Defining moments in the graphing calculator solution of a cubic function task

Jill Brown and Gloria Stillman

A case study investigated cognitive, mathematical, and technological processes undertaken by senior secondary students as they searched for a complete graph of a difficult cubic function using a graphing calculator. Intensive qualitative macro-analysis identified several defining moments in the solution process. Those related to use of scale marks and identification of key function features are presented. Students' understanding of scale marks varied and this impacted on the efficiency and elegance of their solution. A range of calculator features was used in identifying key feature coordinates. These were not always used successfully or with an understanding of the mathematics underpinning their operation.

Understanding what constitutes a complete graph of a function, that is, a graph showing "all the relevant behaviour" (Demana & Waits, 1990, p. 216) has always been important for senior secondary students (Leinhardt, Zaslavsky & Stein, 1990; van der Kooij, 2001). The examiner's report for the first *pure mathematics paper* in the University of Melbourne matriculation examination for 1948, for example, lamented that "the graphing of functions was rarely done correctly" (Teese, 2000, p. 123). The introduction of graphing calculators has had the potential to impact on this understanding according to several authors (e.g., van der Kooij, 2001; Leinhardt et al., 1990). The students in the study described here were undertaking a course of study (VBOS, 1999) where graphing calculator use was expected and use of CAS was not permited in examinations. Some researchers suggest the use of graphing calculators adds to student difficulties as they tend to accept the view presented by the calculator

Jill Brown, *Australian Catholic University*
Gloria Stillman, *University of Melbourne*

Brown, J. & Stillman, G. (2006), Defining moments in the graphing calculator solution of a cubic function task. *Nordic Studies in Mathematics Education*, 11 (3), 57-84.

(Cavanagh & Mitchelmore, 2000; Steele, 1995), need to interpret the graphing output (Ruthven, 1995), need to recognise that only a portion of the graph is displayed in the viewing window (Goldenberg, 1987; Ruthven, 1995; Steele, 1995), and, often for the first time, experience graphs with different scaling on the axes (Goldenberg, 1987; van der Kooij, 2001). According to Mitchelmore and Cavanagh (2000), these difficulties are related to shortcomings in the curriculum that exist in both a graphing calculator and non-graphing calculator environment. Van der Kooij (2001) suggests such difficulties are simply more obvious in a graphing calculator environment which highlights different areas of student misunderstanding due to its focus and the need to understand the differences between local aspects and the global nature of a function.

A graphing calculator environment implies more than having access to the technology. To improve understanding, teaching must change simultaneously to capitalise on alternate ways of doing things and access to new opportunities such as ”allowing manipulation of the viewing window”, ”provid[ing] the power of zooming” (Dick, 1996, pp. 32, 44), ”visual[ising] the effects of scale changes”, understanding the independence of the scale of the two axes and the effects of scale marks (Goldenberg et al., 1988, p. 36). The study described in this paper responds to the plea of Arnold (1998) and others (e.g., Zbeik, 2003) that ”if we are to learn to use these tools effectively, it becomes vitally important that we study the ways in which individuals make use of them within mathematical learning situations” (Arnold, p. 174). Whilst the recent publication edited by Guin, Ruthven, and Trouche (2005) has added considerably to our knowledge of the use of CAS in traditional curriculum, there is still much to be known about how students actually proceed with challenging tasks such as those in this paper, or in ”analyzing quantitative conditions in realistic problem contexts” (Fey, 2006, p. 352) in a technological environment.

Although now some secondary classes are using computer algebra systems (CAS), Zbiek (2003) indicates that ”graphical representations seem to abound in CAS-using classrooms” (p. 208) and ”many studies of the use of CAS depend highly on the graphing component of the tool” (p. 210). Currently many more students have access to graphing calculators than CAS calculators. Hence, the graphing calculator continues to be a technology needing further research.

Functions are ”multi-faceted” (Lloyd & Wilson, 1998, p. 250) and cannot be fully understood within a single representation environment. Being able to make links between representations is crucial to the underlying concepts of functions (Even, 1998). Graphing technology use provides students with the opportunities to make these links although they do not necessarily do so.

The easy creation of graphs in a technological environment allows a large number to be observed and provides easy access to myriad function types. In addition, observing multiple views of a single function can, but does not necessarily, add to the development of a broad ”concept image” (Vinner & Dreyfus, 1989, p. 356) of the prototypical graphical representation of a particular function type. For example, the graphical representation of a quadratic function has two possible ”shapes” depending on the type of stationary point. However, when using graphing technology only a portion of the graph can be seen, and hence, a quadratic function can also appear linear (with positive, negative, or zero gradient) if the viewing window of the technological tool is focused ”closely” on a part of the graph. An understanding of the effect of changes of scale, including where each axis has a different scale (Zaslavsky, Sela & Leron, 2002), is essential for successful graphing calculator use and also helps develop these concept images. As Vinner and Dreyfus state, a student's concept image is ”the set of all the mental pictures associated in the student's mind with the concept name, together with all the properties characterizing them” (1989, p. 356). The use of technology is one way to broaden students' experiences with the function concept and some of the concept image held by students working in a technological environment will have formed as a result of being taught in such an environment and being active users of the technology.

With the mandating of graphing calculator use in schools and assumed access to them in high stakes examinations, expertise in using a graphing calculator to find a global view of a function is essential for senior secondary mathematics students (Anderson, Bloom, Mueller & Pedler, 1999). The study forming the basis of this paper investigated approaches student pairs undertook in finding a complete graph of a cubic function in a non-routine situation. Student understanding of function is receiving international attention in research studies according to Zbiek (2003). The findings of the present study add to this literature.

Several research questions were identified for the study. Those to be addressed here are:

1 What understandings do students have of the effect of the scale marks on the graphical representation of a function?

2 What use do students make of their concept images and particular features of the graphing calculator to identify key features of a function?

The study

An instrumental case study (Stake, 1995, p. 171) was used as it was considered most appropriate for achieving the goals of the study. Rather than the case itself being of primary interest, in an instrumental case study "the case ... serves to help us understand the phenomena or relationships within it" (Stake, p. 171), as is the situation in this study. This case study describes the results of practice in the classroom of the first author and a colleague as evidenced by a snapshot of student responses to a problem task.

Merriam (2002, p. 179) states that the case can be a person, process, or community. The case here is two classes of students studying functions in a graphing calculator learning environment as part of their study of mathematics in their final two years at a particular Australian secondary school. This case was selected "because it [exhibited] characteristics of interest to the researcher[s]" (Merriam, 2002, p. 179). Within this case student pairs were *purposively selected* (Merriam, 2002, p. 20).

