Conceptual understanding of the dot product

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The aim of this study is to investigate whether it is possible to illuminate the development of conceptual understanding of the dot product through analyses of small-group dialogues. In the study we will focus on language, i.e. on the nature of the argumentation that develops. The article presents a rationale for conceptual learning and collaborative learning from a socio-cultural perspective. The article focuses on four sequences that make intelligible how the students use mathematical language and show a highly coordinated thinking-together mode. In spite of inaccurate mathematical formulations, the problem-solving process evolves and the students understand each other. The sequences also show how the students' argumentation evolved, how it changed because of the listeners' contributions, and in which way definitions are understood, used and applied.

Bauersfeld (1980) calls for in-depth studies of the social dimensions of the classroom in order to understand how mathematical knowledge is generated. In this spirit, the aim of this article is to analyse collaborative learning in mathematics within a small-group context. The students we will be observing have been characterised as high-achieving in an upper secondary school. Consequently, the term high-achieving reflects the fact that the students have received high grades in mathematics. This study follows a group of students through their problem-solving process when working with the dot product. The focus of the article is twofold. The firs part of the analysis concerns mathematics learning and problem solving in relation to the particular issue of appropriating and understanding the dot product. The second ambition is to report some observations

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on communication and interaction in a small-group environment. This dual focus implies that we want to study how the students communicate mathematically about the concept of dot product, how they use definitions, and how they argue and manage to apply the concept in the context of the problems given. In this sense the study focuses on language as a mediating tool (Vygotsky, 1975). After a presentation of the theoretical framework, we will introduce four sequences that illuminate this approach. The following research question has been formulated.

> Which aspects of conceptual understanding, exemplified by words, definitions, argumentation, and applications, are identifiable in small-group discourses?

The empirical material comes from a developmental project in the last year of upper secondary school. The idea behind the project was to differentiate and adapt the mathematics teaching using homogeneous small groups accommodated in separate rooms as an ordinary component of the teaching. This project has been established and developed locally by a mathematics teacher (Carlsen, 2002). The empirical material consists of fieldnotes and audiotapes from the group of high-achieving students.

Theoretical framework

In this study we will apply a socio-cultural perspective on learning and development (Säljö, 2001). The basic assumption is that learning is viewed as resulting from social and interactional processes in which students actively participate and contribute with ideas and arguments. The research on collaborative problem solving is extensive and so it is not possible to go into all the details of the various approaches. However, we would like to review some of the research in the areas of conceptual learning and collaborative learning.

Conceptual learning: the case of the dot product

How should one understand conceptual learning, conceptual understanding, and conceptual knowledge? Hiebert and Lefevre (1986) discuss *conceptual knowledge*. They report that this concept has been intensely discussed through years of research in mathematics education. According to Schoultz et al. (2001), conceptual knowledge generally is conceived as something that lies 'under' or 'behind' human performance in concrete social practices, and it is a performed manifestation of something more fundamental than communication and thinking. However, from a socio-cultural point of view, reality is mediated in different ways in various social practices (Säljö, 2001). It is therefore futile to argue for the occurrence of standard situations where conceptual resources and basic cognition might be ascertained. In the different communicative settings that constitute reality, it is problematic to claim that one particular kind of reasoning is more fundamental or more precise across various contexts. "Performance and reasoning are best understood as situated and as relative to circumstances" (Schoultz et al., 2001, p. 214). In a socio-cultural perspective conceptual learning is viewed as part of a communicative process where conceptual knowledge and conceptual understanding are mediated between participants in social practices (Säljö, 2001). Thus, it is interesting to scrutinise the performance of students sharing thoughts and ideas through communicative interaction in a small group.

Little research has been done to investigate students' understanding of the concept of dot product. Wiliam (1986) and Manes (1996) have inquired into this area but they have approached it differently from what will be the case in this study. The background for Wiliam's (1986) research was the assertion that students meeting the dot product from a traditional coordinate approach get bogged down in practising techniques and lose sight of the geometrical aspects. Wiliam reports having employed a geometrical approach to the dot product when teaching vectors. For instance, one problem was about deciding the condition for two given lines to be perpendicular, and then the linkage to the coordinate expressions was established. When using this geometrical approach, he found that the students experienced the usefulness of the dot product as a tool for geometry. To the students this approach encouraged the employment of dot product as a means of expressing geometrical insight. According to Wiliam, students did not lose sight of the purpose of dealing with the dot product when following this approach, and they hardly struggled when practising technique because this geometrical approach provided them with an intuitive background. In a socio-cultural perspective the students' conceptual understanding of the dot product improved in the particular communicative social practice. The concept of dot product was mediated to them by the geometrical approach.

