Assessment of university students' understanding of abstract binary operations

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This paper presents the results of a study on the use of interactive online tasks to assess students' conceptual understanding of abstract binary operations in a first-year linear algebra course. The assessment consists of recognition, identification and production tasks and uses verbal, graphic and symbolic representations of binary operations in numerous point set contexts. The aim of the study is to directly assess the students' understanding of binary operations and – more indirectly – to identify different profiles for the students' procedural and conceptual knowledge levels. A latent class analysis revealed different levels in students' conceptual understanding. Implications will be drawn for teaching abstract binary operations – and other similar concepts. Finally, some suggestions about conceptual qualifications for mathematics teacher education will be discussed.

Considering the heterogeneity in the mathematical background and motivation of university freshman students today, the necessity to adopt new tools and policies for tertiary mathematics education has become urgent. Acquiring a deep understanding of abstract concepts is perhaps the most challenging task of university level mathematics, since modern mathematics and its research rely heavily on abstraction, conjecturing, proof and creating theories. A solid structural understanding of the relationships between mathematical concepts and processes cannot be achieved without detailed knowledge of elementary concepts. Such central concepts are for example the *function concept* and its special case *binary operation*.

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It is evident that students must have a viable understanding of the function concept before they can understand binary operations. Here, a function means a rule (or relationship, correspondence) $f: A \rightarrow B$ which connects to each element x in a set A a unique value $f(x)$ in a set B. The set A of possible *variable* values x is called the *domain* of f and the set B of all potential function values is called the *co-domain*. The set of all *images* $f(x)$ is called the *range* of f. An *internal binary operation* in a set A is a two-variable function o: $A \times A \rightarrow A$, connecting to each pair (x, y) in $A \times A$ a unique value $xoy = o(x, y)$ in the same set A. An *external binary operation* in a set A is a function $K \times A \rightarrow A$, where K is a set (of scaling elements).

Since one important aim in the linear algebra course is to introduce an axiomatic system and develop theory upon it, an exact approach to binary operations is necessary: for example, it does not make sense to talk about associativity of o if we cannot be sure that all results xoy are defined, unique and again belong to A.

In order to emphasize the abstract nature of general binary operations, the set A in which the operation should constitute a function $A \times A \rightarrow A$, varies greatly from abstract character sets and Venn-diagrams to number and vector sets and their subsets, and similarly, a variety of sets appear as the scaling set K.

Many educational researchers have considered these concepts from slightly different points of view. Most studies on learning or teaching the function concept focus on the process of learning and difficulties in this process (Breidenbach et al., 1992; Carlson, 1998; Pesonen et al., 2002, Sfard, 1991; Tall, 1992; Tall & Bakar, 1991; Vinner & Dreyfus, 1989; Vinner, 1991; and a review in Carlson & Rasmussen, 2008). Less attention has been paid to the concept "binary operation" in an abstract sense. Brown et al. (1997) investigated the genetic decompositions of some notions in abstract algebra, including binary operation. Their study approach is qualitative and focuses on the mental constructions that students achieve on the action-process-object-schema scale described in Asiala et al. (1996).

In a quantitative study, Pesonen et al. (2005) showed that even university students have difficulties in recognizing a binary operation when it is given in a verbal form rather than in a symbolic or graphic form. Surprisingly, not much research has looked for different patterns of conceptual abilities that occur among tertiary level students with regard to functions or binary operations. Since binary operations in linear algebra are two-variable functions whose variables are usually multidimensional vectors they cannot be represented with traditional static figures. Therefore, the emphasis of the graphic representations is on interactive "living" figures, in which for example vector variables in the two-dimensional Euclidean plane can be changed continuously by dragging their endpoints using the computer mouse. According to the predefined function, which may be hidden from the user, the image vector changes in real time.

In our case, these figures are based on dynamic geometry Java applets (*Javasketchpad* and *Geometria*, see Pesonen, 2001; Ehmke, 2002), which we shall call *dynamic sketches*. A dynamic sketch can contain text, figures, geometric elements (points, lines, rays, segments, circles and more advanced constructions) and control tools: buttons and sliders can be used for showing, hiding, moving and animating the sketch elements. Dynamic sketches allow – and mostly require – the user to interact with the figures by mouse dragging or by using control tools. This engages the students with the content and problem setting and they get a "feeling" for the dependencies between the given parameters. Pesonen et al. (2002) report our earlier experiences of using dynamic sketches that represent functions. Some results concerning the use of dynamic sketches with the same or parallel study data have been reported in Ehmke et al. (2005) and Pesonen et al. (2005, 2006).

