The case of Brandon: the dual nature of key ideas in the classroom

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This paper looks at proof production in the midst of classroom interaction. The setting is a college level geometry course in which students are working on the following task: Prove that two parallel transported lines in the plane are parallel in the sense that they do not intersect. A proof of this statement is traced from a student's idea, through a small group discussion, to a large class discussion moderated by a teacher. As the proof emerges through a series of increasingly public settings we see ways in which the key idea of the proof serves to both open and close class discussion. We look at several examples of opening and closing, showing how not only the key idea, but also the warrants and justifications connected to it, play an important role in the proof development.

The purpose of this paper is to better understand what facilitates and hinders proof production. The idea for this paper came from watching video-data of a class discussion of a proof in a college geometry class. What was compelling to us in the data was the nature of the discussion between a student named Brandon and the other participants in the class. Brandon stood resolutely by his initial idea for the proof, only tweaking small parts of the argument on occasion. At the same time several class members, including the teacher and the teaching assistant, were asking the right questions – questions that would lead to a more complete and correct proof – but they initially backed down as Brandon repeated his argument.

As we examined the classroom interaction more closely, we noticed the centrality of the key idea of the proof to this process. The key idea,

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which will be discussed more below, can be seen as the linchpin of the argument of a proof. Earlier work on key ideas highlighted the positive contributions of the key idea to proof production. People who had key ideas were able to write proofs, while people who did not have key ideas were not (Raman, 2003, 2004).

In contrast, our work here indicates that key ideas have a dual nature. There is still a positive side – if a person has a key idea, she or he or someone else can further the production of a proof by asking for justifications and warrants, thereby opening discussion. However there is also a negative side – if a person has a key idea, she or he might be so convinced of the truth of the argument that she or he (temporarily) shuts down the production of the proof, thereby closing the discussion.

What we gain from this investigation is a better sense of how proofs evolve (or fail to evolve) in the context of a group discussion. Through numerous examples of a proof discussion which at times opens up and at times closes down, we see the two ways in which the key idea operates in relation to the proof discussion. With the appropriate justifications, or a desire to search for such justifications, the discussion moves forward. Without such justifications, the discussion threatens to close, before a full proof has been formed.

This paper is divided into two parts. In next section we contextualize the study, both in terms of the main theoretical ideas and the relevant details about the data collection and analysis. Then we trace the trajectory of Brandon's proof, as it goes from being less justified to more justified, looking at the role of key idea in the opening and closing of the discussion surrounding this proof.

Background and context

Theoretical background

One of the central issues in proof research has been how to connect the more informal aspects of thinking (hunches, intuitions, pictures) with the formal (rigorous arguments, logic) (e.g. Fischbein, 1987; Schoenfeld, 1991; Viholainen, 2008). The reason this is central is that those connections turn out to be very hard to make. As Thomas Hales, the mathematician who proved the long-standing Kepler conjecture commented, "The hard part is going from the intuition to rigorous math to prove it" (Templeton, 2007).

One attempt to help bridge the gap between the informal and formal, was the identification of the key idea as the bridge between the two (Raman, 2003, 2004, 2006). The key idea, roughly put, is the idea that the proof hinges on, a sort of essence of the proof, which is often an intuition or some sort of informal characterization of the argument. It is key in two ways – it provides a way, if one's technical skills are adept enough, to create a formal argument, and it connects back to informal notions and ideas that give a sense of why the argument is true. A key idea gives the sense that "now I believe it", not necessarily a sense of how the proof should go forward. The key idea is a property of a proof (a proof has a key idea), though we often attribute a person as having or not having a key idea (e.g., we say that a particular person has a key idea if they have identified a key idea of a proof.)

Much work on proof to date has focused on individuals (e.g., Selden & Selden, 2003; Weber & Alcock, 2004). This study joins others like Balacheff (1988) and Zandieh, Larsen and Nunley (2008) that look at proof production in a social setting. While studies of individual and group proof production are obviously linked (see Sfard (2008) for a deep discussion about the possible nature of this link), we chose this social setting because the dialogue appeared particularly transparent in illustrating the various forces at work in shaping a particular proof, and the role the key idea plays in that process.

