaring: ”it's in the text”. All the solutions are shared on the board. Three groups show an arithmetic solution. One group has put x as the sought age: $10 + 3 = x$, $15 + 3 = x$, $18 + 3 = x$. One group shows equations similar to the example in the introduction: $x + 3 = 13$, $x + 3 = 18$, $x + 3 = 33$. Two groups have added the expressions found in the book and come up with solutions seen in figure 2.

Both groups struggle to understand the role of x in the expression. The group who deduces that $x = 0$ says at the start of their discussion: ”that [10], is Osman's age and that [$x + 3$] is someone else's age, it's Mohammed's”. After ten minutes discussion they conclude that x is equal to zero: ”[...] because x is nothing it is just what his age is called.” Ms B does not follow up these students' reasoning in the whole class discussion, but goes back to the task, indicating that they should use the expressions

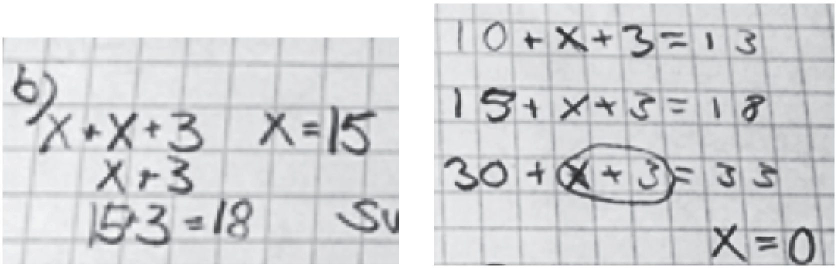

Figure 2. *Two student groups' solutions in Ms B's lesson*

under the pictures. After the introduction the students work with similar tasks translating between words and algebraic expressions. Ms B tells the students to use variables when writing expressions.

Ms B's reason for choosing to share this introduction in the focus group was that she is dissatisfied with the lesson and does not think she gave the students a good understanding of the concept of variable, that she ”only gave them the word”.

Ms C' classroom – ”Investigate how old she will be when I'm forty”

Ms C's lesson is 57 minutes long, of which 12 are spent in whole class interaction. During whole class interaction Ms C uses the term *variable* on seven occasions, *expression* twice and *formula* once. In her introduction Ms C approaches algebra as if it were a tool to model and express functional relations. Ms C starts the lesson reminding the students of equations and introducing the term variable as something that varies (excerpt 3). She describes her family and age relations between the family members (figure 3).

Excerpt 3. Ms C's introduction

T: [00:03:07] We have talked previously about, equations, and we've talked about the equality sign and the like, and today we'll start talking about something that's called variable, and variable reminds us of, variation for example. Thus it is something that varies. So, the word variable has to do with that. [...] And now I'm going to tell you about my family, and how old we are, in my family. Firstly we have my dad, he's called Max.

[T draws the family members and writes their ages, see figure 3]

T: Now then, I will, describe our ages based on a variable. I will describe it with a, with letters and numbers. And I will base it on myself all the time, but one doesn't have to do that. Right now I will start from myself. And I will describe my dad's age right now. And I'll name us in the family after our letter in our name, because we have different letters here. [...] There! My dad's age I'll describe now. So I'm after Max's age. And Max's age, equals [T writes $M =$] and I'm always starting from me now, Jane's age [T writes J] and, is he older or younger than me?

[T calculates the difference 63 – 36 = 27 and completes the formula $M = J + 27$]

T: For me to describe dad's age I'll take my age and then I'll add years, because he's older than me. This, my, since J means 36 right now. So I add 27 there. That means that one can calculate using this formula, if one knows that I'm 36, then, that dad is 36 plus 27, which is 63. And we can also by looking at this understand how old my dad will be when I'm 40. When I'm 40, then the same formula holds, he's always 27 years older. So then you get 40 in there. How old is my dad when I'm 40?