Objectives

The present study focuses on the students' conceptual understanding of abstract binary operations at first year university level. We attempt to find and classify different levels of students' procedural and conceptual knowledge concerning the two abstract binary operations introduced. Therefore, we developed computer-based tests structurally based on the learning phases in concept formation used in the theoretical framework MODEM (see below).

This framework and our modifications are presented shortly in the next section. The following sections contains the description of the study method (sample, design and statistical methods) and the results. The paper ends with a discussion and reference to the www-based appendices.

The MODEM framework

The test worksheets and the study framework are based on MODEM (Model construction for didactic and empirical problems of mathematics education) framework for concept formation in school level mathematics (Haapasalo, 1997, 2003). The leading idea of MODEM is to divide the learning process into consecutive phases and use the three representation forms of a concept in a systematic way. These representation forms are verbal (V), symbolic (S) and graphic in a broad sense (G), including static and dynamic figures, diagrams and tables. The original five phases of concept building in the MODEM framework are orientation, definition, identification, production and reinforcement, and the verbal-symbolic-graphic (VSG) task types are different in each phase.

The *orientation* phase forms the first phase of the systematic concept building. Here the students are offered opportunities to familiarize themselves with the concept presented in verbal, graphic or symbolic form. The students become familiar with the characteristic attributes of the concept.

The role of the concept *definition* is to offer students an opportunity to make their own investigations, to express the investigation results especially in verbal forms in each case, to argue within groups about these results. The phase should finally end with a social agreement of the definition.

The next two phases of concept building utilize the principle of dynamic interaction. The idea is to give students a sufficient number of opportunities to construct concept attributes and procedural knowledge based on them.

In the phase of *identification* the students have an opportunity to train themselves in identifying concept attributes in verbal (V), symbolic (S) and graphic (G) forms. Six different matching tasks are available: IVV, IVS, IVG, ISS, ISG and IGG. Note that for example an IVS task means identifying the same concept given in a verbal and in a symbolic representation form, which implies that IVS tasks are equal to ISV tasks.

In the *production* phase students are offered a possibility to produce a new representation from a given presentation of the concept. The development of production includes nine combinations: PVV, PVS, PVG, PSV, PSS, PSG, PGV, PGS and PGG. The tasks of identification and production should be achievable without complicated processing of information on the student's part.

In the phase of *reinforcement*, the goal is to train the students to utilize concept attributes and to develop procedural knowledge to be used in problem solving and applications, and finally, help them in encapsulating the whole thing to a conceptual entity.

We have reduced the phases of MODEM to suit better for tertiary level education, and have adjusted the task types accordingly. In our revised framework, the orientation and definition phases are replaced by an introduction of the concepts in traditional lectures and an orientation text module in the beginning of the test worksheets, including the definitions of binary operations, a discussion about the reasons for such definition formats, etc., some solved examples and some example tasks on recognizing a binary operation. *Definition recognition task* (Dr) means determining whether the representation features fulfill the requirements of the exact definition; a DrX task type requires the student to recognize the definition validity of type X representation. These tasks will be called recognition VSG-tasks. Here are some some examples of tasks used in this study.

Example 1 (DrV).

Internal: Let $A = \{a, e, i, o, u, y, ä, ö\}$, the Finnish vowels in alphabetical order. The rule o assigns the result to each ordered vowel pair (v_1, v_2) in the following way:

$v_1 \text{ o } v_2$ is the alphabetic characters before the vowel v_1.

Is o an internal binary operation in the vowel set A?

This operation is – for many reasons – far from being a binary operation. The result may be empty or consist of one or more alphabets.

External: Is the operation $*$ an external binary operation $\mathbf{R} \times A \rightarrow A$, when A is the set of real numbers and the result $c * \mathbf{x}$ is the square root of c?

Example 2 (DrS).

