Student teachers' work on instructional explanations in multiplication – representations and conversions between them

ANITA VALENTA AND OLE ENGE

In this study we are analysing student teachers' instructional explanations. The study is based on student teachers' written work on two different tasks about different strategies and properties in multiplication and explaining these. Our research questions concern the type of representation registers student teachers use in their explanations. In explanations where several representation registers are used, we analyse what can be challenges in conversions between representations. Data is analysed using the framework of Duval's cognitive analysis, and analyses and discussions are related to development of mathematical knowledge for teaching.

If we know how much $20 \cdot 5$ is, it is easy to figure out $18 \cdot 5$. Since 18 is 2 less than 20, we can take away two fives, $2 \cdot 5 = 10$, from $20 \cdot 5$ and the result will be $18 \cdot 5$. This strategy is often used in mental calculations. But what is actually happening, how is this strategy working and why does it give the correct answer? How to explain the strategy to pupils in elementary school? Providing an instructional explanation is an essential part of the work of every mathematics teacher (see for example Leinhardt, 2001). Leinhardt, Putnam, Stein and Baxter (1991) define instructional explanations as an activity in which teachers communicate subject-matter to pupils in a way that gives an opportunity for pupils to understand a given concept or procedure. Pupils' opportunity to construct meaning is crucial, and Leinhardt et al. (1991) discuss that a good instructional explanation needs to be more than a description of the steps in performing a procedure. Leinhardt and Steel (2005) emphasize in particular that teachers should use appropriate representations and build on pupils' prior knowledge.

Several studies have highlighted the importance of the ability to represent mathematical ideas in different ways and how these are useful for the

Anita Valenta, *Sør-Trøndelag University College, Norway*
Ole Enge, *Sør-Trøndelag University College, Norway*

Valenta, A. & Enge, O. (2013). Student teachers' work on instructional explanations in multiplication – representations and conversions between them. *Nordic Studies in Mathematics Education*, 18 (1), 31–59.

teaching and learning of mathematics (Kilpatrick, Swafford & Findell, 2001; Ma, 1999). Ball, Thames and Phelps (2008) state in their investigation of mathematical knowledge for teaching (MKT) that an awareness about the existence of different representations of a given concept or procedure, the interplay between them and the opportunities for learning of mathematics each representation can give, are important parts of MKT.

International studies have documented that teachers and student teachers have difficulties in using different representations in their work in mathematics and, in particular, in providing instructional explanations (see Charalambous, Hill & Ball, 2011; Kinach, 2002; Stylianou, 2010).

From studies in Scandinavia we note that Alseth, Breiteig and Brekke (2003) observe that Norwegian teachers rarely use other representations than symbols in their teaching. In her study, Grevholm (1998) reports that student teachers during their teacher education develop an understanding of concepts as more than numerical procedures. Måsøval (2005, 2011) analyses student teachers' reasoning in algebraic generalization tasks and identifies the difficulties with changing representation from natural language to mathematical symbols. More generally, considering student teachers' approach to teaching mathematics, Winsløw and Durand-Guerrier compare teacher education in Denmark and France in their study (2007; see also Durand-Guerrier, Winsløw & Yoshida, 2010). They compare the educational systems, but also student teachers' performances on what they call hypothetical teachers tasks, that is tasks which require an analysis from the mathematical point of view and reflection on how to proceed in a hypothetical situation in a class. The French student teachers analyze the mathematical content more deeply than Danish students, for example they justify a mathematical claim to a higher degree and they typically provide several explanations for pupils. The French students in the study are more concerned with providing justifications of the mathematical results, whereas the Danish students are more inclined to accept things as "rules" and show several examples to pupils (often examples from "real life") in order to illustrate how the rule is working.

In this study we are addressing the question of student teachers' use of representations in instructional explanations on whole number multiplication. Lo, Grant and Flowers (2008) investigated student teachers' reasoning and justification in whole number multiplication and they identified challenges the student teachers experienced. They emphasize the use of two models for multiplication, equal groups and array/area. Equal groups represesents whole number multiplication as a number of sets with an equal number of objects in each set. Array/area represents multiplication through the number of squares in an array or the area of a rectangle. Several of these challenges as "proof by picture", "coordinating area/array interpretation with strategies" and "coordinating equal group interpretation with strategy" (Lo et al., pp. 15–18), can be interpreted as difficulties in connecting different representations used in a justification. While Lo et al. (2008) are interested in student teachers' reasoning, we are concerned with student teachers' instructional explanations. Our area of interest in this article is the kind of representations student teachers use when explicitly asked to justify and explain their computations and reasoning in multiplication to hypothetical pupils. When a student uses several different representations in an explantation, we are interested in how these representations are connected and what can be the challenges in connecting these representations. To address these questions we analyze the work of 140 student teachers, who provided written instructional explanations as part of a homework assignment.

We begin by discussing our theoretical framework and specifying our research questions. Further, we discuss the methods of our study and present the context. In the main part the article, we present our findings, first by providing a general overview of the student teachers' explanations and the representations used, and next by having a closer look at challenges in connecting several representations. Finally, we discuss our findings and consider implications for teacher education.

Theoretical framework

Leinhardt (2001) defined instructional explanations as teachers' explanations that are designed to explicitly teach subject matter, to support pupils' understanding of the content. Several empirical studies give criteria for good instructional explanations (Leinhardt, 1987; Leinhardt et al.; 1991, Leinhardt & Steele, 2005), and Charalambos, Hill and Ball (2011) give a summary of these. In particular, good explanations should explain "the thought process step-by-step without skipping steps", "make the transition between steps clear", "use suitable examples and representations", and "when explaining a mathematical procedure, each step in this procedure is clearly mapped onto the visual representation used" (Charalambos et al., p. 477). The authors point out that providing an instructional explanation is a part of the everyday work of a mathematics teacher. As they discuss, the competence for providing an instructional explanation is difficult to develop and needs attention in teacher education.

Ball and Bass (2003), building on the work of Shulman (1986), introduced the notion of "mathematical knowledge for teaching" (MKT) to describe the kind of knowledge that is special for a mathematics teacher. Based on an extensive analysis of what teachers do when teaching mathematics, Ball et al. (2008) identify several domains comprising mathematical knowledge for teaching. One aspect they highlight in their discussion on MKT is teachers' knowledge of different mathematical representations, of choosing an appropriate representation for a given purpose, and gains and disadvantages of their use.