Given the limited number of students that could be studied in depth, pairs were selected so the phenomena of interest were "transparently observable" (Huberman & Miles, 2002, p. 13) so as to maximise what could be learned. It must be acknowledged, however, that the way in which the students were prepared to participate in the study and expound their ideas could be a result of the situational context (Wedege, 1999) established by the teachers in the classroom where students' voices were expected to be heard. The two mathematics teachers of the students worked closely together using a variety of methods emphasising understanding through exploration, discussion, and collaboration. Their students tended to be accomplished users of graphing calculators and were expected to use them when directed or at their own instigation.

Participants

Five pairs of students, at an inner city co-educational state secondary college, were selected from two mixed ability classes. A pretest was administered to both classes to ensure the students selected had the necessary skills and conceptual knowledge to complete the task. In addition the results of year 11 students on a previous supervised assessment task dealing with the graphing of a cubic function were taken into consideration. Thus, the pairs selected comprised students who were confident in mathematics, worked together in class, and could be expected to solve the problem and articulate their ideas as they proceeded with the task. The experimental setting thus paralleled the classroom situation. Two pairs (1 and 2) were from a year 12 class and three pairs (3, 4, and 5) from a year 11 class.

The problem task

The task was to graph completely the function,

$$y = x^3 - 19x^2 - 1992x - 92\,[1].$$

Binder (1995) used this particular function to investigate a graphical approach to solving an algebraic problem. The task was selected on the basis that no part of the function is visible in the standard viewing window, being $-10 \leq x \leq 10$, $-10 \leq y \leq 10$, on a Texas instruments TI-83 graphing calculator. All students in the study would understand the request to ”graph completely” to mean to produce a by-hand sketch of the function showing the shape of the function, indicating the location of all key features (axial intercepts and local minima/maxima or turning points) and identify these. They would interpret the task as to best represent the cubic function and the particular function would impact on whether exact or approximate ordinates were identified. For this particular function, it was expected that, and only possible for the students to, provide approximate x intercepts. Numerical and graphical analysis provide us with quantitative information and give a readily accessible approach to represent the function and its key features and the opportunity for students to display what they know about the graphical representation of a cubic function and how it might be ”found” using technology. The current concept images of the graphical representation of a cubic function, held by the students, were expected to impact on the task solution, however, these concept images could be modified through their engagement with the task.

Administration of the task

Each task solving session was audiotaped with students asked to articulate their ideas. Students used a graphing calculator, initially set to the standard window, attached to a view screen and overhead projector. The calculator screen output was videotaped via the overhead projector accurately recording the students' actions. Observational notes were also taken. The written task and paper to record working in order to complete a hand-sketched solution were provided.

Case record

Raw data in the form of tape recordings and videotapes of the graphing calculator screen output ensured a permanent record of student interactions with the calculator was made. Protocols of each pair's efforts were produced by matching the combined recordings, student scripts and observational notes. The protocols collectively formed the case record which was used to determine student behaviour. The novel method of recording all calculator screens allowed a more complete record to be assembled than mere videotaping of working pairs would have. The screens capture more of the students' immediate thought processes than the words uttered and recorded. Thus, not only might graphing calculators increase learning opportunities for students, but also they provide opportunity for teachers and researchers to witness more closely the understandings students have as inferred by the results of their actions represented by the graphing calculator screens.

The analysis

The protocols were coded and divided into ”macroscopic chunks” (Schoenfeld, 1992, p. 189) that were classified according to particular behaviours of interest associated with working with functions in a graphing calculator environment. These behaviours are referred to as *episodes* and ”represent periods of time during which the problem solvers are engaged in a particular activity” (p. 189). In addition, time-line diagrams, detailing the length and type of episode undertaken, and the representations being used at any given point in time, were constructed. These were an adaptation by the researchers of Schoenfeld's time-line diagrams (p. 190) used to study problem solving.

Microscopic analysis

The analysis in this study goes beyond the macroscopic analysis of Schoenfeld and was used as a way of looking for *defining moments* in the solution process that were then explored using microscopic analysis to search for explanations. The use of graphing calculator screen data to supplement and enhance the dialogue data in the study allowed a more finely grained classification than was possible by Schoenfeld's (1992) scheme. Protocols were coded using codes devised by the first author that classify actions according to distinct behaviours specific to solving the particular problem task used. The codes referred to the following categories of activities: reading, organising or planning, selecting a viewing window, searching for or identifying a local feature, searching for a global view, adjusting scale marks, evaluating, and recording. To ensure reliability of coding initial coding undertaken by the first author was redone until code-recode consistencies were satisfactory, the second author coded a sample with coder-intercoder reliability calculated to be 86% with discrepancies not appearing to be systematic, and a sample of the protocols was recoded six months later with 97% agreement.

Findings

All pairs eventually successfully produced a sketch of a complete graph of the function; however, the routes to this differed in directness and duration with Pairs 1, 2, and 5 undertaking initial actions that allowed the solution process to become potentially routine (Brown, 2002). No matter how circuitous the route undertaken, each pair persisted until a complete view of the graph was produced. This was contrary to implications of research by others suggesting students would stop after finding *any* graphical representation of the function in the viewing window (Mitchelmore & Cavanagh, 2000; Steele, 1995). All pairs found a global view of the function and identified most, or all, of the key features, albeit to varying degrees of accuracy.

Defining moments

After close analysis of episode and time-line diagrams and the case record where this facilitated macroscopic analysis, several *defining moments* became apparent in the student solutions. The term, defining moments, refers to important or momentous events rather than a particular instant in time. A defining moment was a *circumstance* where some action, cognitive or physical, or a decision (i.e., a metacognitive action), or a series of these, had the potential to facilitate or impede the solution process. Defining moments, therefore, occurred at critical points in the solution process. It was the responses of the pairs to particular circumstances that determined whether they became defining moments for a particular pair's solution. Each circumstance was described in terms of the *situation*, *condition* and *response action* (figure 1). For every circumstance the question was: What situation gave rise to this condition which, in turn, led to this response?