In particular, we will examine the role of the key idea in opening and closing class discussion surrounding the proof in question. By opening, we mean that the proof moves forward in some way, such as investigating the truth of a claim, finding a warrant for it, or working out the details. By closing, we mean that the work on the proof does not move forward, and is prevented from doing so. This can happen appropriately, when a proof is correctly finished, but most instances of closing in this paper involve the discussion shutting down before the proof is finished.

The proof of the theorem

The proof task in this study is the following:

> Prove that in a plane two distinct parallel transported lines are parallel in the sense that they never intersect.

By definition, two lines are parallel transports of each other if there exists a transversal line that cuts each of the original two lines at the same angle. One can think of moving from one line to the other in a way that holds the angle fixed along the transversal. The two lines being parallel transports is the main information that is given and hence should be used in the proof. However, as we will see in the data below, Brandon treats the statement of the problem almost as if it stated that the two lines are *parallel* (rather than parallel transports), an idea for which he has a strong visual image. The notion of parallel transport and the subsequent ideas of symmetry that Brandon and his classmates use in their reasoning about this proof are based on the textbook used for this course, Henderson's (2001) "Experiencing geometry in Euclidean, spherical, and hyperbolic spaces."

As a point of reference for our subsequent analysis, we present here a correct version of the proof that Brandon and the other members of the class seem to be working toward. This proof is similar to proofs of this theorem produced by students in earlier versions of this teacher's course. In particular we give the general argument and point out the key idea of the proof as well as the mathematical relationships of the key idea to the other parts of the proof.

> Proof (by contradiction):
> Assume the two distinct parallel transported lines intersect in one point. If the two lines intersect on one side, they will (by half-turn symmetry) intersect on the other side. However in a plane two distinct lines can only intersect once. So the lines must not intersect.

There are five important statements in this proof that each have a unique connection to the key idea and as such will provide different opportunities for the key idea to be used by participants in the proof production process.

> **Givens**: Two distinct parallel transported lines on a plane.
> **Key idea**: If the two lines intersect on one side they will intersect on the other side.
> **Conclusion**: The lines must not intersect.
> **Warrant 1**: By half-turn symmetry (That is, the reason that we say that the second intersection occurs is because the figure created by two parallel transported lines and the transversal along which they share a congruent angle has half turn symmetry about the midpoint of the transversal segment).
> **Warrant 2**: In a plane two distinct lines can only intersect once (That is, the reason we know that the lines must not intersect is that the class had previously established as a "fact" that two lines on the plane must intersect in 0, 1 or infinitely many points).

Note that Warrant 1 is not fully justified. To do so, one would need to use the given information about parallel transported lines to establish that the figure in question actually does have half-turn symmetry. However, this argument is somewhat lengthy and is never fully engaged by the students in the discussion below, so we have omitted it for space considerations. Warrant 2 establishes the proof by contradiction.

The relationship of each statement to the key idea can be diagrammed using a figure similar to that used by Toulmin (1969) and others (e.g., Inglis, Mejia-Ramos & Simpson, 2007; Rasmussen and Stephan, 2008).

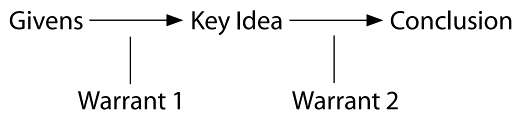


Figure 1. *Relationship between statements.*

From the diagram in figure 1 we can see the central role of the key idea in the proof and the relationships that it has to the connecting statements. Without the warrants, the argument would not be sound, but without the key idea, there would be no argument at all. Throughout the paper we will refer to different parts of this diagram that are either present or missing in the discussion surrounding the evolving proof.

Context of study

The data analyzed for this study comes from a semester-long classroom teaching experiment (Cobb, 2000) conducted in a geometry course of 28 students at a large American university. Data consisted of videotape recordings of each class session, copies of students' written work, researcher field notes, and videotaped debriefing sessions involving all members of the research team. Two video cameras were used, stationed at opposite ends of the room, each focusing on a different small group during group-work and on either the front of the room or the class during whole-class discussion. The data reported in this paper comes approximately two-thirds of the way through the teaching experiment.