[T continues in a similar way to generate $L = J - 13$ and calculates L when $J = 40$

M = J + 27	**M***ax*	**A***nn*-**C***atrin*	**J***ane*	**E***lsa*	**A***nne*	**L***isa*
L = J – 13	*63 yrs*	*60 yrs*	*36 yrs*	*32 yrs*	*29 yrs*	*23 yrs*
			63 – 36 = 27			
			36 – 13 = 23			

Figure 3. *What Ms C wrote on the board during the introduction*

After the introduction the students are asked to describe their own families in a similar fashion (excerpt 4). Some of these are shared on the board, see examples in figure 4.

Excerpt 4. Ms C's first task

T: [00:09:39] I'd like you to draw your family members in your maths books. Make stick figures like these. [...] I'd like you to make up a, that is, you write down a variable and start from your family. For example: mom's age, equals, my age, plus ... Can you write a variable down based on your family?

In this activity students ask questions concerning what letters to use, but not about the role of the letter in the formula. All students write formulas describing a relationship between two variables. In one example the formula has three variables ($M = T + E + 14$) and is only true at one point in time, so the letters represent specific numbers, whereas in all the other formulas the letters are variables that represent a range of positive values. This distinction does not come up in the whole class interaction.

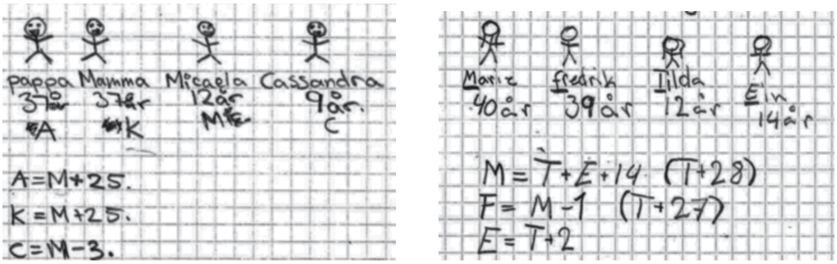

Figure 4. *Two student's work from Ms C's class*

Later in the lesson the students work with textbook tasks where they are asked to translate word statements into algebraic expressions. Here Ms C uses the term secret number instead of variable (excerpt 5).

Excerpt 5: Ms C explains expressions

T: [00:17:20] If I say a sentence, you write that sentence down with numbers and possibly letters. For example like this, "four less than *y*". How does that look, if you write it with numbers and letters and plus or minus or times or divided? Four less than *y*, how would I have written that down? What would I start with?

S: *y* minus four.

T: Do you follow that? Then it is four less than *y*. And *y* can be different things, it's a secret number. This is called expressions.

Ms C's reason for choosing to share this episode in the focus group was her own uncertainty about the concept of variable, since she does not teach this content very often.[4]

Mathematical issues and demands

The teacher's choice of examples highlights the varying aspect of a variable, and the different roles an algebraic letter plays in an expression, an equation and a formula. There is a demand on the teacher to be aware of the range of possible values a variable can assume in various examples and how that range is related to the context of the given examples. The teacher's use of mathematical terminology and algebraic notation highlights the difference between the use of the term variable to mean only a proper variable or the more general use of the term including also unknowns. In the following section these issues will be elaborated, adding teachers' comments on the episodes during the focus group interviews.

Choice of examples: what does varying imply?

Both teachers use the idea of *varying* to describe a variable. We can see from the conclusion about x in figure 2 that students in Ms B's classroom struggle to understand the role of x. Ms B's first example is the expression $x + 2$ describing the statement "a number added with 2". Without further information it is unclear if the number has a specific value (a specific number the teacher is thinking about) or if it represents a general number. To illustrate the idea of varying, Ms B gives three equations: $x + 2 = 5$, $x + 2 = 7$ and $x + 2 = 8$. In each of these equations x has a specific value. In the interview the teachers point out that "the answer" varies along with the value of x. The idea of varying can only be discerned if the three equations are viewed as a whole, as three examples of the same relationship ($x + 2 = y$), where "the answer" is also made explicit as a variable. Likewise, the textbook task in figure 1 describes relationships through expressions without making the dependent variable explicit. When the students deal with the task they either ignore the algebraic expressions or misinterpret the letter x. To identify the art of the incorrect interpretation of x leading to the conclusion that $x = 0$ (figure 2), the teacher needs to be able to see what mathematical reasoning could lead to such a conclusion and how that reasoning deviates from the intended.