Signs, illustrations, language, and different representations in general have a crucial role in mediating and communicating the meaning in all subjects and all learning according to sociocultural theory (see for example Halliday,1978; Saljö, 2004). Different representations have an even more prominent role in mathematics than in other subjects. Duval (2000, 2006, 2008) emphasizes that mathematical concepts are abstract objects and accessible only through their representations. He points out that there are always many possible semiotic representations of the same object (2008; see also Duval, 2006) and that interplay between different semiotic representations characterize all mathematical activity. Duval (2002, 2006) gives four different types of representation registers, i.e. representation systems which open for mathematical processes within the system. The representation registers are:

- Natural language (e. g. verbal associations, arguments from observations)
- Symbolic system (numeric, algebraic, formal language)
- Iconic (drawings, sketch) and non-iconic drawings (geometrical figures)
- Diagrams and graphs

Mathematical activity always involves semiotic transformations (Duval, 2006), either within the same representation register (e. g. from $x+x+1=5$ to $2x=4$, both in symbolic system) or between two different representation registers (e.g. from $2x=4$ in symbolic, to "two times some number equals four" in natural language). Duval (2006) denotes the first kind of transformations, within the same representation register, for *treatments*, while he uses the notion *conversions* for transformations between two different representation registers. He (Duval, 2008) argues that conversions are often the most demanding challenge for learners. They can struggle with trying to understand the connection between two representations in different representation registers, and the conflict can lead to considering the two representations of the same object as being two different objects (2006, 2008). We can recognize this problem of conversion in the study of Lo et al. (2008), where student teachers work with reasoning and mathematical thinking in multiplication. The student teachers used two representation registers (iconic and symbolic system), and several of the challenges Lo et al. discuss can be seen as challenges in conversions between the representations involved. Representations in different representation registers and conversions between them are a critical point in learning mathematics, but it is important both for doing and learning mathematics to use different representation registers (Duval, 2006). None of the many possible representations *is* the given mathematical concept, and different representation registers have different capacities and can give different insights into the concept. This is what Duval calls the cognitive paradox in the learning of mathematics; the use of different representation registers and that the interplay between these is necessary, but is also the most difficult part in learning. Developing an awareness of the existence of different representations for a given mathematical object and the ability to change the representation register when needed is the threshold for learners in mathematics (Duval, 2006).

Duval's analysis of problems of comprehension in the learning of mathematics can be used as an elaboration on aspects of MKT, a mathematics teacher needs also to be aware of the *necessity* and *challenges* of using multiple representation registers and transformations between them in any mathematical activity. Considering instructional explanations, and following Duval's discussion of problems of conversions between different representation registers, we can conclude that it can be difficult for learners to follow an explanation where several representation registers are used. However, instructional explanations should involve several representation registers, and the conversions between them should be visible through the explanation. We can identify this requirement in the studies on instructional explanations mentioned above (Charalambos et al., 2011; Leinhardt, 1987; Leinhardt et al., 1991; Leinhardt & Steele, 2005). Several of the criteria for a consistent explanation and several of the challenges in providing such explanations can be interpreted using Duval's analysis and discussion of representation registers and conversions between them, e.g. that each step in a mathematical procedure must be clearly mapped onto the visual representation (Charalambos et al., 2011).

These considerations lead to our questions:

– What kind of representation registers do student teachers use in their explanations? We search for different types of explanations based on the representation registers used, and we ask what the characteristic properties of the different types are?

– What kind of challenges can appear in conversion between representations in explanations, where several representation registers are used?

We believe that our study contributes to the knowledge of instructional explanations and teaching of mathematics in general.

Methods

Participants and course

The study was conducted at one of the largest university colleges in Norway, on a study program for prospective teachers at 1^{st} to 7^{th} grade in elementary school. We will use the notion *student* for student teacher from here on. One of the compulsory courses in the study program is a course in mathematics and didactics, and students taking this course are the participants in this study. The course is a 30 ECTS course which is distributed over the first four semesters of teacher education, and there are approximately 12 three-hour meetings per semester.

The students participating in the study had just finished their first semester in the teacher education when they were asked to answer the following two problems[1] concerning instructional explanations in multiplication:

Task 1

a) How to calculate $18 \cdot 5$ if you know $20 \cdot 5$? Describe and justify each step in your method.

b) Imagine that you are a teacher in third-fourth grade. How would you adjust your reasoning and explain the relation/procedure to your pupils? Consider use of an appropriate context or/and an illustration.

Task 2

a) Calculate $26 \cdot 18$ in three different ways. Describe and justify each step in all three procedures.

b) Imagine that you are a teacher in third-fourth grade. Choose one of your procedures in a).How would you adjust your reasoning and explain the relation/procedure to your pupils? Consider use of an appropriate context or/and an illustration.

At that point there was a cohort of 143 students, and their written answers to these problems are data in this study. Three of the students did not want to participate in the study, and their written works were removed from the data. Our analysis is based on in depth explorations of the written works provided by the remaining 140 students.

The majority of students were 19 to 22 years old, about 80% of them were females[2]. Very few students had experience from work as teacher or teacher's assistant beside the three weeks of practice they had during their first semester. A large majority of the students had just one year of mathematics from upper secondary school and no other additional courses in mathematics taken before entering teacher education.

The students were divided into four classes in the coursework, and each class had its own teacher, the authors being two of them. The teachers collaborated closely on development of the course. The course was an integrated mathematics and mathematics didactics course with the aim to help students in developing mathematical knowledge for teaching. The underlying mathematical subject during the first semester was multiplicative thinking, and the emphasis was on different strategies, properties and reasoning in multiplication. In our teaching on multiplication we stressed the use of illustrations and contexts as tools for reasoning in multiplication. We also discussed the meaning of such representations for developing pupils understanding of multiplication. This was the first time the students met mathematics as future teachers, and a great deal of time was therefore used to discuss what mathematics is about, and what it means to do, to learn and to teach mathematics.

The study will contribute to the development of our teaching and our course, hence it can be considered as a developmental study as described by Gravemeijer (1994). We develop the course based on our professional knowledge as mathematics teacher educators, and our knowledge of our students. We do research on our practice, which then assists us in developing the course further.

Data

The data we analysed in this study is the students' written answers of the two tasks presented above. It is particularly the b-parts of the tasks we are interested in, the questions concerning providing hypothetical instructional explanations. One can discuss the "realness" of these instructional explanations which are done as a written homework. They are not real in the sense that they are given to real life pupils. On the other hand, working with written hypothetical explanations gives an opportunity to think through the content of the explanation, and try to take account of the complexity of classroom teaching without actually performing the teaching. The use of hypothetical tasks related to teaching mathematics has long tradition in teacher education and mathematics teacher education research (se for example Charalambos et al., 2011; Durand-Guerrier, Winsløw & Yoshida, 2010). Writing is a decisive tool in professional reflection and learning (Berge, 2005; Evensen 2010; Krogh & Jensen, 2008). Through writing, in our case hypothetical explanations, students' thoughts become more structured and more clear, and the language becomes more precise, as discussed in (Jonsmoen & Greek, 2012). By choosing written hypothetical instructional explanations instead of real instructional explanations in a classroom, our study loses some ”realness”, but hopefully we get insight in explanations which are more thoroughly thought through.

The written work was individual homework. The students had 10 days to work on the tasks before submitting their work electronically. Our intention with the chosen setting was to lessen the feeling of time pressure, and make it possible to use notes and literature. Even though the students had to submit individual work, they could cooperate in preparing it. It was important for us that students could deliver a work they were satisfied with.

We had a participating role in the study. We participated in planning of the course and the teaching, and we developed the tasks that students were asked to discuss. We were well aquainted with the discourse in the classes, and this could make interpretation of the data easier in some cases. On the other hand, we were concerned with the fact that our knowledge of the students and our knowledge of the course could also cause some misinterpretations of data, e.g. by seeing what we expected to see from a particular student, or combining a written answer with discussions we had in class earlier. We are aware of these challenges and we have tried to minimalize the influence of our knowledge of the course and particular students on the interpretation and analysis. The data, the written works from 140 students, was depersonalized concerning name, gender and class before we began the analysis.

