Learning mathematics through inquiry

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The role of inquiry in teaching and learning has been discussed for a long time and by many leading educational philosophers and analysts. The purpose of this article is to analyse the assumptions and some of the outcomes of two interrelated and extensive developmental projects in mathematics teaching and learning in Norway. The projects – referred to as the KUL projects (Knowledge, Instruction, Learning) – aimed at introducing the notion of communities of inquiry as a basis for developing mathematics teaching and learning in participating schools, and as a model for organizing developmental work in cooperation between teachers and researchers. In several respects, it seems as if the projects have been successful in the sense that they were accepted by the teachers (especially at lower levels) as a productive mode of engaging in developmental work. In the article, the interpretation of the concept of inquiry in the projects is scrutinized. It is argued that in order to develop our understanding of inquiry processes, detailed analyses of the nature of inquiry in interactional activities in mathematics learning is necessary. It is also argued that the notion of inquiry adopted by the projects is based on a conception where inquiry is seen as a means of learning mathematics better. An alternative conception is to see inquiry as a means of promoting critical thinking in which understanding of mathematics is at the core of the development of more general reasoning skills that play a central role in a democratic society.

The notion of inquiry plays a significant role in educational philosophy and in the development of educational practices, and it has done so for a very long time. In the following, we will discuss inquiry in the context of mathematics education. From this particular starting point, there are at least two sets of issues that we want to explore:

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1 Inquiry can refer to processes of teaching and learning mathematics. One can try to organize these processes in such a way that students can construct and develop their mathematical conceptions and understandings through approaches that are based on inquiry. Thus, inquiry here is conceived as a philosophy for transforming mathematics learning in a specific direction.

2 Inquiry can refer to the active exploration of ideas for developing educational practices. Thus, it is possible for teachers and researchers in cooperation to inquire into new ideas for organizing teaching and learning of mathematics (or any field). In this case, inquiry is conceived as an approach for joint learning about educational activities and for developing the teaching and learning on the basis of the insights generated.

Of course, one can imagine many connections between these two forms of inquiry. In this article, we will pay particular attention to the clarifications of the notion of inquiry with respect to issue (1), although we will keep issue (2) in mind.

Our attempts at clarifying notions of inquiry refer to the projects, *Kunnskap, utdanning og læring* (KUL) carried out between 2004 and 2007. The idea behind the KUL-projects has been to develop teaching and learning practices in mathematics in close cooperation with local schools. The projects have been carried out by scholars and didacticians at the Department of Mathematical Sciences at the University of Agder (Kristiansand. Norway) in cooperation with teachers and staff at a number of schools in the Kristiansand region. We talk about the KUL-projects in plural, since there were two projects, namely *Learning Communities in Mathematics* (LCM) led by Barbara Jaworski (2007a, b) and *ICT*[1] *in Mathematics Learning* (ICTML) led by Anne Berit Fuglestad (2007a). The LCM and ICTML projects are closely related. They share perspectives and approaches, and together they form what is locally referred to as the KUL-projects.

We were invited to provide a formal evaluation of the KUL-projects (see Skovsmose & Säljö, 2007). This evaluation was carried out when the project activities were coming to an end, but while the publication of results was still under way. The evaluation showed that in many respects the projects had been successful, for instance in terms of the acceptance by the staff of the participating schools of the basic ideas for developing mathematics teaching.

The following reflections, however, are not part of the evaluation process, although highly inspired by it, and by the publications appearing from the projects so far. In the Nordic context, the KUL-projects are quite extensive and perhaps also rather unique in their attempts to create a framework for promoting inquiry-based teaching and learning. The systematically worked out theoretical underpinning, the ensuing cooperation between researchers and schools, and the many research activities add to the interest in further discussing the outcomes and their implications. Our aim here is to discuss the notion of inquiry by relating our own work and research regarding processes of inquiry to some features of the KUL-projects. Thus, we intentionally do not go into all the many different features of the projects, and our discussion refers to our own interests in issues of inquiry and learning.

The KUL-projects

The aims of the KUL projects have been to establish close cooperation between a university department and schools, and to test and implement the ideas of developing mathematics teaching and learning through the principles of establishing communities of inquiry and through the uses of information technology. The KUL-projects explored inquiry in both senses described in (1) and (2) above, but, as mentioned, we concentrate on discussing inquiry as a feature of teaching and learning processes in mathematics.