Internal: Let us define $a \text{ o } b = (a - b)/(b + a)$. Is o an internal binary operation in the real number set $\mathbf{R}$?

External: Is the operation $*$ an external binary operation $\mathbf{R} \times A \rightarrow A$, when A is the set of all vectors $\mathbf{R}^2$ with both coordinates strictly positive and

$$a * \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} |a| x_1 \\ |a| x_2 \end{pmatrix}$$

Example 3 (DrG).

Internal: In the dynamic sketch in figure 1 (left side) you see two points x and y and the result $x \text{ o } y$ of an operation o. Is this a binary operation in the disc seen in the figure?

External: In the dynamic sketch in figure 1 (right side) you see a plane vector $\mathbf{u}$, a number c on the real line and the result $c * \mathbf{u}$ of an operation $*$. Is this a binary operation in the whole plane $\mathbf{R}^2$?

The answers can only be found by examining the real dynamic sketches by dragging the variables $\mathbf{x}$ and $\mathbf{y}$ or c and $\mathbf{u}$, and seeing what happens to the images $\mathbf{x} \text{ o } \mathbf{y}$ or $c * \mathbf{u}$. Therefore further examples containing graphic

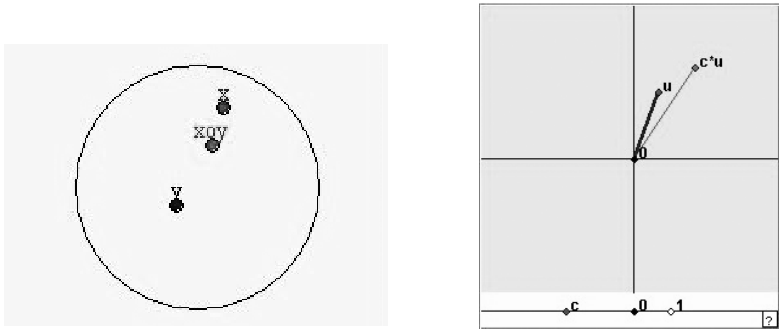

Figure 1. *Screenshots of two dynamic sketches*

representations are found in the www-appendix 1 (the link is given in appendix).

Identification (I) should consist of matching between all pairs of the three representations called *Identification* VSG-tasks IVV, IVS, IVG, ISS, ISG and IGG. We have limited this phase to only IVS, IVG and ISG tasks because of limited amount of test time.

Example 4 (IVS).

Match the verbal description on the left to the corresponding representation on the right in figure 2.

1) The ordinary addition of vectors.	A) $u \circ v = \frac{1}{2}(2u - v)$
2) To the first vector we add the projection of the second one on the horizontal axis.	B) $u \circ v = 2(v - u)$
3) Difference of the first vector and half of the second.	C) $u \circ v = \begin{pmatrix} \lVert v \rVert \\ 0 \end{pmatrix} + u$
4) To the first vector we add a vector parallel to horizontal axis, with the same length as the second vector.	D) $u \circ v = \frac{1}{2}(v - u) + u$
5) Twice the vector between the endpoints of the first and second vector.	E) $u \circ v = v + u$
6) The mean value of the vectors.	F) $u \circ v = \binom{u_1}{u_2} + \binom{v_1}{0}$

Figure 2. *An IVS task entity*

The production (P) phase consists of *Production* VSG-tasks PXY, in which a given representation of type X has to be expressed in form Y. Because of the shortage of time we used only PGV and PGS tasks.

Example 5 (PVS).

Express in a symbolic form this verbal expression concerning real numbers:

The ordered pair is mapped to their distance from each other.

Example 6 (PSV).

Express in a verbal form this rule concerning real numbers:

$a \circ b = (a - b)(a + b)$.

Table 1 summarizes the different task types that are use in this study. The phases on the left correspond to the revised phases in the MODEM framework and the task types indicate the VSG-task type. The number of tasks indicates the total amount of the tasks given to the students, and the last column contains references to the corresponding examples above, which we have taken from the task pool.

