Teacher-assisted open problem solving

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Previous research has developed several problem-solving models and suggested that the teacher plays a crucial role in guiding students' problem solving. However, less is known about the particularities of problem solving and teacher guidance when dealing with open problems which include multiple possible solution pathways. The aim of this study is to understand students' open problem-solving processes and teachers' ways of supporting them. Data collection involved videotaping one 9th grade mathematics lesson with two video cameras and capturing the screens of the students' computers. Seven student pairs worked on an open problem using GeoGebra under the guidance of a teacher trainee. We found that students had various kinds of problem-solving processes and that the teacher had a crucial role in guiding them. We elaborate on 9 ways how the teacher guided students to change between phases in open problem solving.

Previous studies have investigated students' problem-solving processes and described phases that students go through when solving problems (e.g., Mason, Burton & Stacey, 1982; Nunokawa, 2005; Pólya, 1945; Schoenfeld, 1985). Research on problem solving has also suggested that use of technology enhances the exploration of the problem (Healy & Hoyles, 2001). Furthermore, teachers' ways of guiding students' problem-solving processes have been studied (e.g., Anghileri, 2006; Martino & Maher, 1999). However, less research has focused explicitly on students' processes and teacher guidance in *open problem solving*. A problem is said to be open if it has at least one of the following properties: 1) the starting situation is open (Pehkonen, 1997), i.e., the solver has to make selections about what aspects of the problem are to be investigated, 2) the end product is open, with multiple correct answers to the problem (Pehkonen, 1997), and 3) the process is open, i.e., there are multiple correct ways to solve the problem (Nohda, 2000).

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Open problem solving is claimed to support creativity (Kwon, Park & Park, 2006). There is also some evidence that open problem solving enhances students' mathematics achievement (e.g., Boaler, 1998). Some studies suggest that the value of open problem solving is in getting students more involved and enriching their mathematical activity (Pehkonen, 1995; Sullivan, Warren & White, 2000). Fewer studies have explored the particularities of students' problem-solving processes in the case of open problem solving. Cifarelli and Cai (2005) investigated college students' unassisted open problem solving. According to them, students engage in making sense of the problem, posing problems, carrying out solution activity, and reflecting on results. Cifarelli and Cai emphasize that reflection on the result provides opportunities to formulate new problems.

Studies investigating the role of teacher guidance explicitly in open problem solving are rare. One such study is reported by Sullivan, Mousley and Zevenberger (2006). They found that teachers can support students by choosing appropriate open-ended tasks, by preparing prompts to support students experiencing difficulty, by posing extension tasks to students proceeding quickly, and by making classroom processes more explicit than usual. In the case of modelling tasks, which often include openness in the real world situation, Doerr and English (2006) found that teachers supported students in making sense of the task in realistic ways, in evaluating their models, and in creating and explaining representations.

The aim of this study is to understand students' open problem-solving processes and teachers' ways to support them in technology enriched classroom. In particular, we are interested in teacher-assisted students' movements between phases of open problem solving. The research questions that guided the study are: 1) How do students move between phases of open problem solving? and 2) In what ways can a teacher guide students to move between phases of open problem solving?

Theoretical framework

Problem-solving models

The most known mathematical problem-solving model is Pólya's (1945) model which consists of four phases: 1) *Understanding the problem*, 2) *Devising a plan*, 3) *Carrying out the plan* and 4) *Looking back*. Other researchers have further developed the model (Mason et al., 1982; Schoenfeld, 1985). In all of the proposed models there is a phase related to understanding the problem in which the solver has to figure out what actually is asked for in the problem and what conditions are given. In Schoenfeld's

(1985) model, this is part of the analysis-phase, where the solver examines special cases, simplifies the problem and re-formulates the problem as ways to try to make sense of the problem. Most of the changes suggested to Pólya's model, however, concern phases 2 and 3 above, which tend to give a too straightforward image of problem-solving process. Mason et al. (1982) emphasize the nonlinearity of the problem-solving process by claiming that the solver often has to move back and forth between *entry* and *attack* phases as the solver comes up with ideas, tries to implement them but often gets stuck and has to begin a new entry. Schoenfeld (1985) divides Pólya's phase 2 into *design* and *exploration* phases and emphasizes a cyclic movement between these phases. Design means explicit planning and controlling the solution process whereas in exploration phase the solver uses problem-solving heuristics, examines related problems, and might go back to the analysis phase (Schoenfeld, 1985). All the models also include a phase where the solution is checked or reflected upon in the end of the solving process.

A different perspective on modelling problem solving than the models mentioned above is given by Davis and Maher (1990) and Nunokawa (2005). Both problem-solving models emphasize cycling through gathering information from the problem situation and comparing that to the solver's existing mathematical knowledge. Iterative development is highlighted also in the modelling cycle in which students describe a real world situation in a mathematical model, manipulate the model in order to generate predictions, translate the results into the real world, and verify the usefulness of the predictions (Lesh & Doerr, 2003).

In open problem solving, the phase of understanding the problem is particularly important as solvers need to make selections about what aspects of the problem they are going to investigate. Reflection is also emphasized. Because the end product is open, solvers need to examine how reasonable their answer is and because the starting situation is open, solvers also need to reflect on the selections they have made and consider what other kind of selections they could make in the starting situation.

Teacher's guidance of students' problem solving

The problem-solving literature includes advice on how a teacher should guide students' problem-solving activity. Cai and Lester (2010) lists three main discourse factors that can help promote growth of students' mathematical understanding: a) providing students with enough time to work on the tasks, b) avoiding to remove the challenges from the tasks by telling or showing students how to solve tasks, and c) listening and asking thought provoking questions.