Ms C chooses to illustrate the *varying* aspect of the variable by describing age relations in her family through formulas that are valid for a range of values (excerpt 3). A letter represents the age of each family member as in the formula $M = J + 27$ derived from the specific case of $63 = 36 + 27$. The formula is then used to calculate M when J is 40. Making both dependent and independent variables explicit affords an opportunity for the students to see a range of possible values and the letter as representing them all.

In both of the teacher focus groups the aspect of context is brought up several times. It is suggested that the varying aspect of x in the expression $x + 2$ would have been easier to see if the expression had been placed in a context. Concerning Ms C's introduction one teacher comments that the context makes it meaningful to talk about the constant difference and the varying ages. She says it is perhaps not the choice of context as such (age relations), but the way the bridge is built from numbers to a general expression that will enable students to see it as meaningful. These comments indicate a shift from an extra-mathematical context (ages, prices) to an intra-mathematical context (being part of a formula relating two variables). In the formula $M = J + 27$ the letter J has the meaning of being 27 less than M. A subtle difference between the examples given in the two introductions (figure 5) is the fact that Ms B does not supply a context for, or a representation of, the functional relationship $x + 2 = y$, whereas

$x + 2$	
$x + 2 = 5$	$63 = 36 + 27$
$x + 2 = 7$	$M = J + 27$
$x + 2 = 8$	$67 = 40 + 27$

Figure 5. *Ms B's and Ms C's examples of a varying variable*

Ms C provides both context and explicit algebraic representation of the relationship $M = J + 27$ to illustrate the roles of both dependent and independent variable.

When Ms C's introduction is discussed in the focus group, there is some disagreement concerning the example $P = E + 36$. One teacher claims that there are two variables, but Ms C describes only E as a variable since P is the unknown and depends on E. She agrees that P varies depending on E, but does not speak of it as a variable. The terms dependent and independent variable do not appear in the discussion. When the question is posed what students might see as the variable, Ms C reflects that since she is not clear about how she uses the term they could interpret the whole formula as being the variable.

Use of mathematical terminology and notation

Both teachers use the new term *variable* sparsely and somewhat reluctantly. Ms B never distinguishes between a variable and an unknown. Ms C does not explicitly point to the difference between a variable and an unknown, but when students work with expressions in the textbook she talks of a *secret number* rather than of a variable (excerpt 5), indicating a specific number, as yet unknown to those not included in the secret. Nor does the textbook make a distinction between an unknown and a variable. In many of the textbook tasks the role of the letter is undefined, as in ”an expression meaning 5 times x”. This highlights the demand for clarity in the use of mathematical terminology, and the teachers' uncertainty becomes apparent. One teacher suggests that ”In an equation the variable has a specific value”, which would mean that a variable does not necessarily need to vary and an unknown in an equation is also called a variable.