Table 1. *Combined description of tasks in the two binary operation tests*

Phase	Task type	Representation format	No. of tasks	Reference to examples
Definition recognition	DrV	Verbal	16	Example 1
	DrS	Symbolic	22	Example 2
	DrG	Graphic	16	Example 3
Concept identification	IVS	Verbal <-> Symbolic	12	Example 4
	IVG	Verbal <-> Graphic	3	see Appendix 1
	ISG	Symbolic <-> Graphic	3	see Appendix 1
Concept production	PVS	Verbal --> Symbolic	8	Example 5
	PSV	Symbolic --> Verbal	6	Example 6
	PGV	Graphic --> Verbal	6	see Appendix 1
	PGS	Graphic --> Symbolic	6	see Appendix 1

Methods: sample, design and statistical analysis

Sample

The study was carried out in a first-year course on linear algebra in the students' second semester. Most of the students (n = 92) were first-year mathematics or physics majors, but about one fifth of them were 1st to 3rd year computer science students. More than half of the students will qualify themselves as secondary or upper secondary school teachers.

Study design

The course in linear algebra can be divided into three parts: (1) concrete but generalized treatment of the systems of linear equations and the elements of matrix algebra, including matrix inverse and determinant, (2) abstract linear space, subspace, linear functions and their representations, inner product space, (3) eigenvalues and eigenvectors, matrix diagonalization and quadratic forms.

In the beginning of the course, the students' knowledge of one and two variable functions was measured using two computer-based tests designed for other purposes. However, by these tests the students got acquainted with assessment tests of such kind.

Soon after the definition of linear space in the lectures, including the definitions of the internal and external binary operations, the students did two computer-based tests about these binary operations. The test "Binary Operations 1" consisted of a collection of dichotomous yes-no definition recognition tasks (DrV, DrS, DrG). The test "Binary Operations 2" contained the identification and production tasks (IXY, PXY) about internal binary operation, followed by a condensed pool of all three task types about external binary operation. Later in the course, the students performed a traditional paper-and-pencil examination with four tasks. These tasks refer to general conceptual and procedural knowledge. Table 2 summarizes the test instruments used in the experiment entity.

Table 2. *The test instruments*

Test no.	Label	Item type	Test mode	Used in this study?
1	Function test 1	Tasks about functions of one variable	Computer-based	no
2	Function test 2	Tasks about functions of two variables	Computer-based	no
3	Binary Operations 1	DrV, DrS, DrG	Computer-based	yes
4	Binary Operations 2	DrV, DrS, DrG, IVS, IGS, IGV, PGV, PGS	Computer-based	yes
5	Examination	Procedural and conceptual tasks	Paper & Pencil	yes

Statistical analysis

The statistical analysis of the students' responses to the tasks was carried out in three steps. The first step was to combine all items that belong to the same task type to sub scales. By this, we built eight sub scales with regard to the eight task types that are given in column 3 of table 2: DrV, DrS, DrG, IVS, IGV, IGS, PGV and PGS. For each student we calculated his or her mean score for each sub scale, providing us with each student's response pattern consisting of these eight characteristic values.

In the second step, the response pattern of each student was used to group students with similar patterns in classes. The statistical method we used to identify such groups (or classes) was the probabilistic test procedure called Latent Class Analysis (LCA) (Hagenaars & McCutcheon, 2002). This procedure dates from Lazarsfeld and presents a test model for latent categorical classes in which the probability of different response patterns is analyzed. A response pattern consists of the manner in which an individual answers, i.e. the pattern of solving or not solving correctly task types. The aim of LCA is to find out the probability for a person to belong to a certain class if he/she has a specific response pattern. In principle, the person belongs to all classes, but with a different probability in each case. Each person is ultimately assigned to the class for which he/she shows the highest probability of belonging to (Rost, 1996; von Davier & Rost, 1996). We used the WinMira software (von Davier, 2000) for applying the LCA in this study.

In the third step of the data analysis, we evaluated differences between the groups identified in the previous step. Therefore, we calculated the mean scores of eight task types (DrV, DrS, DrG, IVS, IGV, IGS, PGV and PGS) and finally evaluated the mean scores of the final examination for each group as a validation of the results of the latent class analysis.

Results

The main objective of our study was to find and classify different levels of students' understanding of binary operations. Therefore, a latent class analysis was calculated sequentially each time with a predefined number of 2, 3, 4, 5 classes.