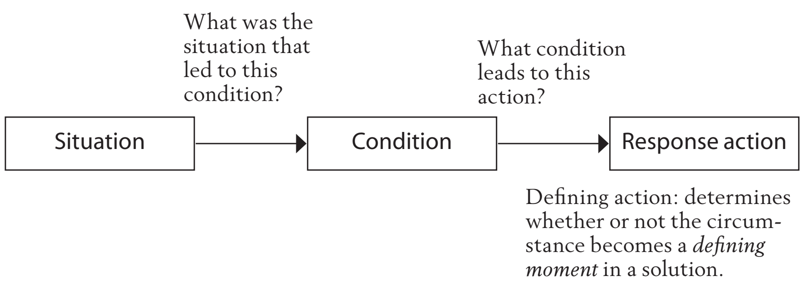


Figure 1. *Identification of a* ***defining moment***

A situation was the context in which students found themselves which could be a specific stage in the solution process or a general situation. The *ongoing search for a global view of the function*, for example, was a situation some pairs found themselves in for a substantial amount of time throughout their solution attempt; whereas, the situation, *global view found*, occurred at particular points in the solution process. One situation could lead to a number of conditions. A condition was the particular state that one or more pairs was observed experiencing or considering such as the sighting of an apparent no view of the function. Differing situations were observed in conjunction with the same condition. In turn, several conditions gave rise to the same response action. The same combination of situation and condition could have occurred in conjunction with differing response actions from different pairs.

The presence of defining moments and students' responses to these impacted on the directness and duration of the solution process and accuracy of key feature identification. Defining moments identified during macroscopic analysis related to the following:

- How students responded to particular views of the function, including apparent no view, apparent vertical lines, and other unusual or unexpected views (Brown, 2003).
- Use of the numerical representation.
- Use of scale marks.
- Use of opportunistic planning.
- Engagement in discussion.
- Identification of key function features.

Defining moments to be discussed in this paper are those related to student *use of scale marks* and *identification of key function features*.

Defining moment: use of scale marks

In addition to setting the viewing domain and range, the WINDOW settings of the TI-83 also include options to adjust the scale marks on either axis. The first set of defining moments to be discussed (see figure 2) is related to the response action *use of scale marks*. This occurred in three situations, namely, the *need to produce a sketch* of the function, the *need to determine the coordinates of the key features* of the function, and the *need to find a global view* of the function. The four conditions in this set of defining moments were *facilitation of the solution sketch*, the *need to eliminate "ugly axes"*, the *need to obtain estimates of the key features* of the function in order to check values to be determined by other more accurate methods, and a *lack of a global view of the function*.

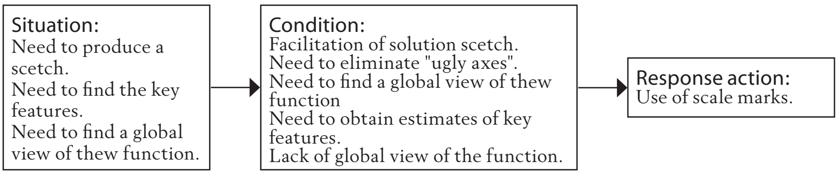


Figure 2. *The circumstances related to the use of scale marks*

Use of scale marks varied in students' intent, consequences, and effectiveness. One pair demonstrated poor understanding of the effect of scale marks as they adjusted them in an attempt to alter the portion of the graph being viewed. Two pairs, in contrast, made effective use of scale marks as they adjusted these to improve their view of the graph.

Interactions between view and scale marks

Judicious *use of scale marks* facilitated students' identification of key function features. Of the three pairs for whom the solution process became potentially routine, only Pair 1, used scale marks to effectively produce a global view that readily facilitated key feature identification. On finding a global view of the function, Linh and Ahmed had no way of estimating coordinates of any key features given the default scale settings of one; so, immediately prior to identification of the key features, they edited the scale marks.

Ahmed: We still have to fix up the scale.
[They edit the scale marks on the *x* axis from 1 to 10.]
Linh: 100 [Talking about the *y* scale marks.]
Ahmed: 5000 because of the large *y* values, and *x* scale of 10.

Subsequently they used the calculator to identify the *x* intercepts accurate to 2 decimal places using the scale marks to first make guiding estimations.

Ahmed: Another *x* intercept would be at around [negative] 30.
Linh: What's our *x* scale?

Ahmed: 10.
Linh: 60, … 50, … 40,
Ahmed & Linh: (simultaneously) It’s x = -36.10375, the second x intercept.
Ahmed: The last one will be around …
Linh: 50 something.
Ahmed: 50.
Ahmed & Linh: (simultaneously) Between 50 and 60.
[Using CALCULATE zero, they enter these scale marks as the bounds of the interval containing the root of the function. But then their judicious use of scale marks allows them to make a quite reasonable estimate of the approximate coordinates of the x intercept.]
Ahmed: Guess, 56 or 55?
Linh: Yes.
Ahmed: x intercept at 55.14.

After identifying all the key features, this pair subsequently used the scale marks to support their by-hand sketching. This informed adjustment of scale marks resulted in a more effective graphical representation of the function (figure 3) as it allowed estimation of approximate coordinates of key features and facilitated identification of key features and production of a written solution (figure 4).

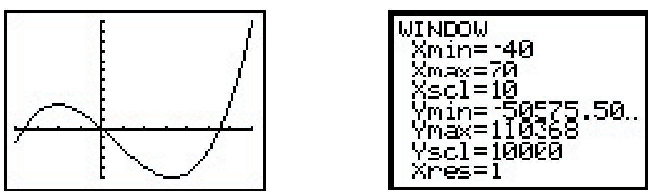


Figure 3. *Use of scale marks in response to condition, facilitation of solution sketch*

Both pairs for whom the solution process did not become routine altered the scale marks, with Pair 3 simply setting them to zero and effectively turning them off. After eighteen minutes engaged with the task, and having finally found a global view of the function, with thick axes, Pete commented as he edited the scale marks to zero for both axes (figure 5), ”No, oh hang on. Wait this is annoying me. Zero. That is better.” Whilst this usually does nothing to facilitate the solution process, it eliminates the ”thick axes” that result from the compaction of scale marks in the viewing window when the domain and range have large magnitudes as

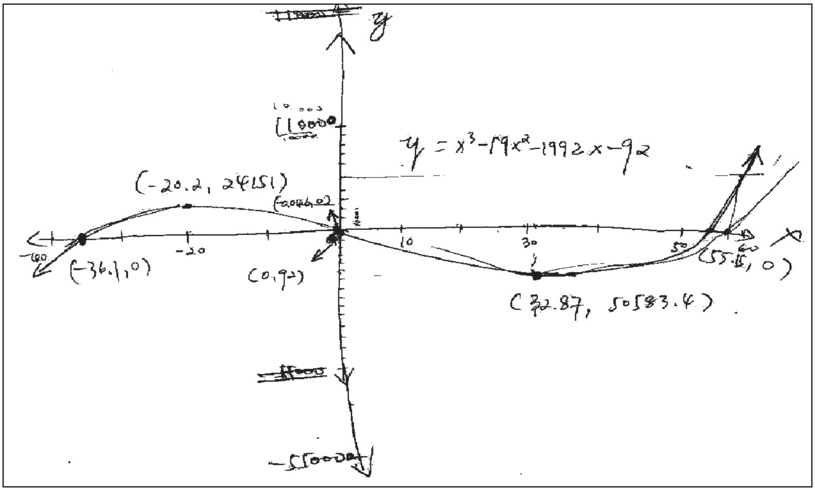


Figure 4. *Solution script of Linh and Ahmed (Pair 1)*

in this task. It appeared that Pete was mindful that the thick axes may obscure some key information given the very large viewing range. With the thick axes eliminated, Pete believed he would be able to perceive the axial intercepts more clearly, and this would have been the case if none of the *x intercepts were very small relative to the others*.