Manes (1996) discusses students' generalising ability through reasoning from dot product problems. Manes interviewed university students who recently had completed a course in linear algebra. The interview questions looked somehow standard as one might find in a mathematics text book, but she had made them somewhat non-standard hoping to initiate student reasoning. She found that students were able to reason from the dot product definitions when working with concrete examples, but they failed when attempting to generalise the results they got from the concrete examples. Here generalising is understood to include the students' attempts to reason about the abstract properties of the dot product working with the related definitions and theorems in social contexts. The interviews established a communicative practice where the students failed to incorporate the underlying general properties of the dot product. The students were not able to apply the concrete findings to abstractly reason about them and extract general results.

Cooperative vs. collaborative learning

Cooperative learning is an umbrella term in which a large diversity of team-based learning approaches is embedded. Characteristically the term is used to describe practices in non-teacher-led classrooms, where classes are divided into small groups (Damon and Phelps, 1989). Davidson and Kroll (1991, p. 362) define cooperative learning as "learning that takes place in an environment where students in small groups share ideas and work collaboratively to complete academic tasks". This definition obviously does not explain everything, but it indirectly presents the concepts of small-group learning, collaboration, and problem solving. It must be seen as a concept that encompasses several different practices (Springer et al., 1999). According to these authors, the research literature distinguishes between cooperative and collaborative learning. Among several aspects cooperative learning is described as a systematic and structured strategy that is characterised by assigning complementary and interrelated roles to the members of the group, establishing common goals for the group, and offering rewards for achieving these goals. In contrast, collaborative learning "... is characterized by relatively unstructured processes through which participants negotiate goals, define problems, develop procedures, and produce socially constructed knowledge in small groups" (Springer et al, 1999, p. 24).

A considerable amount of researchers have in the last few decades studied small-group learning approaches. Webb (1991) has done a meta-analysis of several studies concerning task-related verbal interaction. Webb is focusing on some features of what happens when students give and receive help in the context of group collaboration and how this relates to the students' learning outcome. She argues that the experiences the students get through small-group cooperation influence their learning. The optimal group is one where the members admit what they do and do not understand, and at the same time give each other elaborated explanations about how to solve the problems. In addition they have to give each other the opportunity to express their level of understanding. Collaboration in small groups may be beneficial for the students, because they can give each other immediate feedback and they have a common mathematical language, a mathematical jargon which is understandable for the students involved. In addition, they may develop a common understanding of the mathematical difficulties of others that may be beneficial for their own learning.

Kieran and Dreyfus (1998) analyse interactions in small groups. These authors emphasise the importance of entering another's universe of thought. This means taking part in, and understanding, your partner's universe of thought, and relating this to your own comprehension and understanding. Kieran and Dreyfus, among a peer of high-achieving students, identified five different types of interaction called pragmatic interaction, homogeneous interaction, pseudo-interaction, anti-interaction, and inhomogeneous interaction. These authors identified the interaction as pragmatic when each student was contemplating the problem at hand individually in his own universe. Their thinking was not reflected in their interaction. The interaction was called homogeneous when the students in overlapping parts linked their respective universes. Pseudo-interaction was characterised by no interaction going on at all. It just seemed as if they were interacting. When the students refused to interact and quietly worked on their own, Kieran and Dreyfus identified it as anti- interaction. Inhomogeneous interaction was the interaction connected to the greatest level of learning outcome. The basis in this category of interaction emphasises the differences in the students' universe of thought. You argue for your own way of thinking in a certain manner when considering others' understanding difficulties. This is a strong level of interaction, and it forces considerable mental effort and will to learn. This inhomogeneous interaction relates to a socio-cultural perspective on learning and development. Students participate in a communicative social practice where they argue and act to understand each other and the mathematics (Säljö, 2001).

Vygotsky's (1978) notion of zone of proximal development (ZPD) concerns the distance between what a child is capable of doing alone and what the child is capable of doing in collaboration with more capable peers. The concept of ZPD can be fruitfully used to understand learning in small-group settings since students are put in positions where they may engage in learning with more capable peers. When students interact and express their mathematical understandings, they present different ways of thinking that may be acquired by other group members. In every interactional social context humans have the opportunity to appropriate knowledge from their collaborators (Säljö, 2001). Hiebert (1992, pp. 443-444) argues that by "expressing ideas publicly, by defending them in the face of others' questions, and by questioning others' ideas, students are forced to deal with incongruities and are encouraged to elaborate, clarify, and reorganize their own thinking".