For each analysis, the model fit and the bootstrap results were evaluated. Finally, the three class solution showed the best model fit and was accepted. The average probability of a student to belong to a special group was high: for class 1 $p_1 = 0.90$, for class 2 $p_2 = 0.81$ and for class 3 $p_3 = 0.86$.

Figure 3 represents the profiles of the three student groups identified by using the LCA. The horizontal axis contains the eight task types, and the vertical axis shows the average probability of solving these task types. We labeled the classes as follows: *procedure-bounded* (dashed line), *procedure-oriented* (thick line) and *conceptual* student group (normal line).

These three profiles characterize the student groups according to their response patterns in the eight different task types. These group-specific

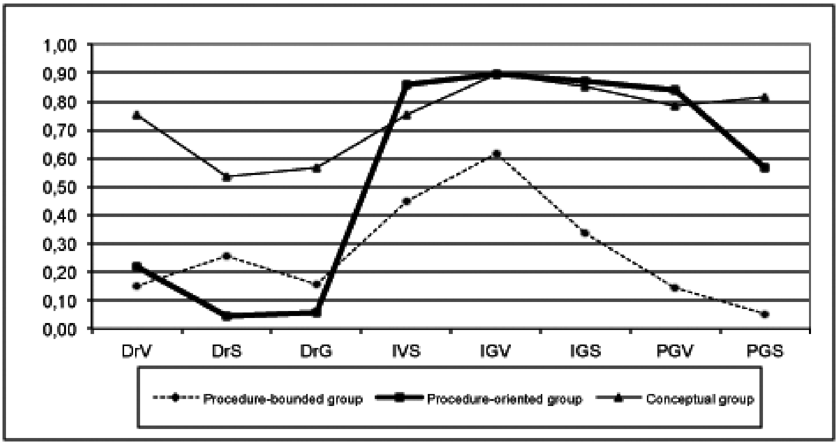

Figure 3. *Three profiles of students' response patterns in the binary operation test*

response patterns describe different levels of understanding the abstract concept of binary operation.

Procedure-bounded students (dashed line, comprising 44 % of the sample) show very low probabilities of solving the recognition and production tasks, but higher probabilities of solving identification tasks. However, students in this group have the lowest solution probabilities compared to the other two groups. Students seem to get stuck to simple procedural thinking procedures like identifying equivalent mathematical expressions given in different representation forms (see table 1).

Students in the *procedure-oriented* group (thick line, 36 % of the sample) can solve identification and production problems very well, but they fail in the recognition tasks. The term "procedural-oriented" refers to our interpretation that students in this group possess more advanced procedural abilities (like the ability to solve production tasks, see table 1). Their good performance in identification and production tasks shows some kind of informal conceptual insight instead of knowing or referring to the exact definition of the concept. One possible explanation for this different behavior was found in a closer look at the test questions, see appendix 2. Solving tasks that belong to the identification and the production phase requires the translation of mathematical expressions between two different representation forms as given in examples 4–6. For these task types, more procedural knowledge than conceptual understanding is needed. Tasks that belong to the concept recognition phase, however, require more conceptual understanding of the definitions of binary operations.

The *conceptual* students (normal line, 21 % of the sample) show a high probability to solve all task types. Students of this group differ from the two other groups in that they can solve not only identification and production tasks but are also good in concept recognition (see figure 3). This indicates that they have procedural and conceptual knowledge on the concepts (Gray & Tall, 1994; Hiebert, 1986).

To validate this classification of students' conceptual and procedural understanding, we compared the results of the three groups in a later examination (see table 3). The examination consisted of four traditional paper-and-pencil problems. Two of them required procedural knowledge and the other two also conceptual knowledge of the course content. Procedural knowledge and skills were sufficient in tasks 2 and 3 (solve a 3 x 4 system of linear equations and find the inverse of a 3 x 3 matrix). In task 1 the students had to "explain shortly and exactly" what is an internal binary operation and what associativity means. In task 4 they had to prove two assertions concerning matrix algebra (see appendix 3 for the original pdf-format exam sheet). For each examination task, the maximum result was 5 points. The mean values and standard deviations of the three groups are given in table 3.