Instead of telling directly what to do, teachers can still introduce new information to support students' reasoning (Lobato, Clarke & Ellis, 2005). Also the idea of *scaffolding* is to support students but fade the support when students can manage without it (Pea, 2004). According to Anghileri (2006), scaffolding through direct interactions between the teacher and students includes traditional teacher explanation as well as reviewing and restructuring. In reviewing, the intention is to encourage reflection whereas restructuring aims to progressively introduce modifications to students' ideas (ibid.). According to Anghileri, scaffolding also focuses on conceptual thinking by developing representational tools, making connections and generating conceptual discourse.

Careful *questioning* can also promote students' reasoning (e.g., Sahin & Kulm, 2008; Martino & Maher, 1999). In particular, Martino and Maher (1999) note that students do not often spontaneously build justifications, and thus, the teacher's questioning is important in ensuring justification. It is also important to ask guiding questions that help students overcome challenges in solving the problem (Sahin & Kulm, 2008). Perhaps the most crucial factor in guiding students' problem solving, however, is listening openly to students (Davis, 1997) and attending to their reasoning (Francisco & Maher, 2011). According to Schoenfeld (1985), a teacher should also scaffold students' metacognitions by helping them to control and monitor their solution process.

Teachers should support students to engage deeper and deeper in mathematical investigations. Hähkiöniemi and Leppäaho (2012) have emphasised this by characterising three levels of teacher guidance: a) In *surface-level guidance*, the teacher guides the student from the teacher's own perspective without noticing a certain essential aspect of the student's solution, b) *Inactivating guidance* means that the teacher reveals the potential path of investigation related to the student's solution to the student, and c) In *activating guidance*, the teacher guides the student to investigate the essential aspect related to the student's solution.

In open problem solving, the teacher's role is particularly crucial in helping students to cope with the open nature of the problem. The teacher may need to help students narrow a multitude of pathways made possible by the openness of the problem situation. If the end product is open, it is important that the teacher guides students to reflect on the reasonableness of their solution.

Use of dynamic geometry software in problem solving

Dynamic geometry software (DGS) is claimed to enrich students problem solving. For example, Hölzl (2001) shows in his case study how students go beyond the use of DGS for verification purposes only.

Furthermore, according to a study by Healy and Hoyles (2001), using DGS "can help learners to explore, conjecture, construct and explain geometrical relationships, and can even provide them with a basis from which to build deductive proofs" (p. 251). DGS can promote students' mathematical problem solving in the same way as in experimental mathematics computers are used in 1) gaining insight and intuition, 2) discovering new patterns and relationships, 3) graphing to expose math principles, 4) testing and especially falsifying conjectures, 5) exploring a possible result to see if it merits formal proof, 6) suggesting approaches for formal proof, 7) computing to replace lengthy hand derivations, and 8) confirming analytically derived results (Borwein & Bailey, 2003).

Arzarello, Olivero, Paola and Robutti (2002) have used what they called *ascending* and *descending* processes to describe students' exploration with DGS. In ascending processes students move "from drawings to theory, in order to explore freely a situation, looking for regularities, invariants, etc." and in descending processes, students move "from theory to drawings, in order to validate or refute conjectures, to check properties, etc." (Arzarello et al., 2002, p. 67). These processes describe how students can use DGS to generate and verify conjectures (ascending) and then find reasons for why the conjectures are true (descending). According to Jones (2000), this way DGS can be used to promote the need for deductive justifications. Yet, despite the advantages of educational software, several studies have pointed out that students still need teacher's guidance to transit from verifying to explaining or from empirical work with software to deductive reasoning (Christou, Mousoulides, Pittalis & Pitta-Pantazi, 2004; Jones, 2000).

Methods

Data collection

The data of this study are a part of a larger study on teacher trainee's implementation of inquiry-based mathematics teaching led by the first author. In the study, teacher trainees were first taught principles of inquiry-based mathematics teaching. For example, the teacher trainees practiced how to guide students in hypothetical teaching situations (see, Hähkiöniemi & Leppäaho, 2012). Then, each teacher trainee implemented one inquiry-based mathematics lesson in grades 7–12. One of these lessons was built around the Amusement Park Problem (modified from Christou, Mousoulides, Pittalis & Pitta-Pantazi, 2005):

> Four towns will build together a magnificent amusement park. Investigate using GeoGebra what would be the most optimal and fair location for the amusement park.

The Amusement Park Problem is an open problem because the students have to think what an optimal and/or fair location means and how the towns could be located (open starting situation), there are several reasonable locations for the amusement park (open end product), and the students may use different GeoGebra tools and ways of reasoning to solve the problem (open process).

The lesson was implemented in grade 9 (age 15) and lasted 45 minutes. The students had computers and access to a webpage (http://users.jyu.fi/~mahahkio/huvipuisto) including a GeoGebra applet, where a new tool was added to GeoGebra. The students could use the new tool to compute the sum of distances from a point to four other points. The lesson followed three phase lesson structure which is general in inquiry-based mathematics teaching (see Stein, Engle, Smith & Hughes, 2008). In the *launch phase* (11 minutes) of the lesson, the teacher trainee introduced the 14 students to the use of GeoGebra software with some examples because the students were using GeoGebra for the first time. In the *explore phase* (23 minutes), the seven pairs of students tried to solve the Amusement Park Problem by using GeoGebra and the teacher circulated guiding them. In the *discuss and summarize phase* (11 minutes), the teacher trainee presented a review of the solutions invented by the students. Different solutions were discussed and evaluated with the whole class.

Data were collected by videotaping the lesson with two video cameras. One camera followed the teacher trainee who had a wireless microphone. The other camera followed one pair of students who also had a wireless microphone. In addition, the seven student pairs' computer screens were recorded using a screen capture software programme and students' written solutions were collected. Altogether, nine videos of the lesson were collected.