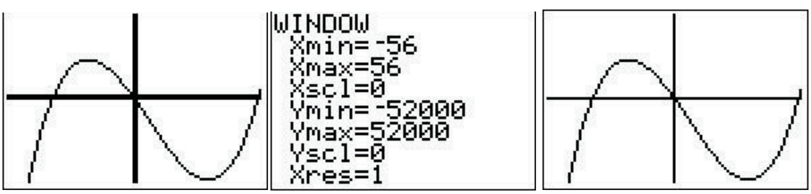


Figure 5. *Use of scale marks in response to condition, "need to eliminate 'ugly axes'"*

At this point Pete and Kate were debating if there was another turning point in addition to the two that were clearly visible on their screen. Kate suggested that they resolve this analytically using "common factors" to determine how many times the function crossed the *x* axis. However, Pete realised that what he was seeing on the screen meshed with his concept image as being one possible graphical representation of a cubic function. "It's got three *x* intercepts, that's all it can have. Forget about turning points, it's got the three x intercepts – one, two, three, a cubic can only have three!" Pete had previously indicated by gesturing that it was a cubic with two turning points, even though at that point all that he had seen on the calculator was an apparent straight line in the fourth quadrant.

The improved view of the function facilitated their subsequent identification of key features and by-hand sketch. Although the calculator screen shows only the slightest thickening of the y axis just above the origin (figure 6), it was sufficient to alert Pete to the possibility of a y intercept which they had noticed at the beginning of the task, but seemed to have forgotten.

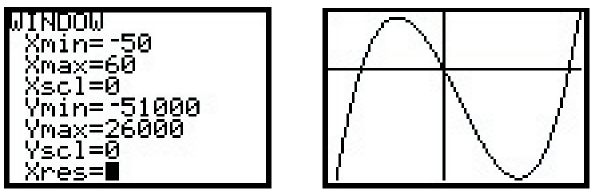


Figure 6. *Pete's view of the y intercept*

Pete:	It has got its minimum, its maximum, and its three x intercepts. [Speaking quickly] Or is that a y intercept?
Kate:	What?
Pete:	There. Calculate. [Uses CALCULATE value to show $x = 0$, $y = -92$.]
Kate:	No, it's still both.
Pete:	No it crosses both [axes].
Kate:	It's not going through zero.
Pete:	No, it crosses both.
Kate:	That's what I am just saying, it is not going through zero [meaning the origin].

Misunderstanding of the effect of scale marks

The other pair for whom the task was non-routine, Reem and Ali, adjusted the x and y scale marks from the homogeneous default settings of 1 to another homogeneous scaling system of 2, as shown in figure 7a and 7b after they were confronted with a thickened y axis indicating the possibility of a section of the graph being present in the viewing window but indistinguishable from the y axis. This adjustment suggests they believed the scale settings could affect the view of the graph but it did

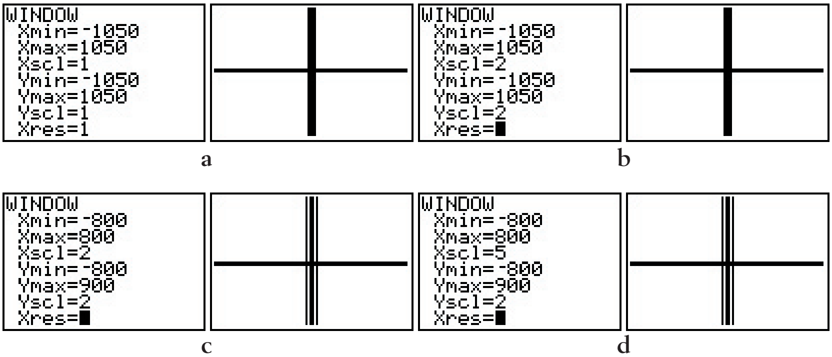


Figure 7. *Use of scale marks in response to condition, lack of a global view of the function*

not as they soon saw. However, they were undeterred by this, possibly believing that the size of the increase was not sufficient to result in a section of the graph being differentiated from the y axis. They made a further adjustment to the x scale to 5 shortly afterwards, as shown in figure 7c and 7d, again with no resulting alteration to the screen but neither commented on this.

Altering the scale marks had no visible effect on the graphical representation of the function in either case. The explicit *use of scale marks* had not been taught to any pairs of students in class. Perhaps, given their lack of success using other items in the window menu and their reluctance to use other features of the calculator, Reem and Ali's foray into exploring the scale marks was the only option left to them.

Defining moment: identification of key function features

The second set of defining moments (figure 8) relate to the condition the *need to identify key features* of the function and the response actions,

a use of dedicated features of the graphing calculator,

b use of TRACE [2], and

c use of the free cursor [3].

This set of defining moments occurred in two different situations, being *the global view found* and the *observing of key features in the viewing window.*

Situation:
Global view found.
Key features seen in the viewing window.

→

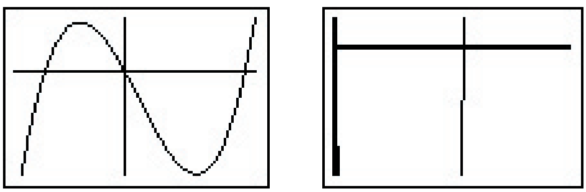
Condition:
Need to indentify key features.

→

Response action:
Use of dedicated features of the graphing calculator.
Use of TRACE.
Use of the free cursor.

Figure 8. *Circumstances related to key feature identification*

The situation, *global view found*, indicates that the shape and all key features of the function were visible in the viewing window as shown in figure 9a. In contrast, the situation, *observing of key features in the viewing window*, indicates that at least one key feature, but not all, was visible in the viewing window as in figure 9b.



a b

Figure 9. *Two situations resulting in key feature identification*

Situation: observing of key features in the viewing window

This second situation and the interactions leading up to it in the case of Pair 3, Kate and Pete, reveal quite a lot about the respective student's concept image of cubic functions. When they first were presented with the task, Kate initially suggested they factorise the polynomial to find the x intercepts. Pete was adamant that they could not do this and must use the calculator.