Task analysis and adaptation of analytical framework

The two tasks we considered are similar, but also different. The first task involves relatively simple numbers and is based on a rather basic use of the distributive property of multiplication, which most of the students could have used as a mental strategy in calculation and have an intuitive understanding of. In the second task the numbers are not so ”nice” as in the first task. Further, contrary to the first task, the students could choose their own strategy to explain. Due to the numbers involved, 26 and 18, it can be an idea to distribute both numbers, for example $(20+6)\cdot(10+8)$ or $(30-4)\cdot(20-2)$. Explanations based on such decomposing of both numbers would be less intuitive and more complicated than the one in the first task. This could lead to more use of several representation registers since students possibly needed them for their own reasoning. On the other hand, more complicated numbers could lead to less use of different representation registers, a reason being that the students might not intuitively use such strategies and representations in their own calculations with these numbers.

What kind of representation registers could be used in the expanations to pupils? The tasks were given in a combination of natural language and symbolic system: *How to calculate $18 \cdot 5$ if you know $20 \cdot 5$?* One possibility can be to use only natural language and symbolic system in the explanation too, as e. g.: "Since 18 is 2 less than 20, we can take away two fives, $2 \cdot 5 = 10$, from $20 \cdot 5$ and the result will be $18 \cdot 5$". In the explanations as this, the natural language is used as a formal, mathematical language (e.g. *less than*, *result*), or it is used to tie up the the numeric/symbolic expressions (*since… we can…*). We therefore consider the explanations of this form as mainly symbolic, and we denote this kind of representation register as *symbolic*. Using Duval's classification of registers (2002, 2006) our symbolic register incorporates symbolic systems with parts of natural language register.

Since the tasks were given in the symbolic representation register, it was natural to expect that symbols would appear in all explanations. In addition, the explanations could for example involve "everyday life" situations as in the following example:

> You have 20 bags with 5 marbles in each bag. That is, $20 \cdot 5$. Then you give away 2 bags, so that is $2 \cdot 5$ marbles. Now you have 18 bags, $18 \cdot 5 = 20 \cdot 5 - 2 \cdot 5$ marbles.

In this example, the mathematical symbols and formal mathematical language are used, but in addition also another type of natural language, which are bags with marbles. We denote this kind of representation register based on everyday situations as *context*. This can be considered as a part of the natural language register as defined by Duval (2002), the part which is about verbal associations.

An explanation can also involve drawing a model of multiplication, e.g. equal groups or array/area, as in figure 1.

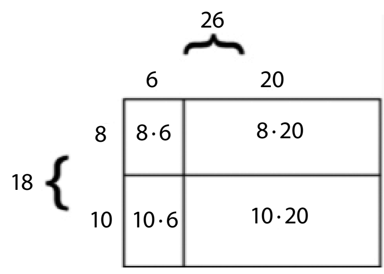


Figure 1. *Illustration used in one explanation*

Again, symbols are used in this explanation, this time together with a drawing, which is another representation register. We denote this representation register as *illustration*. Duval (2006) denotes this type of register as iconic, however we choose to denote it illustration, since we find it more explanatory given the type of tasks we consider in our study.

Data analysis

Having our research questions in mind, we started to analyse the written works concerning the representation registers the students used in their explanations in the two tasks given. The categories we used in this part of analysis are predefined. As discussed above, we considered three representation registers as possible: symbols, context and illustration. Since the tasks were given in symbols, we presumed that symbols would be used in all explanations, and illustration/context could be used in addition, either one or both of them. Our categories therefore were:

- Only symbols
- Symbols and context
- Symbols and illustration
- Symbols, context and illustration

An explanation with no drawing and no reference to everyday situations was categorized as ”only symbols”. If some everyday situation (as children or balls) was mentioned, the explanation was automatically categorized as ”symbols and context”. If some drawing was given as a part of the explanation, it was categorized as ”symbols and illustration” or ”symbols, context and illustration”, depending on whether a context also was used or not. In some cases students wrote ”I would use an illustration” or ”I would use a suitable context” without presenting one. We regarded these explanations as ”not using” a given representation register, but we marked the suggestion in our analysis.

After recording the types of representation registers used, we looked for typical explanations in each category and analysed the characteristical properties of these explanations.

In the second part of the analysis, concerning our second research question, we looked at the cases where several representation registers were used in an explanation, analysing the conversions between the representations. We were interested in what could be challenges in conversions between representations, and in this part of analysis, we followed a grounded-theory approach (Strauss & Corbin, 1998). The categories were developed gradually, step by step, in interplay with the data. We started the analyzis individually and independently of each other, recording the observed challenges first by describing them in depth and gradually more in form of keywords referring to an already observed challenge. After the first part with individual analysis, we discussed similarities and differences between the observations made and the keywords used. Based on the individual analysis and the comparison made, we developed a new set of keywords and categories, and started individually on the second circle of analysis. In between we had discussions on particular explanations to compare the use of categories, define the borders and develop the categories further. Also the analytic tools, categories, interpretations and analysis developed further. We ended up with five categories which describe the challenges we observed in our data.

In our analysis we noticed that some students interpreted "reasoning" as reasoning for general pedagogical choices. They wrote, for example, about letting pupils work in groups, use of blackboard and concrete materials, but did not discuss mathematical details. This interpretation of "reasoning" can be considered as a weak point in our data. If we had some interviews with students and clarified the task, the explanations would maybe be different. It is possible that some students were not aware of the need to be more explicit. On the other hand, their understanding of what is needed to be precise about in questions on reasoning and explaining in mathematics gives indications of their mathematical knowledge for teaching (MKT), and we cannot disregard their explanations as simple misunderstanding.

Findings

This section is composed of two interconnected parts. In the first part we give an overview of representation registers used in explanations and we present typical explanations. Based on these explanations, we discuss, in particular, what characterizes explanations in different categories concerning the representation registers used. In the second part, we take a closer look at explanations involving several representation registers, and we analyse challenges in conversions between representations.

Representations used

As shown in the diagram in figure 2, 27 students used only symbols in task 1. The majority of students used symbols and context, and one third of the students who used context also drew an illustration. Eleven students used symbols and illustration, but no context.

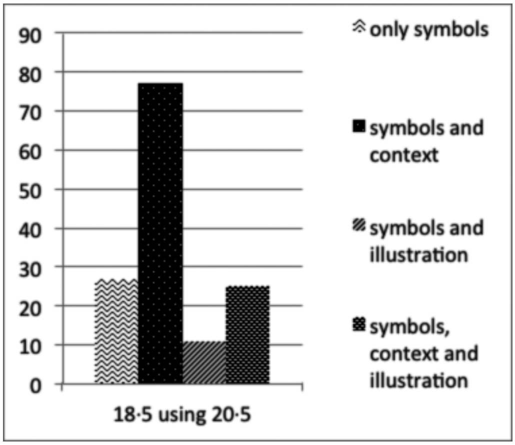


Figure 2. *Number of students using given representation registers in their explanations in task 1*

As mentioned, in the second task the students were asked to choose one way to calculate $26 \cdot 18$ and explain their reasoning for pupils. Six students did not answer the question, and six provided a reasoning which was not mathematically correct. Four students chose repeated addition and two used the associative property, and all six of them used symbols, illustration and context in their explanations. There were mainly three different strategies that the rest of the students chose: the use of standard algorithm[3], decomposing one of the numbers, e.g. $26 \cdot 18 = (20 + 6) \cdot 18$, or decomposing both numbers, e.g. $26 \cdot 18 = (20 + 6) \cdot (10 + 8)$. An overview of the use of representations for these three strategies is given in figure 3.