Data analysis

Data were analysed using Atlas.ti video analysis software. We prepared for the analysis by segmenting the lesson videos for launch, explore, and discuss and summarize phases. We also described the pairs' solutions and solution attempts and transcribed all the episodes in which the teacher discussed with each pair of students. Solutions were defined to result in a suggestion for the location of the amusement park whereas in solution attempts the students' engaged in task-related mathematical activity but did not reach an answer. After this, we repeatedly watched the students' solutions and solution attempts and tried to identify phases described in previous problem-solving models (Pólya, 1945; Schoenfeld, 1985). Although some phases such as exploration (Schoenfeld, 1985) could be found in data, none of the existing problem-solving models satisfactorily explained the students' actions. Thus, in combination of literature review and data analysis, we formulated a new set of phases that described the students' problem-solving processes (see table 1). Preliminary definitions of the phases were modified several times as we coded the students' solutions and solution attempts and negotiated differences among two coders.

Table 1. *The phases of open problem-solving processes*

Framing the problem	Students make selections about what aspects of the problem they are going to investigate. The selections that the students make may not be explicitly stated. In the Amusement Park Problem, the students made choices about possible locations for the four towns or about the criteria for assigning a location to the amusement park.
Exploring the solution	Students engage in task-related mathematical work in search for a conjecture for an answer to the problem. The exploration does not always result in a conjecture. In the Amusement Park Problem, the students, for example, used GeoGebra to draw perpendicular bisectors.
Conjecturing	Students suggest, i.e., conjecture a possible answer to the problem. The conjecture is not necessarily written down. In the Amusement Park Problem, the students suggested the location of the amusement park.
Investigating the conjecture	Students explain how they arrived to the conjecture or examine whether it is reasonable or not. In the Amusement Park Problem, the students, for example, explained how they used GeoGebra tools and how they found the location for the amusement park or measured distances from the conjectured amusement park to the towns.
Justifying	Students explain why their conjecture is a reasonable. Students' explanation can be more or less mathematical, but for them the explanation justifies the conjecture. In the Amusement Park Problem, the students, for example, explained that there is equal distance from the amusement park to the towns.

We summarized students' movement between the phases in diagrams (see figure 1). Based on all the different routes in students' problem-solving processes, we constructed a model for open problem solving (see figure 2). Then, we analyzed the teacher's role in the model. We identified those transitions between phases that were initiated by the teacher's actions. From each of these transitions, we examined the teacher's questions or other actions that might have triggered the students to change a phase. By comparing the teacher's actions in the episodes, we found differences and similarities among them. Through this analysis, we identified nine ways in which the teacher guided the students to change phases in their problem-solving process.

Results

Students' open problem-solving processes

All the students' conjectures for the location of the amusement park are presented in table 2. Altogether, the students presented 20 conjectures. The various conjectures illustrate that the problem really was an open problem. The students arrived at these conjectures through different routes as presented in figure 1.

Table 2. *Students' conjectures*

Conjectures	*f*
Center of a square (5 solutions) or a rectangle (2 solutions)	7
The intersection point of the diagonals when the towns are placed in the vertices of a non-symmetric quadrangle	1
The point where the total distance to the towns seems to be smallest	1
Midpoint of a segment connecting midpoints of diagonals	1
The towns are dragged in order to get the midpoints of diagonals to overlap and the amusement park is placed in this point	1
Intersection point of perpendicular bisectors of diagonals	2
Intersection point of two perpendicular bisectors (equidistance to three towns)	1
The midpoint of a circle that passes through three intersection points of the perpendicular bisectors of the four towns	2
The towns are on a straight line and the amusement park is placed on the midpoint of the outer most towns (1 solution) or on the midpoint of the inner most towns (1 solution)	2
The towns are on a straight line and the amusement park is placed on the perpendicular bisector of the outer most towns outside the segment connecting the towns	2

The students' problem-solving processes were different from each other. Their problem-solving processes did not always proceed linearly in order from phase to phase as in Cecilia's and Carol's solution 2 (figure 1). Sometimes students began directly to explore the solution or they skipped a phase (see, e.g., George & Gabriel in figure 1). In many cases, the students did not justify their conjecture, but continued to build another solution (see, e.g., Alice & Ann in figure 1). Sometimes the students constructed immediately the conjecture on the basis of framing the problem so that they could intuitively see the location of the amusement park (see, e.g., George & Gabriel in figure 1). Also, the students often returned to a previous phase (see, e.g., Ian & Irene in figure 1).

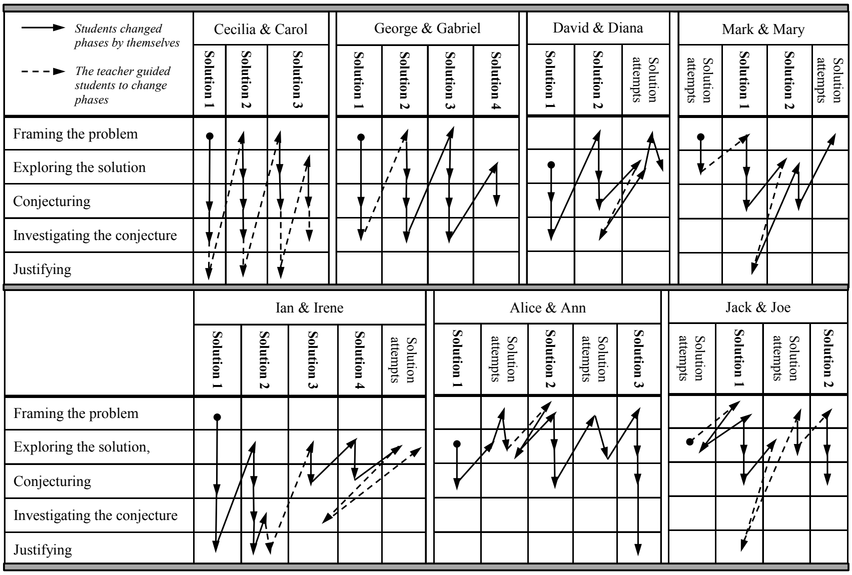

Figure 1. *Mappings of student pairs' open problem-solving processes*

Based on all the different routes in students' solutions and solution attempts, we constructed a model of open problem solving presented in figure 2. In the optimal route (full arrows in figure 2), a solver begins by framing the problem, goes through the phases, and continues to the next solution by framing the problem differently. The model also accounts for the possibility of returning to a previous phase and skipping a phase (dashed arrows in figure 2).