The KUL-projects can be characterized as developmental activities with intense cooperation between researchers (referred to as didacticians in the project), Ph. D. students and teachers (including school leaders) at the participating schools. One central activity has been a series of workshops with all parties involved on various issues connected to the attempts of establishing communities of learning and inquiry. Another activity has been cooperation at schools between local teachers and staff from the projects attempting to experiment with and develop teaching activities (cf. e.g., Daland, 2007; Eriksen, 2007). As an additional data source, longitudinal tests of students' mathematical abilities and attitudes to mathematics have been collected (Grevholm, 2007). Interviews with teachers have also been part of the project activities throughout the process.

In terms of the overall organization of the project activities, the work has followed the logic of a developmental research paradigm, which has some parallels with what is presently referred to as design research (cf. Jaworski, 2004). The notion of a design cycle has been used as a heuristic model for structuring the interventions into the teaching and learning activities in the classroom. Such a design cycle has the following elements: design, action, observation, reflection, and feedback. The logic of this cyclic process is that teachers, in cooperation with researchers from the university, design teaching and learning activities to be tried out in the classroom (cf. Jaworski, 2007a, b). The implementation takes place during the action-phase, which is simultaneously what is observed. After the lesson and the teaching and learning activities, teachers reflect on the activities and compare what has happened with what was intended. During the reflection and feedback phases, the participants analyse the experiences made and modify and develop the ideas further.

The design cycle is also intended as a learning cycle in which ideas and concrete educational experiences provide a basis for continued developmental work and "expansive learning" on the part of participants (Goodchild, 2007). In comparison to most design research, the approach differs in one important respect: the heavy involvement of teachers during all phases of the work. Thus, there were no 'designers' who brought ready-made ideas and solutions to be tested in the classrooms. Instead, the design was a joint undertaking by the teachers and the didacticians, and it was *per se* intended as a learning activity, which is an interesting element in the approach.

An intellectual core of the activities is the notion of 'community of practice' in the sense of Lave and Wenger (1991) and Wenger (1998). Thus, at the outset the project work itself was organized as a community building exercise in which sustainable partnerships between didacticians and teachers were sought to be established and maintained. As an extension of the notion of community of practice, the concept of community of inquiry was formulated (Jaworski, 2005, 2006). The norms of such a community imply that members engaged in activities such as teaching or research assume a stance of critical and analytical questioning of their own practices. This will deepen their understanding and generate new insights for further development of teaching and learning practices. Thus, the KUL-projects establish a close connection between inquiry into mathematics, into mathematics teaching and learning, and into researching the activities of mathematics teaching and learning (Jaworski et al, 2007). These are related, but in important manners different, levels of inquiry, as we have already alluded to.

The idea of inquiry

Over the past hundred years, the term inquiry has played, and continues to play, a significant role in the attempts to reform education and educational practices. The term, and the ideas, are used in pragmatist, sociocultural and various constructivist perspectives as a means of articulating how teaching and learning should be transformed from its traditional reliance on transmission models of teaching and rote learning. The idea of inquiry is an important element of John Dewey's epistemological considerations as well as of his general attempts to reform educational practices along progressivist ideals. He argued that "schools still teach from textbooks and rely upon the principle of authority and acquisition rather than upon that of discovery and inquiry" (1966, p. 280), and, as a consequence, the "statements, the propositions, in which knowledge, the issue of active concerns with problems, is deposited, are taken themselves to be knowledge" (ibid., p. 187). Or, alternatively expressed, the "record of knowledge, independent of its place as an outcome of inquiry and a resource for further inquiry, is taken to *be* knowledge" (loc. cit., italics in original). Also, in his writings on epistemology towards the end of his life, he considered inquiry as an "equivalent of knowing" (Dewey & Bentley, 1949, p. 295), and a characteristic of science is precisely that it is the "conduct of inquiry *as inquiry*" (ibid. p. 283, italics in original). This made science, if properly understood, a model for how learning can be organized in many settings.

Also in constructivist and sociocultural traditions, ideas of inquiry-based learning have been important in articulating how teaching and learning should be developed. Jean Piaget, for instance, considered children's self-guided, curiosity driven and discovery oriented, experimental activity essential for cognitive growth. It was through such engagement with the world, rather than through explicit teaching (cf. Piaget, 1980, p. 715), that children would develop. In the sociocultural tradition, the idea of communication and interaction as means to appropriating knowledge is essential. For instance, Wells (1999) emphasizes that "inquiry does not refer to a method", rather it "indicates a stance toward experiences and ideas – a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" (p. 121). This formulation closely echoes the ideas that have underpinned the KUL-projects.