They began by locating what appeared to be a linear piece of the function in the fourth quadrant. Pete indicated that he was expecting the graph to be that of a cubic with two turning points by tracing an example in the air with his finger. Despite various alterations to the window settings which merely magnified the view of the function in the fourth quadrant, the function still appeared linear; however, both Pete and Kate's concept images of cubic functions were such that they realised that the graph they were seeking would not be linear.

Kate: No, it's still straight, that can't be linear though, it's got ...
Pete: No, it's not linear, it's cubic.
Kate: That's what I mean.

Eventually they found what appeared to be an almost parabolic part of the curve in the fourth quadrant after seeing several straight line views. Kate says, "Trace it, trace it, put it in the middle" so as to identify the location of the turning point. Pete actually used the free cursor to find the apparent minimum of (32.1702213, -51959.23) but this point is actually just below the curve (figure 10).

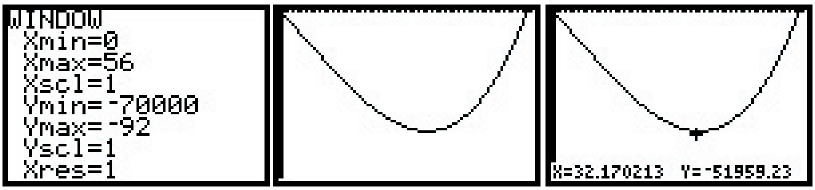


Figure 10. *First non-linear view of the function by Pair 3*

Kate then asked if the function had a point of inflection, meaning a stationary point of inflection as they had not dealt with non-stationary points of inflection to this point in their schooling. Pete was adamant that it could not, and suggested she trace along it herself when she doubted him. It is clear that Pete had developed a much richer concept image of cubic functions from their classroom experiences than had Kate, as the following exchange indicates.

Kate: If it's got the cubed and squared it's a possibility [referring to point of inflection].

Pete: Okay, it's about 52,000 [referring to the absolute value of the *y* ordinate of the local minimum.]

Kate: When does a point of inflection occur?

Pete: A [stationary] point of inflection is where the gradient is zero, but it's not a minimum or maximum turning point.

Kate: Yeah, is there a possibility? Going by this, because you know how graphs have a max, min, and ...

Pete: If it has a squared term, can it? I don't think it can. Like, it has to be on its own like $4x^3$ or $5x^3$.

Kate: No, 'cos the curves thing that we just did [referring to an investigative project exploring the effect of changing parameters of a cubic function on its graphical representation], it's got a point of inflection.

Pete: But that's when it's shifted, like $5x^3 + 7$.

Kate then suggested that Pete use TRACE to identify the coordinates of the turning point. Pete asked if she wanted to know these exactly then proceeded to *use the dedicated feature*, CALCULATE minimum, instead to find the coordinates of the minimum turning point recording these accurately to two decimal places (figure 11). Pete clearly knew that this dedicated feature would result in a more accurate result than the use of TRACE.

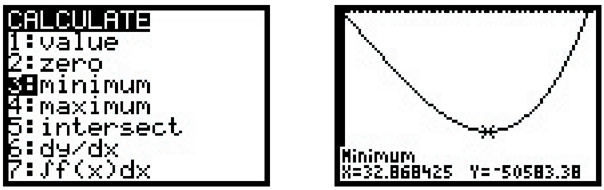


Figure 11. *Use of a dedicated feature in response to the condition, 'need to identify key features'*

At this point the difference in their concept images of cubic functions became even more apparent. Pete stated that their view was ”like what a quarter of the graph, a third of it”. He meant a third of what they might show in their sketch as he had already stated clearly that the function had a ”domain of all real numbers” and insisted later that the ends of the curve in their sketch have arrows to indicate this. Kate said it was half and indicated she expected the function to cut the x axis ”it goes into the top, because it has to cross through”. Pete on the other hand, saw that as it had a minimum, there must be a maximum in keeping with his previous gesture to show the expected shape.

Pete: And there has got to be a minimum turning point, like we've got this part, I reckon when it crosses it just keeps going. But it's ...
Kate: [interrupting] You know how, [that] there's a negative?
Pete: It has a turning point. A maximum turning point somewhere.

Kate questioned this apparently believing that the graph continued upwards on both ends that would result in an image more like a parabola than the various views of cubic functions which she would have experienced previously.

Kate: Do we know that it actually comes down? Or does it go up? It hasn't got a negative in front of the graph, so wouldn't it be going up as positive? Yeah. And you know how we didn't see anything go past the y [axis].

Pete: Let's just go across for a sec[ond].

Kate: Why?

[Pete widens the viewing domain of the window and a section of the graph is shown in the third quadrant.]

Kate: It does come down! How come it comes down?

Even after Pete changed the window to give a global view of the function, showing one of its prototypical shapes, Kate was still not convinced she was seeing all the key features of the graph.

Kate: That can't be all the graph. [Pete increased the x maximum of the window as Kate now seemed to think there would be another turning point.]

Pete: No, it doesn't come back.

Kate: Are you serious?

Pete: How about the other way?

Kate: Check it. [Pete widens the viewing domain in the opposite direction.] No, it just goes like that. That is weird.

Clearly, what she was seeing was somewhat at odds with what she had expected from her memories of previous experiences with cubic functions. Momentarily, Pete too appeared to lose sight of the property that a cubic function can have a maximum of two turning points.

Pete: Is there another turning point? How many can a cubic have?

Kate returned to her notion that they should be able to work out the x intercepts analytically beginning by using common factors and hence determining the number of turning points. This is the point where the scale marks were turned off [discussed in the previous section].

Immediately following the exchange about scale marks, there was a further interaction that reveals more about Kate's incomplete concept image for cubic functions. At this point they have a global view of the function with a good window.

Kate: What if we didn't have that?

Pete: We'd be stuffed. [Australian slang for being in an impossible situation.]

Kate: No, not necessarily, it can always be just worked out.

It is clear that she believed it is possible to find the x intercepts of all cubic functions analytically.

Situation: global view found

Both the choice of calculator feature and its correct use impacted on the accuracy with which pairs recorded key feature coordinates. The effects of choice of calculator feature and the consequences of this for accuracy on the type of key feature being identified are shown in Table 1.