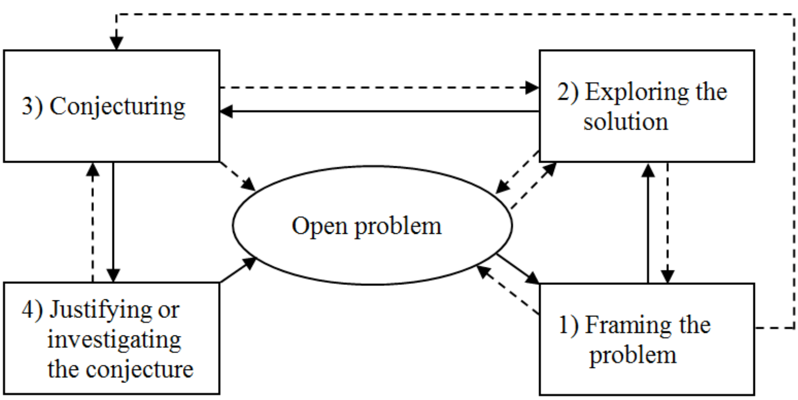

Figure 2. *An open problem-solving model.*

Teacher guides students to change between phases

As indicated by figure 1, the teacher had essential role in guiding the students to change phases in their solution processes. In this section, we elaborate on nine ways in which the teacher guided the students to change phases in their open problem-solving processes.

Teacher guides students to narrow the starting situation

The teacher often guided the students to frame the problem (see figure 1). For example, Mary and Mark had started to explore solutions without thinking how the towns are located. The students seemed to work randomly in the exploring the solution phase, but the teacher guided them to change to the framing the problem phase of solution 1 by asking them to locate the towns so that the problem is easy to solve:

Teacher: How is it going Mark and ...?

Mark: It's really not going at all.

Teacher: What do you have? Do you have some idea what you are trying to do here?

Mark: No. Just experimenting.

Teacher: Right. Well, it's not bad. Would it be a good idea to try something a little simpler? Now you have placed the towns a bit randomly. But if you would first look for, for example, a situation where the locations of the towns in relation to each other are simpler?

Mary: But these [towns] are already [located] quite simply.

Teacher: Move them so that it would be easier to solve first. Drag the points to such locations. [...] Solve first, for instance, some easy situation. And then change it more difficult.

After the teacher's guidance, the students dragged the towns so that they formed a rectangle. They found the location for the amusement park by drawing the intersection point of the diagonals of the rectangle. They justified the solution by writing "Most fair because the distance from the amusement park to each of the towns is the same". So, in this episode, the teacher helped the students return to framing the problem, which resulted in the pairs' first solution to the problem. It should be noted that the teacher did not tell the students how to exactly frame the problem, but let the students to think about this. The teacher guided the students to frame the problem so that it is easy to solve. This means that the teacher helped the students narrow the open starting situation of the problem.

Teacher guides students to widen the starting situation

Another type of guidance happened in situations where a student pair had already built a conjecture and investigated or justified it. Then, the teacher guided them to re-frame the problem differently, i.e., to widen the starting situation. For example, George and Gabrielle had built solution 1 in which the amusement park was placed in the midpoint of a square using the grid. The students had also drawn a circle through the towns and measured individual distances from the amusement park to the four towns. Then the teacher guided them to change to the framing the problem phase of solution 2 by asking them to move one of the towns:

Teacher: What if you move, let's say, the point C [a town]? [Students drag the point outside the circle.] [...] What would now be a smarter location? Can you find a point which would be equally far away from all of those [towns]?

Gabrielle: No. Or maybe you can. But ...

Teacher: Think about it for a while.

In the above episode, the teacher and the students utilized the dynamic character of GeoGebra and visualized by dragging possibilities for framing the problem. Although, in this situation the towns did not form any regular pattern, we considered this as framing the problem because the towns are purposefully placed in these locations.

Teacher guides students to use a reasoning-based strategy

In solution 2, Ian and Irene drew four perpendicular bisectors for four pairs of towns and placed the amusement park on one of the intersection points of these perpendicular bisectors. The students' explanation for selecting one of the intersection points of the perpendicular bisectors was based only on visual appearance of the figure. The teacher did not ask them any further about the criteria for selecting this location for the amusement park. Instead, the teacher guided them to change to exploring the solution phase of solution 3 by asking them to think about another solution method:

Teacher: Okay. Good. You are on the right track. Well. Could you find a way, sort of, to always find the point, the certain right point? Like, basically, these can locate, like this could be one centimeter toward this direction. Then, how would you invent the point in a handy way? Could you find a sort of general way of solving the problem?

Irene: Well, I don't know.

Ian: Well, if you put the perpendicular bisectors to each and every place.

Irene: Yeah.

Ian: And then the intersection point.

Teacher: Do it. Let's see what it will be. [...] [Ian draws perpendicular bisectors to five pairs of towns.]

Teacher: Now we have a problem. They all don't intersect in the same point.

Irene: So, it should be placed there, in the middle [points to the interior of a triangle formed by the three intersection points of the perpendicular bisectors].

Teacher: How do you get that?

Ian: With some circle.

Irene: Yeah.

Teacher: Try it. This is. Hey, this is amazing.