Typical class sessions consisted of a brief introduction of the problem by the teacher (the second author), followed by small group-work on the problem and whole-class discussion of students' reasoning, interpretations and solutions. Data was coded independently by the two authors for instances of the key idea and examples of opening and closing of class discussion. There were 16 instances of the key idea in 48 minutes. It turned out that all instances of opening and closing were centered around utterances of the key idea. We look at these instances in more detail below.

Data and analysis

In this section we trace Brandon's proof through three stages, looking at ways in which the key idea plays a role in opening and closing discussion. Throughout this section we will bold parts of the text in which a speaker states the key idea, to make it easier to follow the role the key idea plays in the discussion.

Episode 1: Brandon shares his proof with his small group

Episode 1a: A first pass at the argument

The class begins with students forming groups to discuss their proofs. Brandon's group decides fairly quickly to focus on the proof of the following statement, discussed above.

> Prove that two parallel transported lines in the plane are parallel in the sense that they never intersect.

Brandon mistakenly thought the homework was due that day so he has written up what he thinks to be a correct proof of this claim. What follows is the first time he articulates his argument to his group.

Brandon: How about number one? I want to see if my proof works [group members indicate agreement].

Brandon: So, let's say if you have two parallel lines here and let's say they do intersect, then it'd be there [draws two parallel lines and shows them intersecting on one side, like one side of figure 2]. But, then by the half-turn symmetry for parallel lines, then that would ... So if you had like two parallel lines and let's say *they do intersect at a point*. So, you assume like not.

Josh: Right.

Brandon: Then by the half-turn symmetry of parallel lines, *you'd know they'd have to intersect over here, too*. But that's impossible because we know two lines can't intersect at two points [finishes drawing figure 2].

Josh: On a plane.

Brandon: On a plane. So it's a contradiction because this is just for a proof on the plane is all we're supposed to do for number one isn't it?

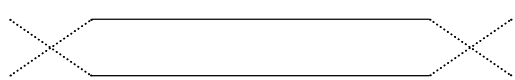


Figure 2. *Brandon's drawing*

Brandon opens the discussion by stating his proof, including the key idea, for the first time. Note that the contradiction structure is correct, and all five main parts of the proof are at least hinted at, with the key idea itself being what is most clearly and centrally specified. Brandon has misstated the givens as two parallel lines and not two parallel transported lines and hence Warrant 1 is similarly misstated as ”half-turn symmetry of parallel lines” instead of ”half-turn symmetry of parallel transported lines”.

The students react to Brandon's proof by asking him for warrants to establish that the argument is sound. A few students are concerned about the argument for half-turn symmetry:

Valerie: So this might be a stupid question but we know that parallel lines have half-turn symmetry? I mean, it seems obvious, but ...

Andrea: Well we know lines do. What are we rotating about, where they intersect?

Previously the class had established that straight lines have half-turn symmetry about any point on the line, but the half-turn symmetry of a grouping of two parallel lines had not been discussed. Brandon responds with a more articulated version of his ”key idea,” but he ignores Andrea's question about the center of rotation:

Brandon: Yeah, just that they should have that half-turn symmetry that they're identical on the other side. So, *if they intersect here then they should have the exact same symmetry over here*. So, I don't know. That's why I was asking. It semi works for me.

This is the first full statement of the key idea, and it seems useful to look at it closely. Note the ”just” in the first sentence, perhaps signifying that the argument boils down to this. Brandon seems to essentially be saying – what's important about this argument is this fact ”if they intersect on one side they intersect on the other”. He does not seem so concerned with how that fact gets established. This confidence in the key idea coupled with a lack of concern about its connections to other parts of the proof serves as a temporary closing of the discussion. However, his phrase ”It semi works for me” indicates that he is not yet convinced that the proof is good enough and this gives a possibility for a reopening of the discussion.