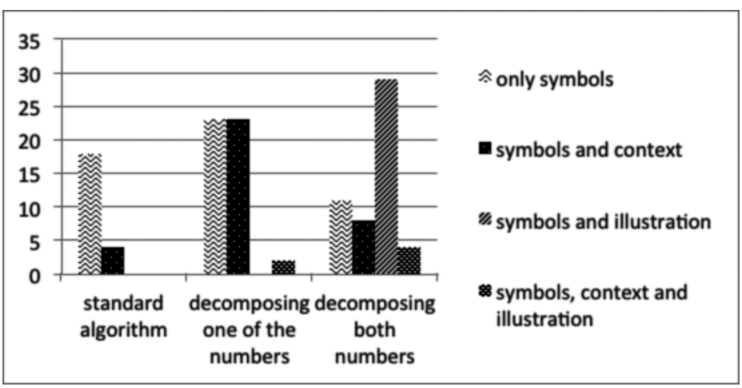


Figure 3. *Number of students using given representation registers in different strategies they choose in task 2*

Explanations based on only symbols

There are many students who chose to explain a standard written algorithm. In many cases the students wrote that they chose it because it is: "the usual way to calculate"; that it is "efficient" and "proper"; and also "easy once one remembers all the steps"; "even if it can take some time to remember the steps", as some write. The majority of students who explained the standard algorithm used only symbols in the explanation.[4] A typical explanation of this type is the following.

> *Explanation A*
>
> $$\begin{array}{r} \underline{\quad 26\cdot 18 = 468} \\ 208 \\ \underline{26} \\ = \; 468 \end{array}$$
>
> First I would have written the arithmetic problem as above $(26\cdot 18)$. Here I start at the very back of the problem and multiply 8 with 6. Get 48 and write the number 8 below 8 as shown above. Number 4 ends on the other hand in "carrying" and I put it above the number 2. Then I take 8 times 2, which is 16, and then add on to the 4, which I "carried" and end up with 20, which I put before 8. After that I continue to the next number, which is 1. Here I do as before and multiply 1 with 6 and get 6. Since I now have started with a new number, I jump down a row and one space to the left. Thus, the number 6 ends up below 0. Then there is just one number left to multiply, and then it will be 1 times 2, which is 2, and get the place before the number 6. Now all the numbers have found their place, and it is just to drag the numbers down to get the answer, like you do in an ordinary addition problem.

In this explanation there is no reference to the value of the digits involved, for example "Get 48 and write the number 8 below ...". The digits are to be manipulated by some rules of placement. All explanations of the standard algorithm were more or less of this form, a step by step instruction of what to do with almost no attempt to explain why. The explanations of this form appear as simple instructions.

Another frequently used strategy was to decompose one of the two numbers, i.e. $26\cdot 18=(10+10+5+1)\cdot 18$ or $26\cdot 18=26\cdot(10+8)$ or $26\cdot 18=26\cdot(20-2)$. Almost half of the students who chose this strategy used only symbols. A typical explanation is given in explanation B.

Explanation B

Step 1. First, I go from 26 to 20. I could also go from 26 to 30, but I find it easier to add than subtract later.

Step 2. I split 20 in 10 and 10. It is because lower numbers are easier to multiply.

Step 3. Multiply $18 \cdot 10$ twice. Add them together to get $180 + 180 = 360$. Now we have multiplied 20 by 18, and we were supposed to multiply 26 by 18.

Step 4. We lack 6 times 18. I split 6 in 5 and 1 because it is easier to multiply a round number (5). This gives two parts: $18 \cdot 5 = 90$, $18 \cdot 1 = 18$. We add those two and get 108.

Step 5. At the end we add the product from step 3 with the product from step 4 and get 360 (from $20 \cdot 18$) + 108 (from $6 \cdot 18$) which gives $= 468$.

Similar to explaining the standard algorithm, the explanations of this strategy, where only symbols are used, are instructions of what to do rather than explanations of why multiplication is distributive, and why and how this can be used as a strategy in calculations. For example, the question of *why* one should multiply *all* parts of 26 with 18 is not argued for. Even though there are some similarities between these two types of explanations, we notice some differences also. In the explanation above concerning distribution of one of the numbers, the student refers to some relations between numbers (e.g. "I split 20 in 10 and 10"). In the explanation of the standard algorithm, the student refers only to the digits with no further description of the relation between the digits and the numbers.

Explanations based on symbols and context

In task 1, the majority of contexts used in the explanations were based on equal groups. A typical context is about 18 pupils having 5 items each. The students who chose to decompose one of the numbers in task 2 and who used a context in addition to symbols, had explanations that are similar to those in task 1. They used contexts based on equal groups, like the following one using the strategy $26 \cdot 18 = (10 + 10 + 6) \cdot 18$.

Explanation C

> Each pupil in a class of 26 was supposed to pay 18 kroner for a ticket to the cinema. On the day they were supposed to pay, they worked in 3 groups, 10, 10 and 6 pupils per group. So, 10-group had $10 \cdot 18$ kroners, the other 10-group had also $10 \cdot 18$ kroner, the 6-group had $6 \cdot 18$ kroner. Totally, it was $10 \cdot 18 + 10 \cdot 18 + 6 \cdot 18 = 26 \cdot 18$ kroner.

In this explanation, a context is used together with symbols, and the context is used to explain why it is possible to decompose one of the numbers, a class of 26 pupils *can* be divided in groups of 10, 10 and 6. Further, the student uses the interplay between the context and the symbols to show why each part needs to be multiplied by 18. Each of the 26 pupils pays 18 kroner, a total of $26 \cdot 18$. If they pay while they are in groups, it will give $10 \cdot 18 + 10 \cdot 18 + 6 \cdot 18$, and this must be the same amount of money as if the whole class paid as one group.

We note that some students used a context where the order of the numbers is changed in the multiplication compared to the symbols. For example, in the first task they use a contest like 5 bags with 20 marbles in each, thus leading to $5 \cdot 20$ instead of $20 \cdot 5$ as the task asks for. The following is an example of this.

Explanation D

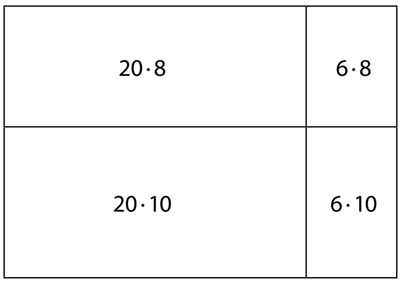
> You have 5 bags with 20 marbles in each bag. So, it is $5 \cdot 20 = 100$ marbles. To get $5 \cdot 18$ marbles, you have to take away 2 marbles from each bag, so $2 \cdot 5$ marbless are taken away. Altogether, 10 marbles are taken away, and you have to take away these 10 marbles from the 100 you started with. $100 - 10 = 90$. $18 \cdot 5 = 90$.