In this episode, the teacher asked the students to think about a solution strategy which is not dependent on these particular locations of the towns. Thus, the teacher tried to guide the students toward a solution strategy which is based on reasoning instead of empirical experimenting. The teacher also pointed out that there is a problem because the perpendicular bisectors do not meet in a single point. However, the teacher did not notice that it is very difficult to find a mathematical criterion for drawing the circle suggested by the students. Instead, the teacher could have tried to guide the students to investigate, for example, when all the perpendicular bisectors would intersect in a same point which could lead to rich mathematical reasoning.

Teacher guides students toward more precise solution

In their solution 3, Cecilia and Carol had used the tool which gave the total distance from the amusement park to the four towns. They had approximated a point where the sum of distances seemed to be a minimum when the teacher asked them to justify why they selected that point. Then the teacher guided Cecilia and Carol to change a phase from justifying to exploring solution by asking how the students could get the location of the amusement park precisely:

Teacher: Why this point?

Carol: Because we moved it and there it was smallest. [The sum of distances from the amusement park to the towns is 18.717. See figure 3a.]

Teacher: All right. Okay, good. But, how could you get it exactly? Now, it seems that you have estimated that it is there. How could you use your reasoning to get it just precisely to that point? [...]

Carol: Could we do the intersection of lines [intersection of diagonals of the quadrangle]?

Teacher: Try it.

Carol: It would be approximately there.

Teacher: Try it. [Students draw the diagonals. The amusement park is not in the intersection point.] Try now. Try now. [Students drag the amusement park to the intersection point.]

Students: It is! [The sum of distances from the amusement park to the towns is 18.687. See figure 3b.]

Teacher: Hey. Absolutely amazing!

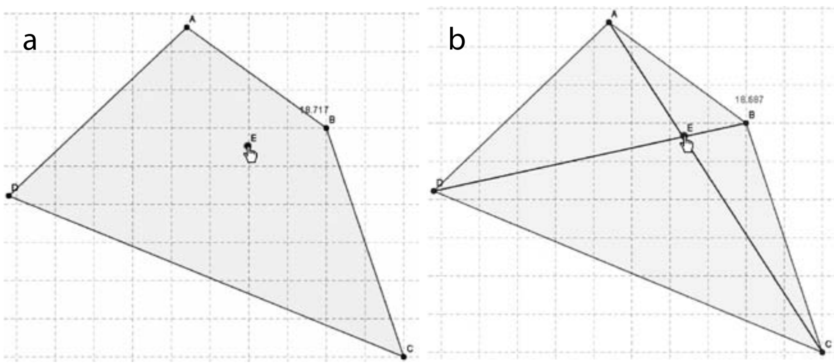

Figure 3. *Screen captures of Cecilia's and Carol's preliminary* (a) *and more precise* (b) *conjecture in solution 3*

In the above episode, the teacher changed the students from empirical justification to exploring the solution by guiding them to find a more precise location for the amusement park instead of just a rough estimation. Visual estimation gave the students the idea of drawing the diagonals because the amusement park seemed to be close to the intersection point of the diagonals. Using GeoGebra, the students found that the total distance was even smaller in the intersection point of the diagonals.

Teacher guides students to test a conjecture in different situations

The teacher also guided the students to change to investigating a conjecture by guiding them to test it in a different situation. As shown in the previous section, Cecilia and Carol had arrived to the conjecture that the amusement park is placed at the intersection point of the diagonals in which the total distance to the four towns is minimum (solution 3). Then the teacher guided Cecilia and Carol to change a phase from conjecturing to investigating the conjecture by asking them directly to "investigate whether it is always like that, if you drag this". After the teacher left the students they dragged one of the towns in different location, then dragged the amusement park to different locations and noticed that the total distance is again minimum in the intersection point of the diagonals (see figure 4). Cecilia and Carol repeated this testing with two more different quadrangles.

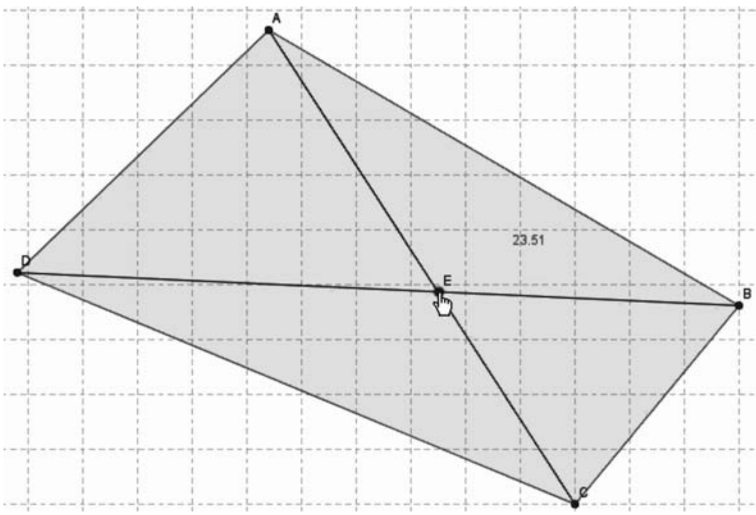

Figure 4. *One of the quadrangles that Cecilia and Carol used to test their conjecture*

Finally, based on the same result with four quadrangles, which all happened to be convex, they wrote their answer: "The amusement park is in the intersection point of the segments connecting the towns and the sum of distances from the towns is as small as possible. It works with every quadrangle." Thus, by the teacher's suggestion the students utilized the dragging feature of the software and empirically tested their conjecture.

Teacher guides students to explain their solution

Often when students had built a conjecture the teacher asked them to explain how they arrived to their conjecture. For example, in solution 3, Ian and Irene had placed the amusement park in the midpoint of a circle that passed through three intersection points of the perpendicular bisectors of the four towns. Then the teacher guided them to change a phase from conjecturing to investigating the conjecture by asking them to explain and draw what they did:

Teacher: What do you have? Yeah. Okay. You drew a circle and you took the midpoint of the circle, which is the amusement park [reads written answer]. Yeah.