Conclusion

The teachers in this study have enough common knowledge of variables (CCK) to move flexibly between the interpretation of an algebraic letter as an *unknown*, as in $5 = x + 2$, and as a *variable* representing a range of possible values, as in $y = 2 + x$, without specifying which interpretation they use in each particular moment. They do not make mathematical mistakes or fail to solve the tasks included in the textbooks. However, the explicit distinction between the two interpretations, and the strategic choice of examples that will highlight the two different roles of the algebraic letter are issues that the episodes described here bring forward as demanding specialised content knowledge (SCK). Likewise, the distinction between an *equation* with one unknown and a *formula* describing a relation between two or more variables is an example of SCK that the teachers discuss and that comes to the fore when contrasting the two lessons. Learners will eventually realise that a formula like $y = 2 + x$ is transformed into an equation with one unknown as soon as one of the variables is set at a particular value, but in the process of learning about it learners need to be able to distinguish the two concepts explicitly and teachers therefore need to choose appropriate examples for that purpose. Connected to this distinction is the use of the term variable and *the varying aspect of a variable*. If the definition of variable is that something varies (as in both of these teaching examples) then the art of that variation and the range of possible values becomes important issues that the teachers in this study did not always seem to be aware of and never brought to the fore in the lesson. The varying aspect of a variable could be focussed in a discussion of the different roles of x in the expression $x + 3$, the equation $x + 3 = 8$ and the formula $x + 3 = y$.

The importance of *context* to give expressions meaning and to see the possible values a variable can assume describes another aspect of SCK. What extra-mathematical contexts will produce formulas with variables that vary and what contexts will produce equations with one unknown? And what role does the variable play when an expression is appears in an equation and in a formula?

Since the focus of the presented analysis was to identify SCK, other domains of MKT found in the material have not been presented. The focus group discussions brought up several issues related to the PCK domains showing that these teachers were well aware of some of the student and teaching issues described in previous research. For example the concern that the choice of letters as abbreviations in the examples given by Ms C could take the attention away from the letter as representing a value (Küchemann, 1981). All teachers described experiences of students' reluctance to "accept lack of closure" (Collis, 1975) and several teachers pointed out the affordances of expressing general statements using words before introducing algebraic notation (Russell et al., 2011).

The MKT framework proved useful to identify and make visible mathematical issues and demands for teaching variables in Swedish grade 6 classes and has helped to outline and describe aspects of SCK. These aspects are not surprising, see for example the discussion and conclusion in Phillip (1992), but have not previously been described as part of teachers SCK within the MKT framework. The inclusion of teachers' reflections in focus group interviews to complement the observed teaching validated the interpretations made concerning teachers' mathematical knowledge necessary for teaching. Aspects of SCK outlined here form an essential part of the body of Mathematical knowledge for teaching variables. This body of knowledge is undoubtedly much larger and differs according to the age and range of the students, for example in connection to the inclusion of letters as parameters in higher grades. The two lessons and focus group interviews form a small-case study but represent teachers with appropriate qualifications and a sufficient amount of teaching experience to have a reflective stance towards their teaching. The results can therefore serve as an indication that these aspects of SCK need attention in teacher training as well as in curricular development.

References

Asquith, P., Stephens, A., Knuth, E. & Alibali, M. (2007). Middle school mathematics teachers' knowledge of students' understanding of core algebraic concepts: equal sign and variable. *Mathematical Thinking and Learning*, 9 (3), 249–272.

Ball, D., Charalambous, C., Lewis, J., Thames, M., Bass, H. et al., (2009). Mathematical knowledge for teaching: focusing on the work of teaching and its demands. In M. Tzekaki, M. Kaldrimidou & H. Sakonidis (Eds.), *Proceedings of the 33rd conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31–47). Tessaloniki: PME.

Ball, D., Thames, M. & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of teacher Education*, 59 (5), 389–407.

Bednarz, N., Kieran, C. & Lee, L. (Eds.). (1996). *Approaches to algebra: perspectives for research and teaching*. Dortrecht: Kluwer Academic Publishers.

Bush, S. & Karp, K. (2013). Prerequisite algebra skills and associated misconceptions of middle grade students: a review. *The Journal of Mathematical Behavior*, 32 (3), 613–632.

Cai, J. & Knuth, E. (Eds.). (2011). *Early algebraization. A global dialogue from multiple perspectives*. Berlin: Springer.