Since muliplication is commutative, $20 \cdot 5$ is equal to $5 \cdot 20$, but the context of 20 bags of 5 is different from 5 bags of 20. This change of order may be a challenge for pupils, both depending on how they view multiplication and what they have been working with regarding multiplication. In our tasks we are not explicit about pupils' prior knowledge. However, we see that this can be considered a challenge in providing an instructional explanation.

Explanations based on symbols and illustrations

The strategy based on distributing both numbers, $26 \cdot 18 = (20 + 6) \cdot (10 + 8)$, or $26 \cdot 8 = (30 - 4) \cdot (20 - 2)$ was used by many students, as shown in figure 3 above. While illustrations were seldom used in the other strategies, and if used they illustrated an equal groups thinking, the majority of students who decomposed both numbers also drew an illustration of area. A typical explanation and illustration is the following.

> *Explanation E*
>
> Using this method, I multiply part by part in the quadrangle and in the end add all answers I found. Like this:



$20 \cdot 10 = 200$
$20 \cdot 8 = 160$
$6 \cdot 8 = 48$
$6 \cdot 10 = 60$
$\overline{}$ 468

> This is a nice and clear method which is more practical than the others, because it is made as a figure.

The student wrote further that she would draw the quadrangle on the blackboard and show the pupils ”how they could divide it in parts as above, in order to make it easier to calculate”. Afterwards she would give several similar exercises ”so that they can practice the method”. Many students, as the one providing the explanation above, referred to the illustration as a ”method”. Thus, the explanations are mainly about *draw*, *do* and *get* the answer. The connection between a quadrangle and multiplication is not discussed; the students do not explain why it is possible to do the different steps, and why we can do them. Even though several representation registers are used, we note that the explanations of this type are still just instructions.

Explanations based on symbols, context and illustration

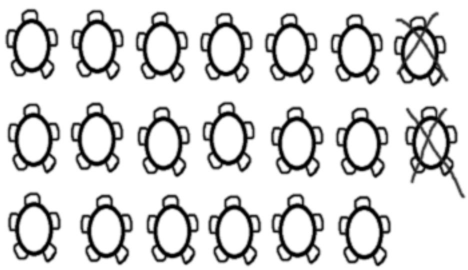
In the following explanation given in task 1, the student used symbols, context and illustration.

> *Explanation F*
>
> It is important for pupils' understanding to use a proper context in explanations of this type. It gives meaning to the numbers and the operation, and it helps them to follow the steps. It can also be

useful to use an illustration to deepen the understanding – pupils can see more clearly what is happening and why. I would explain the strategy to my pupils as for example:

All teachers in some town were supposed to participate at some course. The organizers found that they needed 20 tables. 5 teachers can sit around one table, 100 teachers were supposed to come. So we have $20 \cdot 5 = 100$ teachers coming.

But, during the course the organizers noticed that only 18 tables were occupied, and there were 5 teachers sitting around each table. So, $18 \cdot 5$ teachers came. How can we find out how many teachers came?



As we can see on the picture, two tables are empty ($20 - 18 = 2$). Two tables with 5 chairs, it is $2 \cdot 5 = 10$ teachers who did not come. 100 teachers were supposed to come, 10 did not came, so $100 - 10$ came. So we have $18 \cdot 5 = 20 \cdot 5 - 2 \cdot 5 = 100 - 10 = 90$.

I would at the end have empasized for my student this strategy, that it can often be useful in multiplication.

The student intertwined the three representation registers in her explanation. She started with a context, but referred to symbols frequently, for example in "So we have $20 \cdot 5 = 100$ teachers coming". Her illustration reflected the context and is also actively used in the explanation, as in "As we can see on the picture, two tables are empty ($20 - 18 = 2$)". She justified her thinking by a repeated interplay between the representation registers, and she also summarized her explanation by emphasizing symbols and strategy. This interplay makes the connection between the corresponding elements in different representations explicit, e.g. number of tables and 20, and it makes the steps clear and follows the same order in all representations; first 20 tables, so 18 tables, first 20, so 18.

Challenges in conversions between representations

We now take a closer look at explanations where several representation registers are used, and analyze challenges in conversions between different representations. We point out and discuss challenges by characterizing the explanations where the given challenge appears.

Conversion stops after the first step

One of the students gives the following explanation in task 2.

> *Explanation* G
>
> I would go further and explain to pupils the usual method [5], where you put the numbers under each other and multiply number by number. I would have shown on the blackboard several arithmetic problems, and how to do it this way – you start here, multiply that with that, write it here, remember that, ... I could make a context as for example. 18 plates with 26 apples on each plate, so that it is more fun for students to compute.

In several explanations, as in the one above, the context is used to illustrate the arithmetic problem, here $26 \cdot 18$, and is not used further in explaining the strategy. The conversion stopped after this first step, the explanation continued using just symbols and there was no referring to the context anymore. In approximately half of the explanations where a context was used, it was used more or less in this way, whatever task given and strategy used. In particular, this was the case in all explanations of standard algorithms where a context was used. In the example above, the student made it clear that the context was not to be used to explain the strategy. Some students justified the use of context and/or illustrations by stating that the use of such representations would make the calculation more motivating for pupils. The students did not reflect on the importance of using several representations for developing pupils' understanding. In most cases it was not so clear whether a context would be used or not. Several students wrote that they would use a context, and they gave an example of one, but they did not use it in the rest of the explanation. In the explanations of this type, where conversion stops after the first step, the additional representations are not used activly to justify the strategy; the explanation is mainly in symbols and appears as an instruction.

Conversion to symbols is not explicit

The majority of the students who used an illustration of area did not connect explicitly the symbols, as $26 \cdot 18$, to the area of a quadrangle. The typical explanation is as the one presented earlier, where the student just wrote that she would draw the quadrangle on the blackboard and show the pupils how they could divide it in parts. In the following example, explanation H, the student teacher was more conscious about this conversion.

> *Explanation H*
>
> I find an area model very useful in multiplication. It gives visualization, and makes it possible to split numbers and multiply part by part. It is easy to have an overview of what to do and why. But there is no point in just "giving" the method to the pupils. They need a context which invites them to think, to draw, to find out on their own. They need to use the strategy before we discuss it in general. I think that it is important to begin with some context, so that the problem can be understandable and real for pupils. It is also important to have a main purpose in mind – that the pupils can experience this illustration and the strategy as useful. It is not certain that the pupils are confident with notion of area, and that they associate multiplication with area, so I would use a context as for example:
>
> *A playground behind our school is rectangular, 26 meters long and 18 meters wide. We need to cover it by artificial grass squares which have dimensions 1 x 1 meter. How many squares do we need?*

The student wrote that she would let her pupils work with the problem a while, and that she later would introduce four different parts of the playground to emphasize the strategy of decomposing both numbers in multiplication.