Thus, the idea of inquiry as a model for how to organize teaching and learning appears as a central element in different theoretical perspectives. Even though the definitions of what inquiry implies may differ slightly, there are still commonalities which concern the nature of activities that students are supposed to engage in as learners: learning follows from an active and (in most cases) collaborative engagement with the world in which knowledge and insight emerge from joint interactional practices. In order to be more specific in our discussion about inquiry, we will address the following issues:

- We will characterize inquiry through *processes of communication* that take the forms of dialogue (or multilogues). On this basis, we will consider to what extent inquiry processes have been documented and analysed within the KUL-projects.
- We will characterize inquiry in terms of *issues that are being addressed* through the inquiry process. We will do so by describing different landscapes of investigation, which can be addressed in inquiry processes in mathematics. Furthermore, we will consider the diversity of landscapes of investigation that have been presented as part of the KUL-projects.
- We will address the very notion of inquiry and consider in what sense this concept can be a part of different conceptions of mathematics.

We will do so with particular reference to research we have been involved in, and which in our opinion may contribute to a discussion of inquiry as part of processes of teaching and learning mathematics. Thus, we take some of our own interpretations of inquiry-based learning and connect them to what has been published within the KUL-projects.

Inquiry and communication

In *Dialogue and learning in mathematics education*, Alrø and Skovsmose (2002) develop theoretical notions and perspectives which make it possible to connect 'dialogue' and 'learning'. Dialogue is presented as an *inquiry process*, which aims at bringing about new insights: "Dialogue in this sense is different from instruction, order, and persuasion. Dialogue implies a willingness to question one's understandings and pre-understandings and to examine what is new and different but also what is considered knowledge already acquired. Entering into a dialogue means taking ownership of the process of investigation" (Alrø & Skovsmose, 2004, p. 41).

According to this definition, a dialogue has no pre-defined direction, and the results can never be predicted in any absolute sense. Dialogues take place in an open space "between what is already known and what one might come to know" (ibid, p. 41). This implies that the interlocutors do not always know where the paths of the dialogue "will lead and what might be encountered on the journey" (ibid., p. 41). In other words, the "travel is risky, and so is a dialogue" (ibid., p. 41). As normally understood, the concept of risk has positive and negative connotations similar to the notion of chance. One can gamble, and one might win. One can take a chance when one buys a used car, and it can turn out to be a bad decision. A dialogue also includes *risk-taking* in the sense of engaging in communication which to some extent is unpredictable. It can bring about new insight; it might also bring about unpleasant challenges. A dialogue can run out of control. It is possible that one will come to see things in new and different ways, and in this way a dialogue is a part of a collective learning process.

A dialogue in this sense is based on *equality* and a respect for diversity. This does not mean that a dialogue presupposes symmetry in all respects between the interlocutors (Linell, 1998). On the contrary, differences in opinion and divergent interpretations are productive elements of dialogues. Asymmetries are the foundations of human interaction and of human learning (Jakobsson, Mäkitalo & Säljö, in press). We learn from each other not by completely sharing perspectives but by noticing differences and by profiting from them (Rommetveit, 1974). Thus, when talking about dialogues, we are ideally speaking of equality, complementarity and mutuality in the context of a communicative project but not of complete symmetry or overlap in perspectives or opinions. In an ideal dialogue there should be no use of overt power or force, no up-front persuasion of the other, and no-one should win at the expense of the other. The purpose or outcome of a dialogue should not be defined or decided on through authority.

To be productive, a dialogue implies developing a dynamic interactive process between partners. This applies to a dialogue in general, but also when a dialogue is taking place in an instructional setting. Even when the teacher is the more knowledgeable and competent partner in the dialogue, classroom conversations can be dialogic in the sense that the parties take an interest in the other person's ideas and assumptions. The roles can be different, and so can the competences that emerge. The pupil may learn about mathematics, but the teacher may learn about what makes the learning of a particular mathematical content difficult. Thus, following Alrø and Skovsmose, we understand a dialogue in terms of its qualities of inquiry, risk-taking and equality.