Table 3. *Mean and standard deviations of the examination tasks in the student groups*

	Group 1 Procedure-bounded		Group 2 Procedure-oriented		Group 3 Conceptual	
	Mean	SD	Mean	SD	Mean	SD
Task 2 (Procedural)	3.71	0.90	3.58	1.01	3.95	1.00
Task 3 (Procedural)	4.06	1.32	4.15	0.95	4.18	1.11
Task 1 (Conceptual)	1.49	0.78	1.85	0.95	2.82	1.43
Task 4 (Conceptual)	1.77	1.46	2.40	1.39	3.24	1.34

The results of the group comparison validate our interpretation of the latent class analysis. There are significant differences between the three groups for the two conceptual examination tasks. The students in the conceptual group have significantly higher scores in the two conceptual tasks than students in the procedure-bounded group. Concerning the two procedural examination tasks, there are no significant group differences.

Conclusions and discussion

In this study, we used computer-based tests to assess first-year linear algebra course students' conceptual understanding of abstract binary operations. The assessment consisted of recognition, identification and production tasks using verbal, graphic and symbolic representations of binary operations in numerous point set contexts. The data analysis separated the population into three groups who showed quite different levels of expertise in the abstract concept "binary operation". This classification was in apparent accordance with their success in a later more general examination.

The students in the procedure-bounded group and in the procedure-oriented group were showing a lack of knowledge on the concepts. That can prevent a student from reaching higher levels of abstraction in learning further abstract mathematics. Also, for students who will qualify themselves as mathematics teachers it is a question if a process-bounded level of knowledge is sufficient enough. We should make sure that secondary school mathematics teachers possess viable conceptual understanding of the basic mathematical concepts like functions and binary operations. Therefore, our first recommendation based on the results is that a stronger focus on students' understanding of mathematical concepts is needed.

A second conclusion is to adjust the theoretical framework towards tertiary level mathematics context. Since the concept recognition phase appeared to be very difficult and the best predictor of students' future abstraction ability, it should be postponed to a later phase after identification and even after production, as a part of reinforcement phase in learning abstract concepts. On the other hand, a diagnostic pre-test of concept recognition could be carried out in the beginning, since the identification of the different levels of conceptual and procedural understanding provides us with an appropriate basis for planning adaptivity criterions. Then those students who already succeed in the pre-test can skip further training in concept recognition and move to more challenging topics. For those who fail in the pre-test supplementary learning exercises should be given.

Limitations of this study can be seen in the small sample size and that only one university is involved. Therefore, this classification should be validated in other samples and other universities. On the school level the study of Humberstone and Reeve (2008) could identify four profiles of arithmetics-algebra competence, that are somehow comparable to the profiles that we found in this study. However, their students' sample was much younger. Also, there are several slightly different ways to characterize levels of mathematical understanding according to various frameworks, e.g. Asiala et al. (1996), Gray & Tall (1994), Sfard (1991), Carlson & Rasmussen (2008, chapter 3) for an overview of the research literature. Here, it would be interesting to compare how similar the classifications of a population would be when made according to each framework.

The focus of our further research is in identifying the students' failures in mastering the concept of binary operation more precisely. Therefore, we want to develop recognition tasks that are more systematically constructed. These tasks require the student to identify violations against the definition of a mathematical function like missing images, non-uniqueness, and inconsistencies between the domain, co-domain and the expression of the rule. By this systematic assessment approach misconceptions that put obstacles in the way of understanding the function or binary operation concept could be more deeply identified.

References

Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D. & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In J. Kaput, A. H. Schoenfeld & E. Dubinsky (Eds.), *Research in collegiate mathematics education II* (pp. 1–32). Providence: American Mathematical Society.

Breidenbach, D., Dubinsky, E., Hawks, J. & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics*, 23, 247–285.

Brown, A., DeVries, D., Dubinsky, E. & Thomas, K. (1997). Learning binary operations, groups, and subgroups. *Journal of Mathematical Behavior,* 16 (3), 187–239.

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In J. Kaput, A. H. Schoenfeld & E. Dubinsky (Eds.), *Research in collegiate mathematics education III* (pp. 114–162). Providence: American Mathematical Society.

Carlson, M. & Rasmussen, C. (Eds.) (2008). *Making the connection: research and teaching in undergraduate mathematics education*. Washington: Mathematical Association of America.