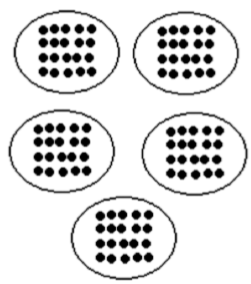
This step of connecting symbols, as $26 \cdot 18$, to a new representation is crucial also in conversions to contexts and other illustrations (as equal groups). In some explanations using contexts, the students were conscious about this step in conversion and made it explicit, as for example "**the question is about 26 · 18, and we can think of it as** 26 boys having 18 marbles each; the question is how many marbles they have altogether" [6], but in the majority of explanations such a connection is not explicit. Typically, the given context is similar to the one above but without a statement as the one we have bolded above.

When the arithmetic problem is not clearly connected to the other representation in the beginning, this deficiency usually continues throughout the whole explanation. One example is explanation I, where no explicit conversion to symbols is done.

Explanation I

I would have presented the following problem for pupils:

Anders has 5 bags with 18 marbles in each bag. Pia has 5 bags with 20 marbles in each bag, and she knows that she has 100 marbles in total. How can Anders find out how many marbles he has, when he knows that Pia has 100?



I would use concrete bags with 20 marbles in each when explaining, and I would draw it also. My explanation would go on taking 2 marbles from each bag, so that there are 18 left. The pupils could count that 10 marbles were taken out, and that Anders has 10 marbles less than Pia. So, he has 90.

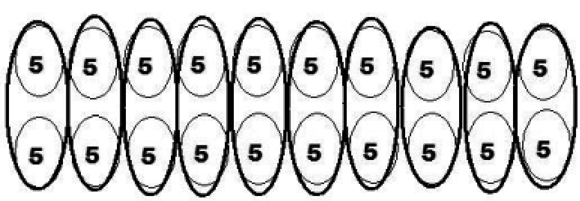
Here we see that the explanation becomes just a story problem with no accentuation of the given property of multiplication and strategy.

Conversions where order of steps is changed

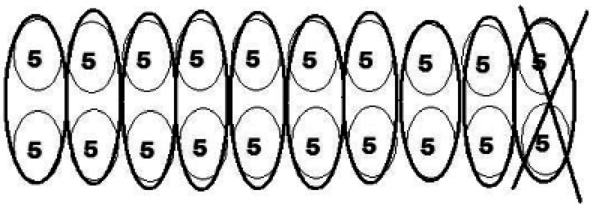
In some explanations the order of steps in symbols was not synchronized with the order in context/illustration. Explanation J is an example of this type.

Explanation J

Trudy has bought 100 marbles which were packed in 20 bags, 5 marbles in each bag. 20 bags with 5 in each, so $20 \cdot 5$ as we usually write it. Two bags together contain 10 marbles, as we can see on the illustration. And we already know that it is 100 in total.



> She gives 2 of the bags to her friend Lisa, so now Trudy has 18 bags, $18 \cdot 5$ marbles, as illustrated. She gave away 2 bags, $2 \cdot 5 = 10$ marbles, so she has $100 - 10$ marbles. So, $18 \cdot 5 = 20 \cdot 5 - 2 \cdot 5$.



In the explanation above, both the context and illustration started with $20 \cdot 5$ marbles, then $2 \cdot 5$ marbles were given away and Trudy was left with $18 \cdot 5$ marbles. Symbolically, this can be written as $20 \cdot 5 - 2 \cdot 5 = 18 \cdot 5$, so the order of steps is changed in the context compared to the symbols, where the order is $18 \cdot 5 = 20 \cdot 5 - 2 \cdot 5$.

Some elements are not converted

In the case above, all elements (that is $20 \cdot 5$, $18 \cdot 5$, $2 \cdot 5$) of the strategy were present in all representations, and it was just the minor interchanging in the order of steps that appeared in the conversion. In other explanations which can be characterized as "interchanging the order", there are elements missing in the conversion too.

One student gave the following explanation on how to calculate $18 \cdot 5$ using $20 \cdot 5$.

> *Explanation K*
>
> In a class of 20 pupils, each pupil has 5 pens. It is 100 pens. One day, two of the pupils were ill and did not come to school. Now there were $100 - 10$ pens in the class.

In this context the main element, $18 \cdot 5$, is missing, and the context is not emphasizing the given strategy. The context refers to $20 \cdot 5 - 2 \cdot 5$, while the question is about a strategy of calculating $18 \cdot 5$ by using $20 \cdot 5$. The mathematical content is consequently changed.

Inappropriate representation makes conversion impossible

In some explanations where several representation registers are used, we notice that the chosen representation is inconvenient and that it makes the conversion to symbols difficult or even impossible. One example of this type is explanation L.

> *Explanation L*
>
> Let's say that we have a farm with different types of animals. We have 5 cages, 20 hens in each, so $20 \cdot 5$ hens. We also have sheep, 5 pens with 18 sheep in each, $18 \cdot 5$. I would point out for pupils the difference between cages with 20 hens and pens with 18 sheep, and how they can use this relation to calculate $18 \cdot 5$ easily. We find out that there are 2 more animals in cages, so the difference is 2. Since there are 5 pens with sheep, and there are 2 animals less in each pen, we have to add $2+2+2+2+2$, or $2 \cdot 5$. It is 10. Then pupils will know that there are 10 sheep less than hens. We can say that we know that there are 100 hens. Then there are $100 - 10$ sheep.

The explanation appears as an intricate word problem, where $20 \cdot 5$ is contextualized as hens in cages, while $18 \cdot 5$ is contextualized as sheep in pens. The strategy to be explained is based on the relation between $20 \cdot 5$ and $18 \cdot 5$, but the numbers are given different meanings in the context. The result is that the explanation ends with $100 - 10 = 90$, where 100 is the number of hens, 10 is the number of animals, and 90 the number of sheep. The context is inappropriate, the numbers have different meanings, thus making the conversion between symbols and context difficult to accomplish. The two distinct types of animals are elements of the context, which is impossible to converge to symbols in this situation. The number 5 has two different meanings, cages and pens, while it is supposed to be "the same 5" in symbols.

Conclusions and implications

In this study we asked what kind of representation registers student teachers use in their instructional explanations of strategies in multiplication, and what the characteristic properties of explanations are when different registers are used. First we note that explanations given with only symbols are more instructions than explanations. However, there are some variations concerning this point, as discussed in the example with symbolic explanation of the standard algorithm, explanation A, versus symbolic explanation of distribution of one of the numbers, explanation B. Second, using another representation register in addition to symbols can create an opportunity to justify the strategy, as shown in explanation C where a context is used to explicate the distribution of one of the numbers. In the example of the explanation where symbols, context and illustration are used, explanation F, we can see how the intertwining of representations can emphasize the justification of the strategy. However, the use of several representation registers does not automatically make the explanation more than an instruction, as we discussed in the example of the use of an area illustration, explanation E.