Irene: So, first we did the perpendicular bisectors to all of these and then it was something like that. And then we adjusted a circle to all the three intersection points and then the mid...

Teacher: Hey, would you draw it? If we would took a look at it? [...] [Students go backward with GeoGebra until the solution 3 is on the screen.] This would be the optimal point?

Irene: Yes. But I'm not sure.

Teacher: Well.

Ian: It could be a bit more to this direction.

Irene: Higher.

In this episode, the students explained that they had drawn perpendicular bisectors, selected three intersection points of the bisectors, drew a circle through the three intersection points and took a midpoint of the circle. The teacher also asked them to show the corresponding GeoGebra figure. In this case, the students also started to be critical about their conjecture. It would be hard to justify mathematically why the amusement park should be placed in the midpoint of the circle and the teacher started to guide them toward another solution method but then ran out of time.

Teacher guides students to reflect critically on a conjecture

The teacher also tried to initiate David and Diana to reflect critically on their conjecture (solution 2), in which the towns are on a straight line and the amusement park is placed on the perpendicular bisector of the outer most towns outside the segment connecting the towns:

Teacher: But is it then optimal, in a way, all the citizens have to travel there?

Diana: It is because it is fair to all of them.

Teacher: Well, we come to the thing that what is most optimal. But would it nevertheless be nice if all would perhaps have a short distance to there? So is it reasonable to build it there so that everybody has to [travel]?

Diana: It is.

David: Like Ideapark [shopping centre] was built in the middle of some forest.

Teacher: Well hey, this is ok solution, but then you can, if you don't want to look for other solutions, then what about the other situations then, if the towns are not in a line?

In this episode, however, the students held on to their conjecture and the teacher let them continue exploring another solution.

Teacher guides students to return to justification

Often students did not spontaneously justify their conjectures (see figure 1). Thus, the teacher had a crucial role in guiding the students to justify their conjectures. In some cases the teacher even returned the students from building another solution to justifying their previous conjecture (Jack & Joe and Mark & Mary in figure 1). For example, in solution 1, Jack and Joe placed the towns in the vertices of a square, then took the midpoints of the both diagonals, and placed the amusement park to the coincident midpoints. Then Jack and Joe continued to explore another solution but the teacher returned them back to the solution 1 by simply asking "What is your argument that this is the most optimal?" In this case, the students' argued that all the towns had equal distance to the amusement park. Similar thing happened when the teacher returned David and Diana as well as Ian and Irene to investigate the conjecture (see figure 1).

Teacher launches justification by listening and trying to understand

Another type of guidance happened when Cecilia and Carol initiated themselves a discussion with the teacher about their solution 2:

Students: How do we explain this [solution 2]?

Teacher: [Reads the written answer.] Hey, hey, amazing. So the park would be in? [...] How did you, hmm?

Carol: We did the segment, the segment and then the perpendicular bisectors for them and there is the intersection point of the perpendicular bisectors [points to the figure].

Cecilia: And then these have the same distance [points to the two towns connected by a segment] and these have the same distance [points to the other two towns connected by a segment].

Teacher: Okay. Would you draw them? Let's see. [The students draw their solution, see figure 5.]

Carol: We calculated the distances. To these two it is the same and to these two it is the same. Then it would be a kind of fair to all of the towns.

Teacher: Yeah, okay. Alright, so in your opinion it would be the most optimal location because all of them would have equal distance.

Carol: No. These two have smaller distance than these. But anyway, none of the towns has a longer distance in its own.

Teacher: Uhm. Uhm. Yes, I think that I understood. I think that I understood. [...]

Cecilia: But how do we explain this situation?

Teacher: Just like you explained to me before. [The students write their justification when the teacher asks them to write it.]

In this episode, Cecilia and Carol had a conjecture and presumably the teacher had difficulties to understand their conjecture. The teacher asked Cecilia and Carol to explain and draw their solution 2. The teacher even proposed the incorrect justification, but Cecilia and Carol did not accept this and explained again their justification. In this episode, the students and the teacher seemed to be equal "mathematicians" and the teacher valued the students' explanation by admitting that "I think that I understood". This episode illustrates the teacher's role in asking and listening students' explanations. Cecilia and Carol had already investigated the conjecture by measuring the distances shown in figure 5. However, the teacher's presence launched justification of the conjecture even though the teacher only had to listen to them and ask questions to understand their justification.

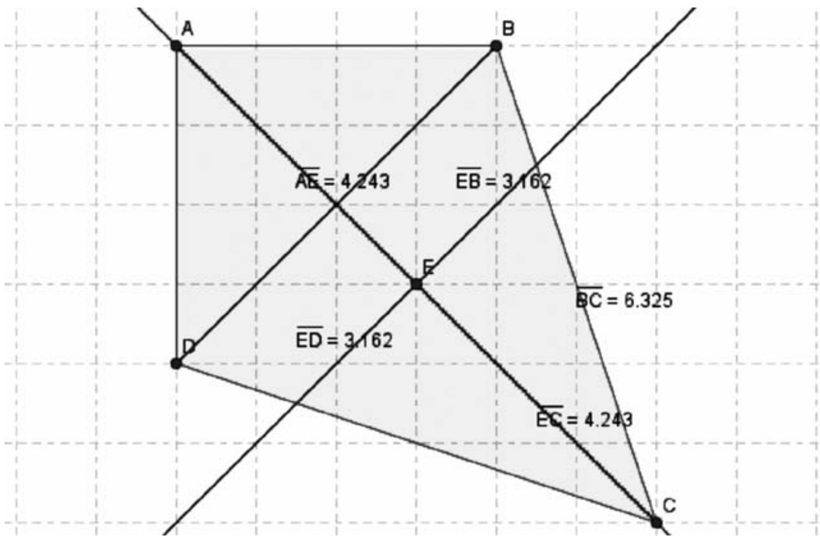

Figure 5. *Screen capture of Cecilia's and Carol's solution 2. The perpendicular bisector of BD happened to coincide with the diagonal AC*