Valerie's first reaction is to let the discussion remain closed, indicating that she likes the idea and is not pushing for details. However, Brandon keeps the possibility of it reopening.

Valerie: It's a very clever idea.

Brandon: I know there could be questions.

In this section we see Brandon introducing his proof tentatively. He is not sure the proof works, he questions his own argument by contradiction, and he responds to a compliment of his idea by saying there could be questions. At this point, the proof discussion seems fairly open, in that the one who suggested the proof does not seem convinced himself that he is really done.

Episode 1b: A second pass at the argument

Josh does not yet understand Brandon's argument, and continues the discussion by asking Brandon to clarify the argument.

Josh: Yeah. Okay, umm, you've got to prove that, uh ... are we given that the lines on a sphere have, the lines on a plane you're saying have half-turn symmetry.

Brandon: See, it's a very ... ohh ...

Josh: Or do you have to prove that it has half-turn symmetry and then use it.

Brandon: I'm just saying by half-turn symmetry of lines that they should.

Josh: I mean, I understand what you're saying and I think that's right. It's just that like, you know ...

Josh's last comment seems to indicate that he understands the key idea, but it seems that perhaps he, like the women above, thinks more justification is needed for Brandon's half-turn symmetry argument. Brandon, however seems concerned only with the fact that the half-turn symmetry, however you establish it, would produce the contradiction that he wants.

In the next excerpt Brandon gets a chance to restate his argument. Significantly, he gives a different justification for his key idea (Warrant 1), shifting from the symmetry of the figure, which he refers to as "half-turn symmetry of parallel lines as a group," to the symmetry of the individual lines.

Brandon: Well, I guess you could say that if they are parallel then by half-turn symmetry this line would just be this line [traces his pencil over the bottom line], this line would just be this line still [traces his pencil over the top line]. *So everything that happens over here on this line has to happen over here on this line as well.* So, I'm going by half-turn symmetry of the lines. So, everything that happens, let's say if you just take some area that you may think is like parallel whatever – where it's supposed to be parallel – if somewhere off in here they do intersect, then by half-turn symmetry the exact same thing that we know lines have half-turn symmetry.

Josh: Right. I understand.

Brandon: So, it's not so much that this half-turn symmetry of parallel lines, as a group. It's just by half-turn symmetry of a line. That if you do have these two lines and they are parallel but then they do intersect ...

Josh: Then they're not parallel.

The move Brandon makes here, which we will refer to as Brandon's "fix," away from thinking about the symmetry of the entire figure to the symmetry of a line seems to be in reaction to the questions he got from the group about the half-turn symmetry. This way he does not have to state the center of rotation nor explain why one half of the figure will land on another. Providing another version of Warrant 1, if it had been correct, would pin down the key idea and thereby help to close down the discussion.

Note that what Brandon has *not* mentioned (nor anyone else in the group) is the fact that these lines are parallel transports (Givens). In fact, Brandon (as well as other class members) consistently says "parallel" instead of "parallel transports." As a result, he ends up proving a tautology: The lines are parallel because they are parallel. By ignoring this given, Brandon fails to gain crucial information about the figure (that parallel transports give equal angles) which in turn would help provide a correct warrant for the key idea. Instead he provides an incorrect warrant, which has potential to shut down the discussion prematurely.

Another thing to mention is that Brandon seems not to care so much *why* the half-turn symmetry works as *that* it works, so that he can establish his contradiction. The part of his argument that he holds onto is the key idea, while he is quick to change the warrant for it. It seems that having the warrant in place is more important for Brandon than being sure that the warrant is correct.

Discussion

The main role of the key idea in this episode is to keep the discussion focused on the main argument, not the details. This focus, in turn, helps Brandon – to the extent that he can – convince the other group members of his argument. After Brandon first states his idea, the discussion is open in the sense that students ask good questions to push him to justify the half-turn symmetry. This happens again when Josh has had some time to think about the argument. He agrees with the key idea ("it is great") but still pushes on the details.