Even though there are many students who used only symbols, the majority used several representation registers in their explanations. Our study indicates that providing instructional explanations utilizing several representation registers is demanding for students. For example, context and illustrations are used in some explanations only as "starters" where conversion stops after the first step and the rest of the explanation is given only in symbols. Almost half of the students who used context in addition to symbols used the context in this way. Some students who used only symbols or in addition used context only as a starter, stated that they find it difficult to explain their procedure, but still they did not try to use another representation register besides symbols. Some even stated that using context or illustration could confuse the pupils. Duval (2006) discusses that one of the challenges in learning mathematics is that the mathematical object can be mistaken as one of its semiotic representations. It can be that some of our students consider symbols as the only way, or the only proper way of representing the multiplication. If so, it constitutes a severe hindrance in the development of mathematical knowledge, since multiple representations provide opportunities for conceptual understanding (Duval, 2006). The use of multiple representations is also a part of mathematical knowledge for teaching (Ball et al., 2008), and using several representations is an important tool for developing such knowledge (see for example Charalambos et al., 2011; Lo et al., 2008).

Another challenge we observed in our data when several representation registers were used together with symbols, is a non-explicit conversion to symbols. We observed this both in the use of contexts, as in explanation I, and in the use of illustrations, as in explanation E. Often when a context is used with no direct connection to the symbols, the context appears as a word problem which "lives its own life". The explanations where illustration of area is used, and the conversion between symbols and illustration is not explicit, appear just as a method and an instruction of what to do rather than *why*. This challenge is not so obvious, and not so problematic when students use illustrations of area in their own reasoning, as they do in the study of Lo et al. (2008). However, when explaining their reasoning to pupils this becomes a critical factor. Duval (2008) discusses the learners' challenge of recognizing the same mathematical object in several representations, and he points out that the result can be to consider the different representations as different mathematical objects. We notice that the explanations with non-explicit conversion to symbols can be challenging for pupils concerning this aspect.

In the explanations where several representation registers are used, we observe that sometimes the order of steps can be changed. The consequence can be that the mathematical content is changed to some degree, e.g. the strategy to be explained can be less explicit as in explanation J. A change of the mathematical content is rather prominent in the explanations with some of elements missing in the representations used, as we discussed in explanation K where the main element of the mathematical content ($18 \cdot 5$) is absent in context. The last challenge we observe is use of inappropriate representations, as in explanation L. Usually it was contexts with many details which were impossible to converte to symbols. Based on our analysis of the explanations of this type, and similar explanations where contexts are used, we suggest that contexts used in explanations need to be as simple as possible. It is demanding to provide an explanation where several representation registers are used, and there is no need to make it even more difficult by including details which are not significant for the mathematical content.

Charalambos et al. (2011) analyse factors that influence students' learning to provide instructional explanations, and they highlight the use of different representations as an explanatory tool as one of the main factors of such learning. The majority of our students uses several representation registers in the explanations, thus the learning process had begun. There are many possible challenges in the use of multiple representations in instructional explanations, as we discuss in this study. The knowledge of such challenges will help us as teacher educators in our work with providing opportunities for our students to develop their mathematical knowledge for teaching.

Finally, we discuss two observations we have made in our analysis which are not directly connected to our research questions, but could be interesting to study further. The first is that there is a rather clear difference in the use of representation registers in task 1 and task 2. The number of students using only symbols is almost doubled – from 27 in task 1 to 52 in task 2. As shown in figure 3, there is also a noticeable difference in use of representations depending on the chosen strategy in task 2. For example, illustrations are overwhelmingly used in explanations where both numbers are decomposed. The strategy to be explained in task 1 is a variant of "decomposing one of the numbers"-strategy in task 2, but even so explanations following this strategy in task 2 were based only on symbols to a greater degree. Almost half of the teacher students who chose this strategy in task 2 used only symbols, while one fifth used only symbols in task 1. One possible reason for this difference can be that the numbers involved in task 1 are "nicer" than in task 2. The students might therefore find it more natural to use the given strategy in their own mental calculations. Further, the students might find it easier to choose a context in task 1 than in task 2, since the numbers or the strategy here are less "nice". It is also possible that smaller numbers, as in task 1, make students intuitively think of younger pupils, and teacher students may feel that they need deeper explanations for young pupils, while pupils who already are working on some larger numbers possibly do not need so comprehensive explanations. These are just possible explanations we can think of, new studies are needed to illuminate the question further.

Another interesting observation is that many students chose to explain the standard algorithm. They justify their choice by saying it is the usual way to do multiplication, that all pupils should know this strategy, and although it is hard to master for pupils they will eventually learn it by extensive practice. One student wrote:

> I am sorry for the rather strange and unnatural explanation, but it was somewhat difficult to write down how I thought in this situation. I do not think that I would have made a word problem or an illustration or something similar to go along with this, it is enough to keep you busy just to make the computations. Pupils might think this is a difficult way to do it, but gradually it will be better, if they are allowed to use this method several times or all the time, they will manage rather easily.

The language used by the students who tried to explain the standard algorithm is pervaded by references to their own experiences as learners of mathematics; what they learned, how they learned, and what is important to learn when learning multiplication. Using Skott, Larsen and Østergaard's (2011) concept of *pattern of participation* we can say that they engage in teacher practice by "re-engagement in prior practices" (Skott et al., 2011, p. 52). As teacher educators we need to find ways to strengthen students' engagement in their future practice as teachers of mathematics. One way to address this problem might be to integrate the students' practice with our work on instructional explanations, and repeat the process of the planning of instructional explanations, enacting these explanations in school practice and evaluate the outcome several times. This is also in accordance with other studies which emphasize the proximity to school practice in the development of competencies for teaching mathematics (see for example Empson & Jacobs, 2008; Nicol, 1999; Niss & Højgaard, 2011; Zaslavsky, Chapman & Leikin, 2003). We believe that the integration of school practice would promote reflection of what multiplication is, how it can be represented and what it means to provide instructional explanations in mathematics.

Another question which merits further studies in relation to our research questions and findings, is to illumintate the use of different representations in explanations applying the different domains of MKT as an analytical tool. One question would be to discuss where does this knowledge of the use of representations belong, which domain of MKT is it a part of and how is it related to other domains.

References

Alseth, B., Breiteig, T. & Brekke, G. (2003). *Endring og utvikling ved R97 som bakgrunn for videre planlegging og justering: matematikkfaget som kasus*. Notodden: Telemarksforskning.

Ball, D. L. & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In E. Simmt & B. Davis (Eds.), *Proceedings of the 2002 annual meeting of the Canadian Mathematics Education Study Group* (pp. 3–14). Edmonton: CMESG/GCEDM.

Ball, D. L., Lubienski, S. T. & Mewborn, D. S. (2001). Research on teaching mathematics: the unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed.) (pp. 433–456). Washington: American Educational Research Association.

Ball, D., Thames, M. H. & Phelps, G. (2008). Content knowledge for teaching. What makes it special? *Journal of Teacher Education*, 59 (5), 389–407.

Berge, K. L. (2005). Skriving som grunnleggende ferdighet og som nasjonal prøve – ideologi og strategier. In A. J. Aasen & S. Nome (Eds.), *Det nye norskfaget* (pp. 161–188). Oslo: Fagbokforlaget.

Charalambos, Y. C., Hill, H. C. & Ball, D. L. (2011). Prospective teachers' learning to provide instructional explanations: how does it look and what might it take? *Journal of Mathematics Teachers Education*, 14, 441–463.