Davier, M. von (2000). *WINMIRA 32pro. A program system for analyses with the Rasch model with the latent class analysis and with the Mixed Rasch model*. Kiel: IPN.

Davier, M. von & Rost, J. (1996). Die Erfassung transsituativer Copingstile durch Stimulus-Response Inventare. *Diagnostica*, 42 (4), 29–48.

Ehmke, T. (2002). Geometria: a tool for the production of interactive worksheets on the web. In M. Borovcnik & H. Kautschitsch (Eds.), *Technology in mathematics teaching. Proceedings of ICTMT5 in Klagenfurt 2001* (Vol. 25, pp. 137–140). Wien: öbv & hpt.

Ehmke T., Haapasalo L. & Pesonen M. (2005). Building a bridge between school and university – critical issues concerning interactive applets. In H. Ruokamo, P. Hyvönen, M. Lehtonen & S. Tella (Eds.), *Teaching-studying-learning (TSL) process and mobile technologies – multi, inter-, and transdisciplinary (MIT) research approaches* (Publications in Education 11) (pp. 39–47). Rovaniemi: University of Lapland.

Gray, E. & Tall, D. (1994). Duality, ambiguity and flexibility: a proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*, 26 (2), 115–141.

Haapasalo, L. (1997). Planning and assessment of construction processes in collaborative learning. In S. Järvelä & E. Kunelius (Eds.), *Learning & technology – dimensions to learning processes in different learning environments* (pp. 51–66). University of Oulu. Retrieved August 30, 2010 from http://herkules.oulu.fi/isbn9514248104/isbn9514248104.pdf

Haapasalo, L. (2003). The conflict between conceptual and procedural knowledge: Should we need to understand in order to be able to do, or vice versa? In L. Haapasalo & K. Sormunen (Eds.), *Towards meaningful mathematics and science education. Proceedings on the IXX symposium of the Finnish mathematics and science education research association* (Bulletins of the Faculty of Education 86) (pp. 1–20). University of Joensuu.

Hagenaars, H. J. & McCutcheon, A. L. (Eds.) (2002). *Applied latent class analysis*. Cambridge: Cambridge University Press.

Hiebert, J. (Ed.) (1986). *Conceptual and procedural knowledge: the case of mathematics*. Hillsdale: Lawrence Erlbaum.

Humberstone, J. & Reeve, R. A. (2008). Profiles of algebraic competence. *Learning and Instruction*, 18 (4), 354–367.

Pesonen, M. (2001). *WWW documents with interactive animations as learning material*. Paper presented in the Joint Meeting of AMS and MAA, New Orleans, January 2001. Retrieved August 30, 2010 from http://www.joensuu.fi/mathematics/MathDistEdu/MAA2001/index.html

Pesonen, M., Haapasalo, L. & Lehtola, H. (2002). Looking at function concept through interactive animations. *The Teaching of Mathematics*, 5 (1), 37–45.

Pesonen, M., Ehmke, T. & Haapasalo, L. (2005). Solving mathematical problems with dynamical sketches: a study on binary operations. In E. Pehkonen (Ed.), *Problem solving in mathematics education. Proceedings of the Promath meeting, June 30–July 2, in Lahti* (pp. 127–140). University of Helsinki.

Pesonen, M., Haapasalo, L. & Ehmke, T. (2006). Critical look at dynamic sketches when learning mathematics. *The Teaching of Mathematics*, 9 (2), 19–29.

Rost, J. (1996). *Lehrbuch Testtheorie Testkonstruktion*. Göttingen: Huber.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1–36.

Tall, D. (1992). The transition to advanced mathematical thinking: functions, limits, infinity and proof. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 495–511). New York: Macmillan.

Tall, D. & Bakar, M. (1991). *Students' mental prototypes for functions and graphs*. Retrieved August 30, 2010 from http://www.warwick.ac.uk/staff/David.Tall/pdfs/dot1991f-mdnor-function-pme.pdf

Vinner, S. (1991). The role of definitions in teaching and learning. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65–81). Dordrecht: Kluwer.

Vinner, S. & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, 20 (4), 356–366.

Appendix

The appendices 1–3 can be retrieved from http://wanda.uef.fi/mathematics/MathDistEdu/Animations2MentalModels/ConceptualProfiles/Appendices.htm

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