However by the end of the discussion, after Brandon produces his individual line argument about half-turn symmetry, the discussion closes up. The small group is still positive towards the main argument ("clever idea") and seems mollified by Brandon's fix. It is as if the fact that the key idea is in place lowers the concern about the details. Once Brandon is convinced that he has the key idea, he is willing to compromise on the justification.

Episode 2: Teaching assistant (TA) and teacher look at proof

Episode 2a: TA comes to look at proof

There is a short exchange between the TA and the members of the group about the validity of the proof. The TA overhears Brandon and Josh discussing whether the lines are parallel. She asks a few questions to be sure that the students can complete the argument (Warrant 2).

TA: Why are they not parallel? Because *you've got one intersection point and then you do a half-turn about ...*

Josh: *And then you get another intersection point.*

TA: So why does that mean they're not parallel?

Brandon: Because you can't have two lines – no, it just proves that if two lines are parallel then they don't intersect.

While Brandon has the TA's attention, he wants to run the whole argument by her, including the key idea:

Brandon: Because that's what one says isn't it? Two parallel lines are parallel lines in that they never intersect. So I was assuming that they do intersect.

TA: Okay.

Brandon: And then by that – you know, by the half-turn symmetry of the *line* (emphasis added) that *whatever goes on let's say over here in this arbitrary side would happen over here as well*. So you get that two lines intersect at two points.

Josh: But two straight lines can only intersect at one point.

TA: Okay, that's what I was like, that little point. That's all I wanted to say. I think you had it. You just hadn't stated it probably. Just for more clarification. Okay.

Brandon has not given a justification of half-turn symmetry (Warrant 1) nor has the TA asked for it (because her concern is with Warrant 2). The TA seems to accept the key idea that "whatever goes on in this arbitrary side would happen over here as well" without a justification. So she does not catch at this point that the proof is faulty since in fact the students do not have a proper justification for this claim. The students go on to ask the TA if they are done and she says yes, they should write it up. What seems to matter is that the group has the key idea, which is easy to communicate, and they can draw from it the contradiction needed to end the proof.

Episode 2b: Teacher comes to look at the proof

A few minutes later the teacher comes over to examine the students' proof. There has been some discussion about an appropriate diagram for the proof, which will go on the large sheet of paper to be viewed by the class, and the students decide on figure 3.

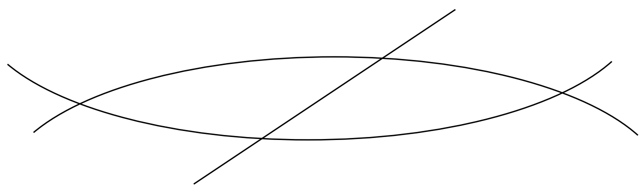


Figure 3. *Students' diagram*

For now we are not interested so much in why this particular figure was drawn, but in how the teacher responds to a proof with this picture. Basically the teacher comes over, looks at the proof and says that the students need to justify their half-turn symmetry argument (Warrant 1), which is what Valerie and Josh were concerned about in Episodes 1a and 1b. Brandon tries his modified symmetry argument, of the lines, not the whole figure:

Teacher: Umm, good, okay, so the next thing I want you guys to do to spiff this up – you don't necessarily have to write it on this piece of paper because you don't have any more room – is umm the proof of how you know that this figure has half-turn symmetry. And you should just state where the center of the symmetry is.

Valerie: Okay. Can I see the black [pen]?

Teacher: And you can put that [center of rotation] on here. Umm, state where it is in this [Valerie puts dot at center of transversal].

Brandon: Wouldn't it just be – so all it would be would be the half-turn symmetry of lines though right?

Teacher: Well the lines ...

Brandon: *So whatever happens, let's say you have a point P here, so whatever happens on let's say the right side of the line also has to happen on the left side of the line* because we know *lines* have half-turn symmetry. So, if this line is intersecting here then it's got to do the same thing here and this line is intersecting and it's got to do the same thing there.