Discussion

The students had various kinds of routes in moving between phases in their open problem-solving processes. They often skipped a phase or returned to a previous phase, which highlights the cyclic and recursive nature of problem solving (cf. Nunokawa, 2005; Davis & Maher, 1990; Schoenfeld, 1985). The results illustrate how open problem solving includes framing the problem in the beginning of a solution process. Thus, the students not only try to understand the problem but also participate in setting their own questions, an important aspect of mathematical thinking emphasized by Cifarelli and Cai (2005). In addition, after solving the problem, students continue by framing the problem differently. This highlights the very nature of open problem solving in which students are supposed to pose more general problems based on their previous solution (Nohda, 2000). Similar to modelling tasks (Lesh & Doerr, 2003), students had several different ways to approach the problem. Mousoulides (2011) used the same task in a modelling activity and found that students built models that resemble the solutions presented in this article.

According to the results, the teacher had a crucial role in guiding or scaffolding the students through the problem-solving process. Particularly, the teacher guided the students to change between phases. For example, the teacher helped the students change to framing the problem by helping them narrow or widen the starting situation. Narrowing the starting situation is similar to the well-known problem-solving strategy of solving an easier problem or looking for special cases, techniques that help problem solving (Schoenfeld, 1985). Narrowing also resembles to giving enabling prompts to students experiencing difficulty with the task, which Sullivan et al. (2006) found effective type of guidance. Both help students to start working with the problem. Sullivan et al. (2006) also described posing extension tasks to students proceeding quickly. This is similar to widening as in both cases students extend their previous thinking.

The teacher also helped the students change to exploring the solution by guiding them to use reasoning-based strategies and to look for a more precise solution. Using the terms of Anghileri (2006), the teacher tried to scaffold students' problem solving by restructuring their mathematical thinking. The teacher also had a crucial role in pushing the students to justify or investigate the conjecture by guiding them to explain their solution, reflect critically on a conjecture, test the conjecture in different situations and justify the conjecture. Some students were even in exploring another solution when the teacher invited them to return to justify their previous solution. This is further evidence that students need teacher guidance to transit to justifying their conjecture (Martino & Maher, 1999). Especially when DGS is used, a teacher needs to make sure that students also explain why their conjecture is true instead of just relying on empirical observation (Christou et al., 2004; Jones, 2000).

In most cases, the reported teacher's ways of guiding affected positively to students mathematical activity. Narrowing the starting situation helped the students build their first solution and to continue from that. Guiding to widen the starting situation, to search for more precise solution, and to test a conjecture all helped the students expand their thinking. Different ways of guiding to investigate or justify a conjecture resulted in students to look back on their solution and conceptualize their thinking. On the other hand, when the teacher guided the students to use a reasoning-based strategy and reflect critically on a conjecture, he was not successful in actually promoting students reasoning. Although we do not have empirical evidence, we believe that in other conditions also these two ways of guiding could affect positively to students' activity. All this is further evidence of the importance of teachers' scaffolding role in promoting students' problem solving (Anghileri, 2006; Martino & Maher, 1999; Sullivan et al., 2006).

The observed ways of guiding are related to the open nature of the problem. Guiding to frame the problem is specific to open starting situation in which students need to make selections. Open process emphasizes multiple strategies, and thus, is related to guiding to exploring the solution. When the end product is open, it is particularly important to examine how reasonable the achieved solution is. Similarly, Doerr and English (2006) found that certain teacher moves were related to specific features of modelling tasks.

We developed the open problem-solving model to describe students' movements between phases and to help examine teacher guidance specifically in open problem solving. We did not include at all the phase of devising a plan unlike Pólya (1945) and Schoenfeld (1985) because in the observed classroom conditions the students seemed not to create a plan and then implement it. In this aspect, our model is similar to the model by Mason et al. (1982). Instead of devising a plan and implementing it, the students in our study seemed to engage in unstructured exploration (cf. Mason et al., 1982; Schoenfeld, 1985). In particular, the use of DGS emphasized exploration. As previous research has shown, DGS makes exploration easy and fast and students can easily create conjectures (Arzarello et al., 2002; Healy & Hoyles, 2001). Similar to Arzarello et al.'s (2002) ascending and descending processes our model emphasized the work before (exploring the solution) and after (justifying or investigating the conjecture) creating the conjecture. Like the modelling cycle (Lesh & Doerr, 2003) also the proposed model emphasizes iterative development of more general solutions. The open problem-solving model can be used to study how teachers orchestrate students' problem-solving processes. Furthermore, the study also offers some concrete ways how teachers can do this. As an implication for practice, we believe that the model helps teachers to conceptualize open problem solving and prepare for guiding the students.

This paper reported a rare study addressing the particularities of students' problem solving and teacher guidance when dealing with open problems. Yet, these issues require further research. For example, the study described specific ways in which the teacher can guide students' open problem solving. More research could suggest other ways and the particular conditions they could be used. Also, in different conditions, students may move differently between the phases. In this study, students' used particular educational software. Further research would help clarify the role of the technology in open problem solving. For example, more insights are needed on the role of technology in open problem solving and how the teacher can guide students in such environments.

References

Anghileri, J. (2006). Scaffolding practices that enhance mathematics learning. *Journal of Mathematics Teacher Education*, 9, 33–52.

Arzarello, F., Olivero, F., Paola, D. & Robutti, O. (2002). A cognitive analysis of dragging practices in Cabri environments. *ZDM – The International Journal on Mathematics Education*, 34 (3), 66–72.