Dialogical approach

In the analyses of data, we have in accordance with Cestari (1997) and Bjuland (1998; 2002), used a *dialogical approach* to understanding and analysing communication and cognition in mathematics classrooms. A dialogical approach follows Markovà's (1990a) characterisation of a dialogue, as an interaction in timely and spatial closeness between two or more persons who are aware of and oriented against each other in an act of communication. Thus a dialogue is characterised by sequentiality. From an analytical standpoint, we agree with the basic assumption of the related term *dialogicality* (Wertsch, 1990). The fundamental supposition of this approach is that "spoken and written utterances can be adequately interpreted only if their interrelationships with other utterances are taken into consideration" (op. cit., p. 63). The dialogical approach is derived from the epistemological approach to the study of cognition and language called *dialogism*. From this point of view "... language and speech originate and develop *through* social interaction and communication. Social interaction and communication, therefore, are absolutely essential to the existence of language and speech as living phenomena" (Markovà, 1990a, p. 4). Viewing language and social interaction as important is central to a socio-cultural perspective on communication (Säljö, 2001). Inspired by Bjuland (2002), we will employ dialogism as a framework for analysing and understanding the small-group discursive practices.

In this literature review we have emphasised knowledge as connected to argumentation and action in social contexts. Learning is viewed as a social process in which appropriation of intellectual tools emerges from diverse communicative practices. Related to the purpose of this article, we study how students actively participate and contribute with mathematical ideas and arguments in a small-group context. Thus the appropriation of intellectual tools relates to the mathematical language and arguments the students employ. It follows from this theoretical stance that it seems interesting to analyse the communication among students when collaborating in small groups.

Method

This is a qualitative study of a group consisting of high-achieving students at upper secondary school. Following an ethnographic approach in gathering data, the students' activities have been recorded, and the audiotapes have been transcribed. The recordings, transcriptions and fieldnotes made serve as the data material.

The analyses

We will analyse what the group members say following a dialogical approach. This is done by paying attention to what is being said ahead of and after each utterance, partly following an Initiative-Response analysis (IR). This is a coding system for dyadic interaction elaborated from the dialogical approach (Linell and Markovà, 1993). Markovà (1990b) asserts that every message is linguistically and contextually embedded and messages are both past- and future-oriented. This means that every message is potentially both retroactive and proactive simultaneously. Markovà (op. cit.) co-ordinates these concepts respectively with response and initiative. Thus IR analysis concerns both the retroactive ties and proactive ties of an utterance. The retroactive, or response, aspects of an utterance are subdivided into five categories (Linell and Markovà, 1993): (1) Locally tied vs. non-locally tied vs. no retroactive tie, (2) tied to others' vs. speaker's own utterance, (3) focally tied vs. non-focally tied, (4) minimal vs. expanded responses, and (5) adequate vs. non adequate responses. The proactive or initiative aspects of an utterance are subcategorised into two types, soliciting and non-soliciting initiatives that request or only invite a following utterance from the other. By employing such an approach, the single utterance will be contextually analysed. In this way we may discover whether ideas and contributions, which are mentioned earlier in the dialogue, relate to the content in single utterances.

Particularly our analytical approach will be focusing on how the students communicate mathematically about the concept of dot product, how they employ related definitions, and how they argue and manage to apply the concept.

Participants

The investigation was done in a Norwegian city, in a class of mathematics students at upper secondary school who had chosen the mathematics course 3MX. This course is the most advanced one at upper secondary school and prepares the students for further studies in mathematics at university level. The mathematics teacher had in this course developed a project which, in short, included lectures and collaboration in small groups in separate rooms with chalkboards, to complete academic tasks (Carlsen, 2002). The mathematics teacher guided the groups. The whole class consisted of 27 students. They were subdivided into five homogeneous groups based on their mathematics grade in 2MX. The group observed consisted of the six students with the highest grades, five boys and one girl. The observation took place during the first five weeks in the autumn semester where small-group collaborative learning was employed. The empirical data consisted of transcripts of four entire group lessons and transcripts of parts of additional four group lessons. The sequences were chosen where the concept of the dot product emerged and became a subject of discussion in the dialogue (Carlsen, 2002).

Students' work with the dot product

In order to illustrate how the students communicate and collaborate while solving problems, we will introduce and analyse four excerpts from the group dialogues. All four sequences have elements of 'unsatisfying' use of language related to the concept of dot product, but use of definitions, argumentation, and applications of the concept are present in the dialogues as well. All excerpts are taken from the seventh of ten observed group lessons because we believe our analytical approach is exemplified in these excerpts. The students were introduced to the dot product after having dealt with vectors and vector algebra for about a week. The dot product was first defined in two dimensions, first without coordinates and then with coordinates (This relates to the two formulae mentioned in sequence 3). After some days, the definitions related to the dot product were extended to three dimensions. The concepts of vector product and parametric equations were now introduced as well.

1 *Addressing the problem*

In the following sequence the students struggle with the task B 172 which was written like this:

> B 172
> A triangular pyramid is spanned by the vectors $\vec{p} = [3, -2, 1]$, $\vec{q} = [-2, 4, 5]$, and $\vec{r} = [1, 6, t]$. Let $t = 2$ and calculate the volume.
> Decide the value of t when the volume of the pyramid is $\frac{86}{3}$.