Teacher: Okay. Point taken. So, you could without proving this *whole figure* has half-turn symmetry, you can still use half-turn symmetry to make this argument. Yeah, so that seems fair to me. Without … so if I ask on the test for example what are the symmetries of this figure? Prove one. Then I would want more than that you know. But that's not what's being asked and you're right for what's being asked you really just need that whatever happens on this side has to happen on that side.

Brandon: So would this be long enough?

Teacher: That's probably going to be okay. I'm glad you talked to me about it.

This is another example of Brandon closing a door with the key idea. When Brandon emphasizes half-turn symmetry of the line, the teacher thinks he means the half-turn symmetry of the transversal. She mistakenly assumes that this can imply a half-turn symmetry of the whole plane (since the transversal lies on itself and the half-turn transformation will move objects from one side of the transversal to the other.) This becomes a ground for misunderstanding in that the teacher does not actually evaluate Brandon's warrant for the key idea (Warrant 1), but Brandon thinks that she has endorsed it. Later, in whole-class discussion, these differences become clear, and the teacher catches her mistake. However, it seems that the fact that the key idea resonates with her own understanding of the proof makes her, at this junction, not push for further details.

Discussion

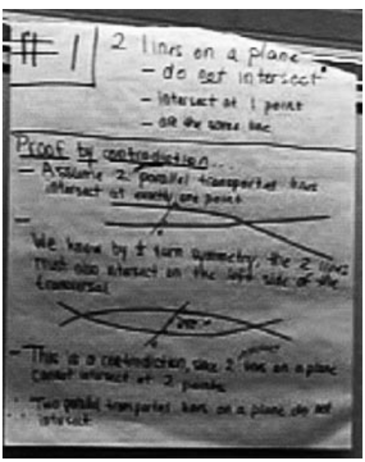
The TA and teacher seem interested in different parts of the argument. The TA is checking for Warrant 2 and the teacher for Warrant 1. However in both cases the fact that a faulty argument gets passed over seems in part due to the fact that the key idea is in place. One reason the teachers may have been fooled by Brandon's argument is that having the key idea may have given him a sense of confidence that he had a correct proof. While he is still uncertain if his proof is "long enough," he does not seem to hesitate about stating his argument and defending the half-turn symmetry of individual lines instead of the entire figure.

Another reason is that, while listening to Brandon's argument, the TA and the teacher had the key idea themselves. In addition they had justifications to back it up. When Brandon said the key idea, he connected with the teachers' sense of what the proof should be. Even though he didn't have the argument to back up the key idea, the teachers did and therefore under the pressures of class time, they did not scrutinize all the details.

Here we see that whether the key idea helps to open or close discussion depends, at least in part, on the knowledge of the discussants. In Episode 1 when Brandon shared his proof with his group-mates, the discussion was opened because they wanted him to provide a justification that they could not yet produce themselves. However in Episode 2 when Brandon is talking with teachers, who could produce a justification themselves, the discussion closed when they falsely assumed he had correct warrants. Unlike the students, who were genuinely curious about what Warrant 1 should be and kept pushing for justifications, the teachers were, at least for the moment, content that the most important parts of the argument were in place.

Episode 3: Whole-class discussion

The students go on to work on other proofs, and at the end of the class, the teacher convenes the class for a whole-class discussion. Students pin their work to a board in the front of the room and the teacher leads a class discussion about each of the proofs. The excerpt below comes from the discussion of Brandon's group's work. Their proof is replicated below (figure 4), with the actual student work on the left and a transcript of what is written on the right.



#1 2 lines on a plane
- do not intersect
- intersect at 1 point
- are the same line

Proof by contradiction ...
- Assume 2 parallel transported lines intersect at exactly one point
- We know by 1/2 turn symmetry the two lines must also intersect on the left side of the transversal.
- This is a contradiction since two non-distinct lines on a plane cannot intersect at 2 points.

∴ Thus two parallel transported lines on a plane do not intersect.

Figure 4. *Students' proof*

Sandy: I'm confused about why the lines aren't straight.

Brandon: It's just a picture problem. They're supposed to be like parallel and then – they're supposed to look like the opposite of the top picture.