According to speech act theory, to speak means to act. Using language is tantamount to acting in a social setting, and there are many ways of acting through language. One can order, blame, question, correct, etc. Such acts can all be interpreted as speech acts, but they cannot be assumed to be dialogic acts in the sense we employ the term here. Dialogic act are interpreted as a particular form of speech act. They are collective acts. One may assume that speech acts can bring about learning: thus instruction and examining are example of speech acts, but we would not think of them as dialogic acts. As part of a learning process, dialogic acts provide learning with certain qualities, and it makes sense to talk about dialogic learning (see also Säljö, 2005).

It might be a simplification to think of a dialogue as a sequence of dialogic acts, i.e. as made up of a stream of atomic dialogic acts. A dialogue is more than an adding up of dialogic acts; it has a life and logic of its own (just as a human body is more than what one gets when adding its organs). And we must not forget that a dialogue is embedded in institutional traditions of what it means to communicate, learn and know in the classroom Säljö & Wyndhamn, 1993). This means that one may better interpret the dialogic acts as indicators of a dialogue taking place.

We may talk about *dialogic learning* when a teaching-learning process includes a rich variety of dialogic acts such as "getting in contact", "establishing a shared context", "reformulating", "challenging" and "evaluating" (cf. Alrø and Skovsmose, 2002, 2004, for a complete description). The dialogic acts can occur in different clusters and combinations, when teachers and students engage in an inquiry type of cooperation. Thus, the density of dialogic acts is an indicator of an inquiry taking place. Or, to put it more explicitly: *an inquiry process is characterized by the density of dialogic acts*. Alrø and Skovsmose (2002) put these different dialogic acts together in an Inquiry-Cooperation Model (the IC-Model), which then serves to characterize an inquiry process.

The publications emerging from the KUL-projects so far include many references to inquiry processes taking place in the classroom. However, it is a surprise to us that there are only a few cases where substantial documentation of inquiry processes or patterns of classroom interaction between teachers and students, or among students, has been provided. We have found some documentation of such concrete inquiry processes in Bjuland, Cestari and Borgersen (2008a, b), but even in these cases the details of the inquiry processes are not fully attended to. Rather, the focus here is on the issue of uses of gestures, and the combination of gesture and talk, in the mathematics classroom. The details of these analyses are interesting, but the authors do not explicitly discuss the multimodality of human communication in relation to inquiry. Even in the section on Inquiry-based activities in classrooms in the major publication to appear from the project so far (Jaworski et al., 2007, section 2), the analyses do not seem to delimit, describe and conceptualize the notion of inquiry processes in any great detail. There are some further indications of what is claimed to be inquiry processes in some of the articles in this publication by Jaworski and her colleagues when the activities of the students are presented. However, the focus is more on presenting student practices rather than further exploring the nature of inquiry in order to theorize it more explicitly (cf. for instance, Borgersen & Bjuland, 2007).

One may assume that there are different explanations for the low priority given to documenting processes of inquiry so far in the publications from the KUL-projects. One reason could be that the concept behind the projects would lead to a conclusion that it is not necessary to focus on patterns of interaction, including dialogues, in the analytical work or in the presentation of the results. But such an assumption, we find questionable. It might also be the case that the teams behind the KUL-projects have not intended to go into depth about inquiry processes as part of the learning of mathematics. But this is equally unlikely, as the inquiry into teaching and learning possibilities, i.e. inquiry of type (2), is a prerequisite for establishing inquiry processes in the classroom, i.e. inquiry of type (1).

In our opinion, inquiry processes must be understood as interactional achievements and as parts of the joint construction of meaning. So if one wants to document that an inquiry process has taken place, in-depth analyses of interactional processes are necessary. Otherwise, inquiry will appear as a black box in the argumentation; it is assumed to take place, but no-one knows what it looks like or what the criteria are for considering a teaching and learning sequence as an instance of inquiry. For instance, there are many forms of interaction which are not likely to bring about inquiry processes. Teachers' quizzing strategy, aimed at making the students produce a specific answer to a given question, will often not support inquiry; instead it generally invites a guess-what-the-teacher-is-thinking strategy, where students often demonstrate a well-developed sense for this kind of guessing. [2]

A project aiming at analysing the role of inquiry in learning in all these different respects cannot, and should not, avoid attending to the details of patterns of interaction in the classroom. It is here that some of the unique contributions of research by people specializing in teaching and learning in mathematics can be made. The IC-Model brings together dialogic acts that characterize an inquiry. Naturally, one could assume that there might be other inquiry-cooperation acts than those included in the IC-Model, but it is difficult to assume that claims about inquiry processes can be made without considering the fine details of interaction and joint meaning-making.