Boaler, J. (1998). Open and closed mathematics: student experiences and understandings. *Journal for Research in Mathematics Education*, 29 (1), 41–62.

Borwein, J. & Bailey, D. (2003), *Mathematics by experiment: plausible reasoning in the 21st century*. Natick: AK Peters.

Cai, J. & Lester, F. (2010). *Why is teaching with problem solving important to student learning*? Retrieved April 15, 2012 from http://www.nctm.org/uploadedFiles/Research_News_and_Advocacy/Research/Clips_and_Briefs/Research_brief_14_-_Problem_Solving.pdf.

Christou, C., Mousoulides, N., Pittalis, M. & Pitta-Pantazi, D. (2004). Proofs through exploration in dynamic geometry environments. *International Journal of Science and Mathematics Education*, 3 (2), 339–352.

Christou, C., Mousoulides, N., Pittalis, M. & Pitta-Pantazi, D. (2005). Problem solving and problem posing in a dynamic geometry environment. *The Montana Mathematics Enthusiast*, 2 (2), 125–143.

Cifarelli, V., & Cai, J. (2005). The evolution of mathematical explorations in open-ended problem-solving situations. *Journal of Mathematical Behavior*, 24, 302–324.

Davis, B. (1997). Listening for differences: an evolving conception on mathematics education. *Journal for Research in Mathematics Education*, 28 (3), 355–376.

Davis, R., & Maher, C. (1990). What do we do when we "do mathematics"? *Journal for Research in Mathematics Education*. Monograph, Vol. 4, Constructivist views on the teaching and learning of mathematics, 65–78, 195–210.
Doerr, H. & English, L. (2006). Middle grade teachers' learning through students' engagement with modeling task. *Journal of Mathematics Teacher Education*, 9 (1), 5–32
Francisco, J. & Maher, C. (2011). Teachers attending to students' mathematical reasoning: lessons from an after-school research program. *Journal of Mathematics Teacher Education*, 14, 49–66.
Healy, L. & Hoyles, C. (2001). Software tools for geometrical problem solving: potentials and pitfalls. *International Journal of Computers for Mathematical Learning*, 6, 235–256.
Hähkiöniemi, M. & Leppäaho, H. (2012). Prospective mathematics teachers' ways of guiding high school students in GeoGebra-supported inquiry tasks. *The International Journal for Technology in Mathematics Education*, 19 (2), 45–58.
Hölzl, R. (2001). Using dynamic geometry software to add contrast to geometric situations – a case study. *International Journal of Computers for Mathematical Learning*, 6, 63–86.
Jones, K. (2000). Providing a foundation for deductive reasoning: students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics*, 44 (1–3), 55–85.
Kwon, O. N., Park, J. S. & Park, J. H. (2006). Cultivating divergent thinking in mathematics through an open-ended approach. *Asia Pacific Educational Review*, 7 (1), 51–61.
Lesh, R. & Doerr, H. (2003). Foundation of a models and modeling perspective on mathematics teaching, learning, and problem solving. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism: models and modeling perspective on mathematics teaching, learning, and problem solving* (pp. 3–33). Mahwah: Erlbaum.
Lobato, J., Clarke, D. & Ellis, A. B. (2005). Initiating and eliciting in teaching: a reformulation of telling. *Journal for Research in Mathematics Education*, 36 (2), 101–136.
Martino, A. & Maher, C. (1999). Teacher questioning to promote justification and generalization in mathematics: what research practice has taught us. *Journal of Mathematical Behavior*, 18 (1), 53–78.
Mason, J., Burton, L. & Stacey, K. (1982). *Thinking mathematically*. Bristol: Addison–Wesley.

Mousoulides, N. (2011). GeoGebra as a conceptual tool for modelling real world problems. In L. Bu & R. Schoen (Eds.), *Model-centered learning: pathways to mathematical understanding using GeoGebra* (pp. 105–118). Rotterdam: Sense.

Nohda, N. (2000). Teaching by open-approach method in Japanese mathematics classroom. In T. Nakahara and M. Koyama (Eds.), *Proceedings of the 24th conference of the international group for the psychology of mathematics education* (Vol. 1, pp. 39–53). Hiroshima: PME.

Nunokawa, K. (2005). Mathematical problem solving and learning mathematics: what we expect students to obtain. *Journal of Mathematical Behavior*, 24, 325–340.

Pea, R. (2004). The social and technological dimensions of scaffolding and related theoretical concepts for learning, education and human activity. *Journal of the Learning Sciences*, 13 (3), 423–451.

Pehkonen, E. (1995). On pupils' reactions to the use of open-ended problems in mathematics. *Nordic Studies in Mathematics Education*, 3 (4), 43–57.

Pehkonen, E. (1997). Introduction to the concept "open-ended problem". In E. Pehkonen (Ed.), *Use of open-ended problems on mathematics classroom* (Research report 176) (pp. 7–11). Department of teacher education, University of Helsinki.

Pólya, G. (1945). *How to solve it? A new aspect of mathematical method*. Princeton University Press.

Sahin, A. & Kulm, G. (2006). Sixth grade mathematics teachers' intentions and use of probing, quiding, and factual questions. *Journal of Mathematics Teacher Education*, 11 (3), 221–241.

Schoenfeld, A. (1985). *Mathematical problem solving*. London: Academic Press.

Stein, M., Engle, R., Smith, M. & Hughes, E. (2008). Orchestrating productive mathematical discussions: five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, 10, 313–340.

Sullivan, P., Mousley, J. & Zevenberger, R. (2006). Teacher actions to maximize mathematics learning opportunities in heterogeneous classrooms. *International Journal of Science and Mathematics Education*, 4, 117–143.

Sullivan, P., Warren, E. & White, P. (2000). Students' responses to content specific open-ended mathematical tasks. *Mathematics Education Research Journal*, 12 (1), 2–17.

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