Teacher: So, he could have drawn the first picture like that [draws figure 5]. I mean, then you would have had the problem that the angles wouldn't look quite equivalent. But you could have made the lines straight. And I think his was instead, let's try to make the angles the same, but that will force me to curve my lines. Is that why?

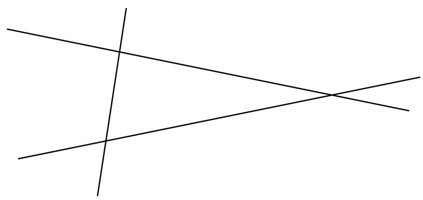


Figure 5. *Teacher's picture*

Several students go on to say how nice Brandon's proof is, and better than another one they have just seen. Next a student makes a comment that turns out to be pivotal to opening back the discussion:

Sue: I just have a question. After you do the half-turn symmetry, you kind of create two new lines don't you? And so you don't have a double intersection of the two same lines.

This comment essentially brings into question Warrant 1. Once it is made, the teacher realizes she misinterpreted Brandon's half-turn symmetry argument. She now draws the class attention to it.

Teacher: Hmmm ... This might bring us back to having to prove that the figure has half-turn symmetry after all. Did you guys get her question? Anybody need to hear it again? Say it again.

Sue: Okay, after you do the half-turn symmetry about the point on line T [the transversal], you've almost created two new lines and so you don't have a double intersection on the same two lines, you have a single intersection on two different sets of lines. So, I don't know how that's a contradiction.

Sue's comment opens the discussion in a major way. The rest of the class time on this question is spent discussing various ways the class tries to patch up the argument. Brandon tries to shift the argument to a different diagram. He draws his figure (figure 6) and restates his key idea, without justification.

Brandon: The picture I was actually looking for was something like this and say you know, for instance, they might intersect over here. Then *by half-turn symmetry of both these lines, then they'd also have to intersect* *like that*. I wasn't even saying so much half-turn symmetry of this line [transversal]. I was thinking more of the half-turn symmetry of these individual lines – not even the transversal so much [goes back to his seat].

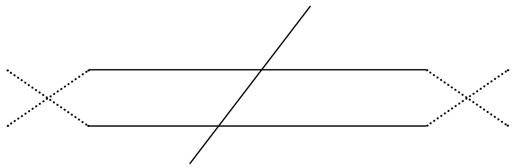


Figure 6. *Brandon's drawing*

Brandon's reasoning here is that the top and bottom lines – individually – would under rotation land on themselves, a fact that had been proven previously in the class. However, this statement by itself does not give a contradiction (if you rotate two lines that intersect once, individually, they will still only intersect once.) Brandon does not acknowledge this problem with his warrant, but rather provides figure 6 as clarification. The class does not accept Brandon's argument, and the students end up devoting considerable class time to trying to find a better warrant.

Eric is one of the students who keeps the discussion of half-turn symmetry open.

Teacher: Okay, so let me see if, I'm going to go back to what ... So, I think Eric was saying suppose you have this picture where you're not making any assumption about the angles. Maybe there's a line that crosses them at the same angle, maybe there's not. You know, either way. So, just in general. Umm, if I do half-turn symmetry for this line about some point and then for this line about some point, am I supposed to then assume that these two lines intersect on the other side?

Eric: Exactly. Like does half-turn symmetry really work in that case – prove it? Yeah.

Teacher: Yeah, is it really kind of doing the job you want it to do?

Eric: I think it does. You just have to prove that it does work.

Eric and Sue use newly created diagrams like figure 5 to pose challenges to the overall argument that has been presented. Brandon seems content to ignore the diagrams that conflict with his argument that symmetry of each individual line is sufficient. He focuses on the fact that the key idea is correct at the expense of not seeing how other diagrammatic depictions show weaknesses in his argument. The other class members, like Eric and Sue, while not throwing out the key idea, push on these weaknesses and demand justification to fill in the holes. The key idea remains fixed, while the desire to establish a warrant, or to think (erroneously) that one already has one, helps open or close the discussion, respectively.