That this has not been documented, however, does not mean that the classroom activities within the KUL-projects did not provide rich examples of inquiry. This might well be the case. Our point is only that *so far* the KUL-related research has not paid enough attention to this obligation to deepen our understanding of inquiry as interactional achievements. The lack of such research documentation and analysis can, however, be repaired as the members of the KUL-projects have been very productive in other respects with a lengthy list of publications. Data from many different situations have been collected, for instance from workshops, seminars, school meetings, classroom settings, interviews, briefing papers, longitudinal tests, exchange of e-mails and other correspondence. The data also exist in varied forms: as video and audio recordings, word and text documents including data reductions (factual summaries of the content of video and audio recordings) and transcriptions of recordings, PowerPoint presentations, copies of texts produced by pupils and teachers, Excel sheets, and objects from software such as Cabri and Geogebra. Such data have been collected by all didacticians. The fundamental principle is that the data are owned by the project as a whole, not by individuals. All data exist in digital format and are stored on the UiA main file server using a standard windows-based filing system. The data base created from the KUL-projects is impressive and is an important resource for further research on the nature of inquiry in both senses mentioned in (1) and (2) above.

Inquiry and learning mathematics

Inquiry not only concerns patterns of interaction, it also concerns the issues and topics inquired into. This observation is especially relevant in the context of mathematics education. To a significant extent this education has settled within what we can refer to as the *exercise paradigm*. This implies that the activities engaged in the classroom to a large extent involve struggling with pre-formulated exercises that get their meaning through what the teacher has just lectured about. An exercise traditionally has one, and only one, correct answer, and finding this answer will steer the whole cycle of classroom activities and the obligations of the partners involved: The teacher has to explain how to solve a particular type of exercise; the students have to try to do so; the teacher must check the students' solutions, as mistakes have to be eliminated from the mathematics classroom; a student could sometimes be asked to present a solution to the whole class, and so the cycle continues.

A preoccupation with pre-formulated exercises is not, however, the only approach to the learning of mathematics. There are many alternatives to the exercise paradigm. The concern for trans-disciplinary issues makes many forms of traditional exercises obsolete. The notion of problem-based learning has had an impact on mathematics education, and the project organization of mathematic education has also brought us far beyond the exercise paradigm.[3] The ambition of promoting mathematical inquiry can be seen as a general expression of the idea that there are many educational possibilities to be explored beyond the exercise paradigm.

A *landscape of investigation* refers to a learning milieu different from those structured through exercises (see, Skovsmose, 2001, 2003a; Alrø & Skovsmose, 2002). Landscapes of investigations can be of very different types, and as an illustration we mention three such types.

Landscapes of investigation can be located within mathematics, for instance within geometry, where one can explore geometric properties using a dynamic-geometric software. It is in fact possible to locate landscapes of investigation within any mathematical domain. As an illustration one can open a landscape within combinatory, accessible also to the youngest school children, by asking: How many different "animals" do you think you can make with centi-cubes? What about using 5 centi-cubes? One can certainly produce a long animal connecting the five centi-cubes in a row. One can also place the four in a row, and put the last one on top of one of the others, so that it looks like a one-humped dromedary. One could put three in a row and put the two others beneath as legs. Well, how many different animals could be made by 5 centi-cubes? And when are such animals in fact different? Would a two humped camel be different from an animal with two legs? Making animals from centi-cubes is a case where one is operating within the frame of a mathematical discipline of combinatorial analysis. There are many issues to explore about animals, and this can be done without gluing the teaching-learning process to a sequence of pre-formulated exercises. Even though we talk about "animals" in these instances, we are dealing with a landscape of investigation clearly located within mathematics, and the variation of the design of such tasks clearly prepares the ground for various kinds of generalizations and abstract forms of mathematical reasoning.