Durand-Guerrier, V., Winsløw, C. & Yoshida, H. (2010). A model of mathematics teacher knowledge and a comparative study in Denmark, France and Japan. *Annales de didactique et de sciences cognitives*, 15, 141–166.

Duval, R. (2000). Basic issues for research in mathematics education. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th conference of PME* (pp. 55–69). Hiroshima: I. Nishiki Print Co. Ltd.

Duval, R. (2002). The cognitive analysis of problems of comprehension in the learning of mathematics. *Mediterranean Journal for Research in Mathematics Education*, 1 (2), 1–16.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61, 103–131.

Duval, R. (2008). Eight problems for a semiotic approach in mathematics education. In L. Radford, G. Schubring & F. Seeger (Eds.), *Semiotics in mathematics education* (pp. 39–62). Rotterdam: Sense Publishers.
Empson, S. B. & Jacobs, V. R. (2008). Learning to listen to children's mathematics. I. D. Tirosh & T. Wood (Eds.), *The international handbook of teacher education* (Vol. 2, pp. 267–282). Rotterdam: Sense Publishers.
Evensen, L. S. (2010). En gyldig vurdering av elevers skrivekompetanse? In J. Smidt, I. Folkvord & A. J. Aasen (Eds.), *Rammer for skriving: om skriveutvikling i skole og yrkesliv* (pp. 13–31). Trondheim: Tapir Academic Press.
Fosnot, C. & Dolk, M. (2001). *Young mathematicians at work. Constructing multiplication and division*. Portsmouth: Heinemann.
Gravemeijer, K. (1994). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education*, 25 (5), 443–471.
Grevholm, B. (1998). Teacher students' development of concepts in mathematics and mathematics education. In T. Breiteig & G. Brekke (Eds.), *Theory into practices in mathematics education. Proceedings of Norma 98* (pp. 139–146). Kristiansand: Høgskolen i Agder.
Halliday, M. A. K. (1978). *Language as social semiotic*. London: Edward Arnold.
Jonsmoen, K. M. & Greek, M. (2012). "Hodet blir tungt – og tomt" – om det å skrive seg til profesjonsutøvelse. *Norsk pedagogisk tidsskrift*, 96 (1), 15–26.
Kilpatrick, J., Swafford, J. & Findell, B. (Eds.). (2001). *Adding it up: helping children learn mathematics*. Washington: National Academies Press.
Kinach, B. M. (2002). Understanding and learning to explain by representing mathematics: epistemological dilemmas facing teacher educators in the secondary mathematics "methods" course. *Journal of Mathematics Teacher Education*, 5, 153–186.
Krogh, E. & Jensen, M. J. (2008). *Portfolioevaluering og portfoliodidaktik*. Fredriksberg: Dansklærerforeningens forlag.
Leinhardt, G. (1987). Development of an expert explanation: an analysis of a sequence of subtraction lessons. *Cognition and Instruction*, 4, 225–282.
Leinhardt, G., Putnam, R. T., Stein, M. K. & Baxter, J. (1991). Where subject knowledge matters. In J. Brophy (Ed.), *Advances in research on teaching* (Vol. 2, pp. 87–113). London: JAI Press inc.
Leinhardt, G. (2001). Instructional explanations: a commonplace for teaching and location for contrast. In V. Richardson (Ed.), *Handbook for research in teaching* (4th ed.) (pp. 333–357). Washington: American Educational Research Association.
Leinhardt, G., Steele, M. D. (2005). Seeing the complexity of standing to the side: Instructional dialoges. *Cognition and Instruction*, 23, 87–163.

Lo, J., Grant, T. & Flowers, J. (2008). Challenges in deepening prospective teachers' understanding of multiplication through justification. *Journal of Mathematics Teacher Education*, 11, 5–22.
Ma, L. (1999). *Knowing and teaching elementary mathematics: teachers' understanding of fundamental mathematics in China and the United States*. Hillsdale: Erlbaum.
Måsøval, H. S. (2005). Student reasoning constrained by the didactical contract. *Nordic Studies in Mathematics Education*, 10 (3-4), 83–99.
Måsøval, H. S. (2011). *Factors constraining students' establishment of algebraic generality in shape patterns: a case study of didactical situations in mathematics at a university college* (Unpublished doctoral dissertation). Kristiansand: University of Agder.
Nicol, C. (1999). Learning to teach mathematics: questioning, listening, and responding. *Educational Studies in Mathematics*, 37, 45–66.
Niss, M. & Højgaard T. (Eds.). (2011). *Competencies and mathematical learning ideas and inspiration for the development of mathematics teaching and learning in Denmark* (English edition, October 2011. IMFUFA tekst no. 485). Roskilde University. (Published in Danish in 2002). Retrieved April 11, 2013 from http://milne.ruc.dk/ImfufaTekster/pdf/485web_b.pdf
Säljö, R.(1999). *Læring i praksis. Et sosiokulturelt perspektiv*. Oslo: J. W. Cappelens forlag.
Skott, J., Larsen, D. M. & Østergaard, C. H. (2011). From beliefs to patterns of participation – shifting the research perspective on teachers. *Nordic Studies in Mathematics Education*, 16 (1-2), 29–55.
Shulman, L. S. (1986). Those who understand: knowledge growth in teaching. *Educational Researcher*, 15(2), 4–14.
Strauss, A. & Corbin, J. (1998). *Basics of qualitative research: techniques and procedures for developing grounded theory* (2nd ed.). Thousand Oaks: Sage.
Stylianou, D. (2010). Teachers' conceptions of representation in middle school mathematics. *Journal of Mathematics Teacher Education*, 13, 325–343.
Winsløw, C. & Durand-Guerrier, V. (2007). Education of lower secondary mathematics teachers. *Nordic Studies in Mathematics Education*, 12 (2), 5–32.
Zaslavsky, O., Chapman, O. & Leikin, R. (2003). Professional development in mathematics education: trends and tasks. In A. J. Bishop, M. A. Clemens, C. Keitel, J. Kilpatrick & F. K. S. Leung (Eds.), *Second international handbook of mathematics education* (pp. 877–917). Dordrecht: Kluwer Academic Publishers.

Notes

1 Note that both the tasks and the students' explanations presented are translated from Norwegian language. In the translation we have tried to keep the exact content as far as it was possible for us.

2 Since the majority of students are females, we will use she-form when referring to individual students.

3 With "standard algorithm" we refer to the most common written procedure for multiplying numbers in Norwegian schools.

4 Four teacher students had a context also, but it was not used actively in the explanation. We will discuss this type of use of context later.

5 Standard algorithm used in Norwegian schools.

6 Not bold in original.

Anita Valenta

Anita Valenta is Associate Professor in mathematics education at Sør-Trøndelag University College in Trondheim, Norway. Her main interest is in developing a knowledge base for mathematics teacher education, research in student teachers' mathematical knowledge for teaching and algebra in the early grades.

anita.valenta@hist.no

Ole Enge

Ole Enge is Associate Professor in mathematics education at Sør-Trøndelag University College in Trondheim, Norway. He is interested in developing a knowledge base for mathematics teacher education, research on instructional routines in mathematics teaching and students' proportional reasoning.

ole.enge@hist.no