Discussion

In this episode we see that the key idea is an anchor or a linchpin that allows a fairly wide-ranging discussion to happen. In the case of Brandon, it fails to push him further because he is unable (unwilling?) to see how his argument could be revised. In the case of other class members like Sue and Eric, the fact that the key idea is in place helps opens discussion. The class as a whole likes the general argument but pushes for clarity about the half-turn symmetry argument. No one seems to be debating the idea of proving by contradiction or even what the contradiction consists in. Rather the debate seems to focus around the justifications for the claim that make the half-turn symmetry argument work.

So what seems to open or close discussion is not just the presence of the key idea, but the extent to which it is connected to soundly reasoned warrants. Having a key idea without trying to determine a sound warrant seems to shut down discussion, and having a key idea while trying to find a sound warrant seems to open the discussion up. This seems to indicate that the key idea is not an isolated player in proof production. It is and must be connected to other essential parts of the proof to help the proof move forward.

Coda

In the end, the students run out of time to find a suitable half-turn symmetry argument themselves. One student asks the teacher to provide an argument and she does, using an argument for half-turn symmetry that had been done in a previous version of this course. However it is clear at this point that the teacher is supplying the appropriate details to the argument, not changing the thrust of Brandon's argument itself. This in a way is the ultimate shutting down of the proof discussion, as the final details of the argument are laid to rest.

Conclusion

In the three episodes above, we have seen the dual nature of the key idea. On one hand the key idea helps open discussion. It promotes proof-building in the sense that it focuses discussion on a critical part of the argument. On the other hand the key idea can help close discussion. It can block proof building in the sense that it makes people, like Brandon, so certain a claim is true that no further justifications are needed.

We have also seen some of the reasons *why* the key idea serves to open and close discussion. The key idea can be seen as an entity which by its nature requires a justification. What is it that makes the key idea hold? While the key idea is the essence of a proof, it is not a full proof or argument without the appropriate justifications. So when a key idea is presented in a group, it is natural for group members to want to be sure the justifications and warrants are fleshed out. As we saw in the data above, students can be quite persistent in searching for these justifications, and that is what appears to us as the opening of a discussion.

However, it can also be assumed incorrectly that a justification is in place, as was the case when each of the teachers in the class listened to Brandon's argument. In such a case the key idea is agreed upon without agreement of the justifications and warrants for the argument, and a false line of reasoning can go undetected. This is what appears to us as the closing of a discussion since the details of the proof have gone unchecked and unquestioned. What seems notable about the closing of discussion is that the key idea seems to give the owner a false sense of confidence that may come across to others as a reason to accept his or her proof.

The question for teaching, then, is how to keep the classroom discussion open. How do you keep students focused on finding justifications and warrants rather than feeling they are done if they find the key idea? There is no easy answer to this question, but this study provides a hint at an answer. That answer lies in somehow balancing the two forces set in motion by the key idea.

One of these forces comes from an inner conviction that one is right. Without this inner conviction, one has no motivation to find a justification nor to make a formal argument. However this force can potentially shut down the proof development, so it needs to be balanced by another force that seeks to pin the argument down with the force of mathematical evidence and argumentation. This is the force that keeps the proof open, until the argument is secured and the proof can be laid to rest. These two counter-balancing forces seem to almost need each other in the process of creating full and valid mathematical proofs.

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Sammandrag

Artikeln beskriver en studie av hur bevis konstrueras i interaktion mellan elever. Under en geometrilektion på ett amerikanskt college arbetar eleverna med följande uppgift: Bevisa att två parallellförskjutna linjer i planet är parallella i meningen att de inte skär varandra. Formuleringen av beviset följs från en idé från en av eleverna, via diskussion i en mindre grupp till en lärarledd diskussion i helklass. Allteftersom beviset utvecklas genom en följd av diskussioner i allt större grupper finner vi olika sätt varpå bevisets "key idea" bidrar till att både öppna och sluta diskussionen. Vi beskriver flera exempel på öppnande och slutande, och visar hur inte bara nyckelidén, utan även de rättfärdiganden och motiveringar som är knutna till den, spelar en viktig roll i utvecklingen av beviset.