Many landscapes of investigation, however, include references to non-mathematical domains, often in the form of some kind of semi-reality. For instance, one can open a landscape of investigation around a taxi-geometry. This is geometry for taxi-driving in a city where the streets are spread out in a network of squares. One can think of New York with streets and avenues crossing each other in right angles. Given such a squared network of streets and avenues, one can start exploring some of its properties. For instance: What is the shortest distance between two given points in this network – a point being defined as the intersection of a street and an avenue, and the distance being measured in units of blocks? Let us say that we have determined the distance between two points to be 7 blocks. One could then ask: How many different shortest distances are connecting two points with a distance of 7 blocks? What would be the number of shortest distances in case the distance is 5, or 7, or 119 blocks? Or the inquiry could take a different direction: One could start from a point and take a taxi-trip in the city, and come back to the same point. How long was the trip? Well 18 blocks. Take a new trip and find out about the distance. It was longer, it was 26 blocks. Take an even longer trip. It became 46 blocks. The lengths of the taxi-trips seem to be even numbered. How could that be? Would it be possible to drive around in the city and return to where one started and make a trip of an odd number length? Through such questions, one can try to open a taxi-geometry for exploratory activities.

There are very many properties to be located in this geometry. One should see such questions, not as preliminary formulations of exercises, but rather as invitations for students to explore a landscape of investigation. In this case we have to do with a kind of semi-reality, as we find references to familiar entities such as streets, avenues, block, and distance within a city. But at the same time there is something artificial about a taxi-geometry. It is not a geometry used by real-life taxi drivers. Nevertheless, it provides a landscape which can be explored. It is possible to establish many such semi-realities as landscapes of investigation.

Landscapes of investigation can also include real-life references. As an illustration we can refer to the project Terrible Small Numbers.[4] The idea of this project was to make students experience mathematical reasoning related to the estimation of risks. We could consider the information that the likelihood of a grave incident at an atomic power plant is very small, like 0.00 ... 01 per day. But what does such a small number in fact mean? And to what extent could one claim that we have to do with a risk that could be ignored? We live in an environment, where we have to consider many different kinds of risks in terms of, for instance, pollution: the levels of certain metals in our drinking water have to be below a certain limit in order not to be a threat to our health. However, behind any such statements certain calculations of very small numbers have been carried out. Such calculations are part of the decision making process, which however also includes many other elements, not least those expressed in economic terms. The estimation of safety limits is part of decision-making in business, industry, and in the context of environmental planning. All these practices include mathematical calculations, which operate in the midst of huge controversies.

Naturally one can think of other types of landscapes of investigation, but the examples illustrate that we have to do with a spectrum of possibilities. A landscape of investigation has to be explored in order to function as a scenario for inquiry processes. It might be that the students are not fully committed, for instance, to finding out about the centi-cube-animals. A landscape of investigation can only serve as such if the students accept the invitation to enter the landscape. It is not possible to force anybody into inquiry processes. One can imagine that the teacher has raised some questions: What if we look at this particular five-animal? Let us produce a six-animal by adding one more centi-cube. How many different ways could this be done? Such formulations include a challenge, but it may very well be that the students do not really pay much attention to the questions. One could also imagine that the students themselves start formulating what-if questions. In this way, they may get involved in the inquiry process, and they might develop a sense of ownership of the process.

The KUL-projects refer to instances of such landscapes of investigations. Let us just mention some of them. One can explore the relationship between the height and the age of person (see Bjuland, Cestari & Borgersen, 2008a, b). Such a task might establish a small landscape of investigation, which, however, easily can be opened to a broader one that concerns measuring the human body. Other examples concern fractions, which might be opened to landscapes of investigations using computers (Fuglestad, 2007b). One can make inquiries into equations including two unknown numbers, and several papers from the KUL-projects refer to how the equation $x + y = 7$ can provide an entrance to a principal mathematical insight (Jaworski, 2007a; Hundeland, Erfjord, Grevholm, & Breiteig, 2008). One may enter a landscape of investigation by asking if the students can find two numbers, x and y, which would fit into the equation. One could also ask what would happen if one of the numbers, say x, was negative. And, what about numbers with decimals? Sure, there could be many sets of numbers x and y that do not fit into the equation $x + y = 7$. What would, in a coordinate system, characterize the set of such numbers x and y that fit into the equation? Different games have also been used in framing landscapes of investigation (see Fuglestad & Jaworski, 2005).