When correctly selected the *use of dedicated features* (e.g., CALCULATE minimum[4] or CALCULATE zero[5]) resulted in accurate recording of key feature coordinates. In contrast, when other features of the calculator such as the *use of free cursor* or TRACE identified the coordinates of the key features this was usually less accurate. However, when TRACE was selected to determine coordinates of the y intercept and the y axis was centred in the viewing window giving a symmetric viewing domain the resulting coordinates were able to be identified exactly.

Pairs 1, 2, and 3 correctly identified the coordinates of all key features, usually to an accuracy of 2 decimal places (what is expected in examinations). All pairs correctly identified the coordinates of the y intercept and the right x intercept. Pair 4 failed to identify the coordinates of the central x intercept with Reem stating at one point, "there are two x intercepts" when three were visible on the screen. Even though Ali questioned her certainty of this, only two x intercepts were recorded later when they had an even better view of the function. The degree of accuracy of their identification of the remaining features was tempered by their using TRACE rather than a more accurate method such as CALCULATE minimum.

The identification of turning point coordinates by Pair 5 was inaccurate for two different reasons. When identifying the coordinates of the minimum turning point, Susan and Jing initially successfully used CALCULATE minimum. However, the window settings resulted in superimposed

Table 1. *Accuracy of methods of determining key function features*

Actions	Consequences for accuracy of		
	y intercept	x intercepts	Turning Points
Using dedicated calculator features[a]	Accurate[b]	Accurate[c]	Accurate[d]
TRACE	Accurate[e]	Not Accurate[f]	Not Accurate
Free cursor	Not Accurate	Not Accurate	Not Accurate

Note. [a] Assuming correct use. [b] CALCULATE value. [c] CALCULATE zero or Equation SOLVER. [d] CALCULATE minimum or CALCULATE maximum. [e] Assuming symmetric viewing domain. [f] Pair 4 used TRACE to determine right x intercept coordinates accurate to nearest integer.

output of the graph and the minimum turning point coordinates which the students could not read. After adjusting the window settings appropriately, they reselected CALCULATE minimum, however, on this occasion poor selection of a search domain resulted in the required point not being included. As a result, the minimum point determined by the calculator was the minimum point of the function in the specified domain. The students did not notice this. The second inaccuracy occurred when, for some inexplicable reason, they did not use the dedicated feature, CALCULATE maximum, instead using TRACE in their identification of the maximum turning point. They clearly had knowledge of CALCULATE maximum as they had used its counterpart previously.

The defining action, *use of dedicated features* of the graphing calculator, facilitated the finding of a complete graph of the function, however, the flawed use of dedicated features, the *use of TRACE*, or the *use of the free cursor* to identify the coordinates of key features reduced the degree of accuracy with which these were recorded.

Discussion and conclusions

Students' understandings of the effect of scale marks differed in this study. Of the five pairs, only three altered the scale marks. The most effective and efficient of the year 12 pairs, Linh and Ahmed, demonstrated a good understanding of the effect of scale marks as they adjusted these with the intention of facilitating their identification of key function features in order to produce a pen and paper sketch of the function. The less efficient and effective solution process of the other year 12 pair, Hao and Abdi, was due in part to the lack of use of scale marks.

One year 11 pair behaved as expected by Williams (1993) as they focused on adjusting the scale marks in order to alter the view seen, not realising this has no effect. A similar finding was reported by Mitchelmore and Cavanagh (2000) who found that students in their study did not understand the nil effect of scale marks on the section of the graph portrayed in the graphing calculator window.

In contrast, the setting of the scale marks to zero by another year 11 pair had a positive impact from their perspective on the solution process. Whilst the action undertaken may seem trivial, anecdotal evidence from teachers confirms that removal of "ugly axes" from the viewing window is perceived as desirable by many students as it is congruent with their notion of elegance of a solution. It appeared that the removal of the scale marks facilitated the students' observation that the function did not pass through the origin. Actions that result in positive feedback for students are likely to boost their confidence and this in turn may result in a more effective and efficient solution path being followed. Thus, this student initiated action contributes to improving the affective component of learning.

If students are to appreciate that scale marks have no effect on the portion of the graph visible in the viewing window then this implies explicit teaching to overcome the possible misconception (Cavanagh & Mitchelmore, 2000; van der Kooij, 2001; Zaslavsky, Sela & Leron, 2002). Learning activities need to be presented that allow students to consider the effect of altering the scale marks in a given viewing window, for a range of window settings, and a range of functions. Students need to consider the effect of altering the scale marks on the axes and on the view of the function seen. This "misconception" with scale marks is probably transient in that it should occur less as teachers become more familiar with how to teach well with graphing calculators.

Students used an extensive range of features of the graphing calculator to determine the coordinates of key features of the function. When correctly selected the *use of dedicated features of the graphing calculator* (CALCULATE value, CALCULATE zero, CALCULATE minimum, CALCULATE maximum, SOLVER) had a direct effect on the accuracy of key feature coordinates so identified. In contrast when the actions, *use of free cursor* or *use of TRACE*, were employed, these were less accurate except in the case of the y intercept centered in the viewing domain.

Use of dedicated features does not necessarily imply appropriate identification. The dedicated features were used unsuccessfully in the identification of key features on four occasions. These were (1) deliberate selection of a feature for the wrong purpose (i.e., CALCULATE intersect to find the point of intersection between the graph and the x axis when its purpose is to identify points of intersection between two functions); (2) inadvertent selection of the opposite dedicated feature to identify local extrema (i.e., CALCULATE minimum for a maximum turning point); (3) use of an appropriate dedicated feature for identifying local extrema but with a view that resulted in unreadable superimposed output from the calculator; and (4) use of a search domain that did not contain the key feature of the function with an appropriate calculator feature for identifying local extrema (i.e., CALCULATE minimum).

The first occurrence of unsuccessful use of dedicated features was reasonable. The action of selecting CALCULATE intersect to identify the coordinates of the point of intersection between the curve and the x axis was logical. This misuse of CALCULATE intersect raises the importance of students being familiar with and understanding the differences between mathematical language and graphing calculator language. In this case, the term, "intersect", has a slightly different meaning on the graphing calculator to when it is used mathematically. Students' thinking about the mathematical meaning of this term could reasonably assume that this feature would allow them to determine the coordinates of the point of intersection between a curve and the x axis. However, on the model of calculator used in this study, this is not the case. This calculator treats the axes as embedded objects rather than lines. CALCULATE intersect only allows the user to determine the coordinates of the point of intersection between functions entered by the user into the calculator. Teachers and students need to be aware of the specific declarative knowledge related to the use of calculator features. The graphing calculator placed limitations on the term, "intersect", that excludes this pair of students' seemingly logical action. As Hiebert and Leferve (1986) comment "procedures ... may or may not be learned with meaning" (p. 8). Where the latter occurs students will, as in this case, apply procedural knowledge external to the domain of the conceptual knowledge.