Cai, J., Moyer, J., Wang, N. & Nie, B. (2011). Examining students' algebraic thinking in a curricular context: a longitudinal study. In J. Cai & E. Knuth (Eds.), *Early algebraization: a global dialogue from multiple perspectives* (pp. 161–185). Berlin: Springer.

Carlsson, S., Liljegren, G. & Picetti, M. (2004). *Matte Direkt Borgen 6B*. Stockholm: Bonniers.

Collis, K. (1975). *The development of formal reasoning*. University of Newcastle, Australia.

Hill, H., Blunk, M., Charalambous, C., Lewis, J., Phelps, G. et al., (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: an exploratory study. *Cognition and instruction*, 26, 430–511.

Hill, H., Rowan, B. & Ball, D. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, 42 (2), 371–406.

Kaarstein, H. (2014). A comparison of three frameworks for measuring knowledge for teaching mathematics. *Nordic Studies in Mathematics Education*, 19 (1), 23–51.

Kaput, J., Carraher, D. & Blanton, M. (Eds.). (2008). *Algebra in the early grades*. New York: Routledge.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York: Macmillan Publishing Company.

Kilhamn, C. (3013). Hidden differnces in teachers' approach to algebra – a comparative case study of two lessons. In B. Ubuz, C. Haser & M. Mariotti (Eds.), *Proceedings of the eighth congress of European Society for Research in Mathematics Education, CERME8* (pp. 440–449). Ankara: Middle East Technical University.

Kilhamn, C. & Röj-Lindberg, A.-S. (2013). Seeking hidden dimensions of algebra teaching through video analysis. In B. Grevholm, P. S. Hundeland, K. Juter, K. Kislenko & P.-E. Persson (Eds.), *Nordic research in mathematics education, past, present and future* (pp. 299–328). Oslo: Cappelen Damm Akademisk.

Kiselman, C. & Mouwitz, L. (2008). *Matematiktermer för skolan* [Mathematical terminology for school]. Gothenburg: National Centre for Mathematics Education.

Küchemann, D. (1981). Algebra. In K. M. Hart (Ed.), *Children's understanding of mathematics: 11–16* (pp. 102–119). London: John Murray.

MacGregor, M. & Stacey, K. (1997). Students' understanding of algebraic notation: 11–15. *Educational studies in mathematics*, 33 (1), 1–19.

Philipp, R. A. (1992). The many uses of algebraic variables. *The Mathematics Teacher*, 85 (7), 557–561.

Rojano, T. (1996). The role of problems and problem solving in the development of algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: perspectives for research and teaching* (pp. 52–62). Dortrecht: Kluwer Academic Publishers.

Russell, S., Schifter, D. & Bastable, V. (2011). Developing algebraic thinking in the context of arithmetic. In J. Cai & E. Knuth (Eds.), *Early algebraization: a global dialogue from multiple perspectives* (pp. 43–69). Berlin: Springer.
Shulman, L. (1987). Knowledge and teaching: foundations of the new reform. *Harvard Educational Review*, 57 (1), 1–22.
Subramaniam, K. & Banjerjee, R. (2011). The arithmetic-algebra connection: a historical-pedagogical perspective. In J. Cai & E. Knuth (Eds.), *Early algebraization: a global dialogue from multiple perspectives* (pp. 87–107). Berlin: Springer.
Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. F. Coxford (Ed), *The ideas of algebra, K–12* (pp. 8–19). Reston: NCTM

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Notes

1 http://cerme8.metu.edu.tr/index.html

2 Authors translation of ”storhet som kan anta värden i en given mängd”

3 Kilhamn & Röj-Lindberg, 2013

4 In Sweden teachers normally follow one class from grade 4 to grade 6, teaching most of the school subjects, thus revisiting introduction to variables every third year

Cecilia Kilhamn

Cecilia Kilhamn is a teacher and a senior lecturer in Mathematics Education in the Faculty of Education at Gothenburg University.

cecilia.kilhamn@ped.gu.se