In the KUL-projects, we see a clear dominance of landscapes of investigations which refer to mathematics domains or to invented examples where real world events serve as background illustrations for mathematical exercises such as in the case of word problems (cf. Wyndhamn & Säljö, 1997). We see only few attempts to use real-life environments as a basis for establishing inquiry processes. In fact, it appears that in the texts we have considered (see our reference list), the only opening to a landscape with real life references is the one which correlates age and height of persons. It appears that the KUL-projects have operated within a rather narrow set of landscapes for mathematics learning. We find this to be a problematic limitation of the scope of the inquiries, which we will discuss in more detail in the following section.

Alternative conceptions of mathematics and of inquiry

The notion of inquiry can be related to conceptions of mathematics in different ways. In this concluding section we will simplify, and, for the sake of discussion, we will refer to a modern conception of inquiry and of mathematics, and a critical conception of inquiry and mathematics, respectively. Our modest aim is one of pointing to the implications of this difference, while accepting that projects cannot achieve everything.

John Dewey (1966) argued that the scientific methodology not only represented a valuable approach for obtaining scientific knowledge, but that it could be seen as a sound way of obtaining any form of knowledge. It is a way of thinking that is crucial to any form of learning process, in fact to any form of systematic human intellectual endeavour. Dewey even went so far as to see it is as a pillar of the proper conception of knowledge of a democratic way of life. Within these overarching conceptions, Dewey developed his ideas of inquiry as a principle of education that is grounded in people's experiences of living in a complex world.

From Dewey there is also much inspiration for making inquiry a significant part of processes of learning mathematics. All this, however, too easily becomes located within a grand trust in mathematical rationality, and the way of thinking that is promoted through mathematics education. The aim of an inquiry-based mathematics education becomes to bring students into mathematics and to make students appreciate mathematics. According to the modern conception of mathematics, teachers of mathematics should serve as ambassadors of mathematics, as it were. This conception dominated the Modern Mathematics Movement, which was initiated during the late 1950s, and which concentrated the teaching and learning of mathematics within the structures of mathematics itself. It also dominated many other approaches to mathematics education, radical positivism for instance. The notion of inquiry that became developed along these lines emphasized the importance of inquiring *into* mathematics.

As an illustration of an alternative, and more critical, conception of mathematics we can look at the following formulation by Ubiratan D'Ambrosio:

> In the last 100 years, we have seen enormous advances in our knowledge of nature and in the development of new technologies. [...] And yet, this same century has shown us a despicable human behaviour. Unprecedented means of mass destruction, of insecurity, new terrible diseases, unjustified famine, drug abuse, and moral decay are matched only by an irreversible destruction of the environment. Much of this paradox has to do with an absence of reflections and

> considerations of values in academics, particularly in the scientific disciplines, both in research and in education. Most of the means to achieve these wonders and also these horrors of science and technology have to do with advances in mathematics.
>
> (D'Ambrosio, 1994, p. 443)

This formulation reflects the observation that mathematical rationality can be used in many different contexts and for a variety of purposes. It can be part of work processes of any kind; it can be part of processes of automatization; it can be part of economic decision making; it can be part of advanced technological practices as well as of a range of every-day practices. The point is that how mathematics is operating cannot be characterized in a uniform manner. When mathematics is brought into action, these actions can be risky, hazardous, attractive, productive, inspiring etc. We experience the whole spectrum from "wonders" to "horrors" when mathematics is put to use, as pointed out by D'Ambrosio.[5]

This observation points to a significant challenge for an inquiry approach to mathematics education. If we follow a modern conception of mathematics, it makes good sense to cultivate inquiry processes with reference primarily to mathematics, since this is the way we will foster a "progressive" rationality. The introduction of digital technology in the teaching and learning process should serve the same purpose. However, if we make a critical conception of mathematics our premise, we have to think of inquiry processes differently, and it might be interesting to consider briefly what this implies. It becomes relevant to develop processes of inquiry with reference to landscapes of investigation that are not exclusively located in a mathematical environment, but which connect to significant everyday events and daily concerns. Thus, the example of Terrible Small Numbers makes it possible for students to be involved in inquiry processes, which concern burning issues of reliability and responsibility. Naturally the point of a critical conception of mathematics is not that each and every inquiry process should open the space for such reflections. But if one should address the "wonders" as well as the "horrors" of mathematics within an inquiry approach to teaching and learning, it is important that learners meet a wide range of different types of landscapes of investigation.