Implications for teaching from these findings include provision of a range of teaching and learning activities that include an emphasis on mathematical language, graphing calculator language, and the differences between these. Students need to experience using a variety of graphing calculator methods to find a complete graph of a function including the dedicated features, free cursor, TRACE, and TABLE (which was not used by students in this study, although Kate suggested the use of TABLE to find "how many turning points and everything it's got"), to identify key features and compare and contrast the results. This will enable them to make sensible decisions about when a particular selection may be pertinent. In addition, changing the viewing window by altering the window settings, or using dedicated ZOOM features to focus on key features, and determining their effect on accuracy in the identification of the key features, using a variety of methods, are further experiences that students learning about function should undertake.

This research positioned students in a situation that mirrored their classroom setting except that there were no opportunities for interaction between student pairs and other students or the teacher. Interactions with other students and particularly the teacher have the potential to facilitate defining moments for learning being seized and acted on to facilitate the solution process and learning. Defining moments occur where some action (cognitive, physical, or metacognitive) has the potential to facilitate or impede the solution process. Students' responses to the presence of a defining moment have been shown to impact on the directness and duration of the solution process as well as on the accuracy of key feature identification and are informed by their concept images (Tall & Vinner, 1981; Vinner & Dreyfus, 1989) which may comprise quite stable or fragile knowledge. Opportunities for additional interactions that occur in the normal classroom environment and particularly those initiated by the teacher can optimise student responses to defining moments in such a way as to facilitate learning. For example, a request by the teacher for Reem and Ali to justify their response action as they unsuccessfully attempted to find a global view of the function by editing the scale marks, may well have resulted in their realising the futility of their actions. Additionally, the questioning of their actions may have led them to consider an alternative, and hopefully more effective, action.

Expertise with the available technology and a strong concept image of cubic functions allowed some students (e.g., Linh and Ahmed) to seamlessly use the available technology to successfully complete the task. For other students, less experienced with the technology and having a less developed concept image due to fewer mathematical experiences with the concept such as the year 11 students, there were some difficulties during task solution. However, it was apparent that whatever the difficulties encountered, the concept image of cubic functions held by at least one member of the student pairs was strong and stable enough for them to persevere and determine window settings that did show a complete view of the given function, unlike the students reported in studies by Cavanagh and Mitchelmore (2000), Goldenberg (1987), Ruthven (1995) and Steele (1995). For some of the students who had fragile knowledge in their concept images that led to momentary lapses in their expectations for desirable properties in an acceptable view of a cubic function (e.g., Pete), experience with the task has brought further experience to strengthen this fragile knowledge in their concept image. For others, experience with the task brought elements of their concept image into conflict with what was seen on the screen and thus there was a further opportunity for their concept image to be modified and extended. However, there were still areas where inadequacies in these students' concept image of a cubic function was not challenged by the task. Kate, for example, still believed at the end of the task that all key features of a cubic function could be found analytically, despite their failure to do this.

How might these students be helped? It is clear that the notion of identifying defining moments in tasks such as this has potential in relation to mathematics teaching for concept formation. Identification of defining moments that enabled students to harness their knowledge of functions and technology in order to efficiently produce a solution to the task, could be used to inform future teaching actions. However, less efficient pathways have also revealed a wealth of experiences that have honed students' concept images. Even those that remain unresolved, such as Kate's belief that all x intercepts can be found analytically, could be used to challenge such notions in a follow up to the task. Explication of all the defining moments for a task such as this serves to give teachers a basis for designing teaching situations that highlight the mathematical and technological focus of such defining moments. Also, knowledge from research such as this where students were deliberately chosen who were expected to be able to complete the task could form a basis for designing learning experiences for students with less experience and/ or confidence in mathematics where student activities are scaffolded through teacher direction of actions to take and orchestrated discussion and debate about particular response actions in various circumstances given certain situations and conditions as identified using the framework in figure 1 and instantiated as in figures 2 and 8.

References

Anderson, M., Bloom, L., Mueller, U. & Pedler, P. (1999). The impact of the graphics calculator on the assessment of calculus and modeling. *International Journal of Mathematical Education in Science and Technology*, 30 (4), 489-498.

Arnold, S. (1998). Technology and mathematics education: From calculators to multimedia. In N. Ellerton (Ed.), *Issues in mathematics education: a contemporary perspective* (pp. 173-198). Perth, Western Australia: Mathematics, Science and Technology Education Centre, Edith Cowan University.

Binder, M. (1995). A calculator investigation of an interesting polynomial. *The Mathematics Teacher*, 88 (7), 558-560.

Brown, J. (2002). A complete graph, a graphing calculator environment and student understanding. In C. Vale, J. Roumeliotis & J. Horwood (Eds.), *Valuing mathematics in society learning* (pp. 365-375). Melbourne, Australia: Mathematical Association of Victoria.

Brown, J. (2003). Defining moments in determining a complete graph in a graphing calculator teaching and learning environment. In L. Bragg, C. Campbell, G. Herbert & J. Mousley (Eds.), *Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia* (MERGA), Geelong, Australia (pp. 168-175). Sydney, Australia: MERGA.

Cavanagh, M. & Mitchelmore, M. (2000). Student misconceptions in interpreting basic graphic calculator displays. In T. Nakahara & M. Koyama (Eds.), *Proceedings of 24th conference of the international group for the Psychology of Mathematics Education* (Vol. 2, pp. 23-27). Hiroshima, Japan: Program Committee.

Demana, F. & Waits, B. K. (1990). The role of technology in teaching mathematics. *Mathematics Teacher*, 81 (1), 25-31.

Dick, T. P. (1996). Much more than a toy: graphing calculators in secondary school calculus. In P. Gómez & B. Waits (Eds.), *Roles of calculators in the classroom* (chap. 5). Retrieved September 9, 2006 from http://ued.uniandes.edu.co/servidor/em/recinf/tg18/Dick/Dick-1.html

Even, R. (1998). Factors involved in linking representations of functions. *Journal of Mathematical Behavior*, 17 (1), 105-121.

Fey, J. (2006). Connecting technology and school mathematics [Review of the book The didactical challenge of symbolic calculators: turning a computational device into a mathematical instrument]. *Journal for Research in Mathematics Education*, 37 (4), 348-352.

Goldenberg, E. P. (1987). Believing is seeing: how preconceptions influence the perceptions of graphs. In J. C. Bergeron, N. Herscovics & C. Kieran, (Eds.), *Proceedings of 11th international conference for the Psychology of Mathematics Education* (Vol. 1, pp. 197-203). Montreal, Quebec: Program Committee.