The point of this discussion is to argue that the landscapes which the KUL-projects are cultivating, and in spite of the appeal to inquiry, have a clear tendency to refer to mathematics or to contexts where the real-life references have the status of illustrations suitable for mathematical exercises. We have found no obvious examples of inquiry processes that present learners with more critical considerations of mathematical operations and their consequences. The notion of inquiry that has been cultivated in the KUL-projects clearly has been strongly associated with the modern conception of mathematics.

If one adopts a notion of inquiry grounded in the modern conception of mathematics, one will be directed towards certain educational possibilities and strategies. In particular, one will try to find possibilities for how students in the classroom can be engaged with mathematical thinking in one of its many forms. Inquiry, within this interpretation, risks becoming means to an end which is already defined and, by and large, unproblematic. In the background one hears the triumphant celebrations of mathematics as the prime source of human rationality. If one adopts a critical conception of mathematics as a premise for understanding inquiry, it is imperative to find educational possibilities that broaden the scope of reflections on mathematical rationality as it is embedded in society. A modern conception of mathematics is based on the assumption that using inquiry in mathematics education will result in a situation where students will learn to appreciate mathematics, while a critical conception of mathematics points to the importance of facilitating critical reflection as part of the processes of learning mathematics. In our opinion, something fundamental is at stake in this tension of how we conceive of inquiry; either it is a means to a given disciplinary end, or it is a productive road to knowing through which human rationality is both cultivated and questioned.

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Notes

1 ICT refers to Information and Communication Technologies.

2 For a discussion of this forms of interaction, see Alrø and Skovsmose (2002, Chapter 1).

3 See, for instance, Vithal, Christensen and Skovsmose (1995) and Christensen (2008).

4 This project has been presented in Alrø and Skovsmose (2002) and in Alrø, Blomhøj, Bødtkjer, Skovsmose, & Skånstrøm (2003, 2006).

5 For a discussion of a critical conception of mathematics, see Skovsmose (2003b, 2005, 2006a).

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Sammanfattning

Begreppet inquiry har spelat, och spelar ännu, en framträdande roll i den pedagogiska diskussionen och i forskares och didaktikers försök att utveckla undervisning. Syftet med artikeln är att diskutera en del av de antaganden om inquiry som utgjort utgångspunkt för två omfattande forsknings- och utvecklingsprojekt inom matematikundervisning som bedrivits vid Universitetet i Agder. Dessa projekt – som lokalt går under beteckningen KUL-projekten (Kunskap, Undervisning, Lärande) – syftade till att introducera en förnyelse av matematikundervisning som bygger på idéer om att etablera "communities of inquiry/learning" mellan universitet (forskare/didaktiker och doktorander) och skolor (lärare, skolledning, elever). Projekten är ovanliga i den mening att de har en väl artikulerad teoretisk bas för utvecklingsarbetet grundad i community begreppet och i ett medvetet försök att arbeta inom ramen för den modell som numera går under beteckningen design research. Intresset var dessutom inriktat både mot att utveckla matematikundervisningen och samtidigt att lära sig om hur utvecklingsarbete mellan universitet och skolar kan utformas. En preliminär utvärdering av de inledande resultaten av projekten, utförd av författarna till denna artikel, visar att arbetet på många sätt varit framgångsrikt. Bland annat framgår att idéerna bakom sätten att utveckla undervisningen, liksom samarbetsformerna mellan forskare och lärare, uppfattades som mycket givande för deltagarna på skolorna (särskilt bland lärare i grundskolan). Projekten har också avkastat många intressanta publikationer och publiceringen pågår alljämt.

I föreliggande artikel diskuteras den tolkning av begreppet inquiry som de båda projekten bygger på. Det påpekas att för att vår förståelse av vad inquiry innebär som beståndsdel i pedagogisk praktik skall utvecklas, så är det nödvändigt att ingående analysera interaktion i undervisningen. Analysens mål måste vara att klarlägga vad inquiry innebär som kommunikativ praktik, och att urskilja vilka slags interaktiva mönster som karaktäriserar det slags aktivitet som kan kallas inquiry. Inquiry är mer än att personer samtalar med varandra. Det påpekas också att den föreställning om inquiry som projekten bygger på innebär att man ser inquiry som ett sätt att förbättra inlärningen av matematik. En alternativ ansats är att se inquiry som en aktivitet som utvecklar människors förmåga till kritiskt tänkande och där förståelse av matematik blir kärnan i utvecklingen av mer generella analytiska förmågor som spelar en central roll i ett demokratiskt kunskapsbegrepp och i ett demokratiskt samhälle.