Goldenberg, E. P., Harvey, W., Lewis, P. G., Umiker, R. J., West, J. & Zodhiates, P. (1988). *Mathematical, technical, and pedagogical challenges in the graphical representation of functions.* (Tech. Rep. No. TR88-4). Cambridge, MA: Harvard University, Educational Technology Centre.

Guin, D., Ruthven, K. & Trouche, L. (Eds.) (2005). *The didactical challenge of symbolic calculators: turning a computational device into a mathematical instrument*. Dordrecht, The Netherlands: Kluwer.

Hiebert, J. & Leferve, P. (1986). Conceptual and procedural knowledge in mathematics: an introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: the case of mathematics* (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum.

Huberman, A. M. & Miles, M. B. (2002). *The qualitative researcher's companion*. Thousand Oaks, CA: Sage.

Kooij, H. van der (2001, December). Functional algebra with the use of the graphing calculator. In H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.), *Proceedings of 12th International Commission on Mathematics Instruction study conference* (Vol. 2, pp. 606-612). Melbourne, Australia: University of Melbourne.

Leinhardt, G., Zaslavsky, O. & Stein, M. K. (1990). Functions, graphs and graphing: tasks, learning, and teaching. *Review of Educational Research*, 60 (1), 1-64.

Lloyd, G. M. & Wilson, M. (1998). Supporting innovation: the impact of a teacher's conceptions of functions on his implementation of a reform curriculum. *Journal for Research in Mathematics Education*, 29 (3), 248-274.

Merriam, S. B. (Ed.). (2002). *Qualitative research in practice: examples for discussion and analysis*. San Francisco, CA: Jossey-Bass.

Mitchelmore, M. & Cavanagh, M. (2000). Student's difficulties in operating a graphics calculator. *Mathematics Education Research Journal*, 12 (3), 254-268.

Ruthven, K. (1995). Pressing on: towards a considered calculator use. In L. Burton & B. Jaworski (Eds.), *Technology in mathematics teaching: a bridge between teaching and learning* (pp. 231-256). London, England: Chartwell-Bratt.

Schoenfeld, A. H. (1992). On paradigms and methods: what do you do when the ones you know don't do what you want them to? Issues in the analysis of data in the form of videotapes. *The Journal of the Learning Sciences* 2 (2), 179-214.

Stake, R. E. (1995). *The art of case study research*. Thousand Oaks, CA: Sage.

Steele, D. (1995). *The Wesley College technology enriched graphing project* (Unpublished masters thesis). University of Melbourne, Victoria, Australia.

Strauss, A. & Corbin, J. (1998). *Basics of qualitative research* (2nd. ed.). Thousand Oaks, CA: Sage.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12 (2), 151-169.

Teese, R. (2000). *Academic success and social power: examinations and inequality*. Melbourne, Australia: Melbourne University Press.

Victorian Board of Studies (VBOS) (1999). *Mathematics study design*. Melbourne: Australia: Author.

Vinner, S. & T. Dreyfus, (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, 20 (4), 356-366.

Wedege, T. (1999). To know or not to know – mathematics, that is a question of context. *Educational Studies in Mathematics*, 39 (1-3), 202-227.

Williams, C. G. (1993). Looking over their shoulders: some difficulties students have with graphing calculators. *Mathematics and Computer Education*, 27 (3), 198-202.

Zaslavasky, O., Sela, H. & Leron, U. (2002). Being sloppy about slope: the effect of changing the scale. *Educational Studies in Mathematics*, 49 (1), 119-140.

Zbiek, R. M. (2003). Using research to influence teaching and learning with computer algebra systems. In J. T. Fey, A. Cuoco, C. Keiran, L. McMullin & R. M. Zbiek (Eds.), *Computer algebra systems in secondary school mathematics education* (pp. 197-216). Reston, VA: National Council of Teachers of Mathematics.

Notes

1 To facilitate understanding of the analysis of the student solutions, the reader is advised to use a graphing calculator to solve the task before reading on.

2 Selection of TRACE shows a cursor linked to a function that displays the coordinates of successive points of the function.

3 Selection of the free cursor displays the coordinates of selected points in the viewing window.

4 Graphing calculator identifies the minimum value of a function within a user specified search domain.

5 Graphing calculator identifies where the value of a function changes sign within a user specified search domain.

Jill Brown

Jill Brown was a secondary mathematics teacher for over two decades and has recently joined the staff at Australian Catholic University where she is a lecturer in mathematics education. Research interests include use of graphing calculators in the teaching of function at the secondary level. Jill is currently undertaking her doctoral studies within the field of technology-rich teaching and learning environments at the University of Melbourne. The research presented in this article was undertaken for Jill's research masters thesis at the University of Melbourne.

Jill Brown
School of education,
Australian Catholic University
Locked bag 4115
Fitzroy, 3065
Australia
j.brown@patrick.acu.edu.au

Gloria Stillman

Gloria Stillman is a senior lecturer of mathematics education in the science and mathematics education cluster within the faculty of education at the University of Melbourne, Victoria, Australia. Research interests include teaching and assessing higher-order thinking through applications and mathematical modelling, metacognition, technology use in mathematics teaching at the secondary level, curriculum change, and ethnomathematics.

Gloria Stillman
Science and mathematics education
University of Melbourne
Parkville
Victoria
Australia 3010
g.stillman@unimelb.edu.au

Sammenfatning

Denne artikel rapporter et case-studie til undersøgelse af kognitive, matematiske og teknologiske processer hos gymnasieelever (11. og 12. klassetrin), der arbejder med at tegne en fuldstændig graf for et vanskeligt 3. gradspolynomium ved hjælp af en grafregner. Intensive analyser af elevernes virksomhed har identificeret en række ”defining moments” (afgørende momenter) i elevernes løsningsprocess, der er bestemmende for forløbet af deres virksomhed og for deres brug af grafregneren. I artiklen præsenteres og analyseres ”defining moments” knyttet til elevernes brug af skalering og enheder ved tegning af funktionens graf samt til elevernes udnyttelse af grafregnerens faciliteter til bestemmelse af koordinater for funktionsgrafens karakteristiske punkter. Der var stor variation i elevernes forståelse af skalering og dette havde indvirkning på effektiviteten og grad af elegance i elevernes løsningsstrategier. Eleverne brugte en række af grafregnerens forskellige faciliteter til bestemmelse af koordinater for grafens karakteristiske punkter, men ikke altid på en succesfuld måde og heller ikke altid baseret på forståelse af den matematik der ligger til grund for deres operationer på grafregneren.