The extract is taken from the beginning of the seventh lesson, and the letter B in the task number tells that it is a difficult task from the text book writers' point of view. The text book tasks are divided into A and B levels with B as the most elaborate ones. The students try to calculate the volume of the pyramid and for this they need to consider the dot product between the vectors [-14, -17, 8] and [1, 6, 2] (The volume of the pyramid equals $\frac{1}{6}\left|(\vec{p} \times \vec{q}) \cdot \vec{r}\right|$, where $\vec{p} \times \vec{q} = [-14, -17, 8]$ and $\vec{r} = [1, 6, 2]$). The discussion starts in the following way[1]:

1 Pål: Eh, yes, how do you multiply two(.) such e:: parameters? I can't remember anything
2 Tor: Two parameters?
3 Tim: That is(.) minus 14 minus 14 multiplied by 1 added minus 17 multiplied by 6 =
4 Pål: Yes it's only that isn't it
5 Tim: = added 8 multiplied by (t, or 2 that ( ))

From Pål's utterance we discover that he uses the words ”multiply” and ”parameters”. The word ”multiply” is employed when the correct word would be ”dot” or ”scalar multiply”. However, it does not seem that this way of employing the mathematical language results in any difficulties for the group. What Tor responds to is the word ”parameters” and not ”multiply” (2). From the context we understand that Pål most likely means ”vectors”, but Tor wonders about this choice of words. Whether Tor still understands what Pål is trying to say, we cannot tell from the dialogue, but it is fairly obvious that Tim understands what Pål is talking about. He starts to explain how to calculate the dot product of two vectors by referring to the actual task they are working with (3, 5). In this perspective Tim's explanation shows understanding of how to compute a dot product. From a dialogical point of view, Pål's initiative (1) seems to request a response because he formulates his utterance as a question. Tor's response (2) is both locally and focally tied to Pål's utterance and Tim's response (3, 5) is also focally tied to Pål's question as well as being an expanded response. The sequentiality and dialogicality seem to be present here as well, exemplified by the turn-taking and face-to-face interaction.

The reason why Pål uses the word ”parameters” instead of ”vectors” is explainable if we consider the teaching and learning that preceded this session. Before this group session the group worked with parameters and how to construct parametric equations. In addition, the task involves the parameter *t*. Pål says that he does not remember anything, but this seems not to be the case. From the utterance in (4) we understand that Pål remembers something. He just needed to be reminded of how to do the calculations.

In this sequence we observe that Pål mixes different concepts. It seems as if the concept of ”parameter” has overshadowed the concept of ”vector” in a way that makes him a bit confused about how to proceed. But from the dialogue we can conclude that Pål was only in need of a small reminder or that his thoughts were confirmed as right. In this sequence we also see that the students give each other immediate feedback in an understandable jargon. Tim obviously understands the meaning of Pål's utterance in spite of his inexact language. This illustrates that lack of precision in spoken mathematical language or terminology does not necessarily harm the understanding of the concept. But because Pål admits his shortcoming, he establishes the possibility for learning to occur. It is possible to understand the content in a concept without an exact internalised language around it.

This sequence not only illustrates talk between collaborating students but is also an example of how the students think together. It is Tor who reacts to Pål's inaccurate words, but it is Tim that gives an answer to Pål's question. In this way Tim does not seem to reflect on, or be bothered by, the wrong terminology, but understands what Pål really means by his utterance. In this communicative process we believe that Tim with his argument mediates conceptual understanding to Pål by using a psychological artefact, the algorithm of executing a dot product. In their collaborative thinking they seem to be concurrent.

Another example of imprecise mathematical language we find in the next sequence. In this excerpt we observe an attempt to argue from a definition related to the dot product as well.

2 *Arguing about the dot product*

The following sequence comes from the group discussion later on in the seventh lesson. It is primarily the first part of the task below that is the object of discussion in this and the next two sequences as well (3 and 4). The students are going to show that the described quadrilateral is a parallelogram. We notice that this is a B-level task too, which is a relatively difficult task at this point in the academic year. The task was written as follows:

> B 173
> In a quadrilateral *ABCD* the vertices are defined as *A*(-3, -1, -3), *B*(2, 5, 1), C(4, 6, 4) and *D*(-1, 0, 0). Show that this quadrilateral is a parallelogram. Find the area of the quadrilateral.
> The point *T* is (4, 5, 12). Decide the volume of the pyramid *ABCDT*. What is the altitude from *T* to the base *ABCD*?

In this task the students were going to show that the quadrilateral is a parallelogram. In the discussion before the following sequence, they debated how to do this. They agreed that they had to show that $\overrightarrow{AB} = \overrightarrow{DC}$. If that was the case, they asserted that $\overrightarrow{AD}$ had to be equal to $\overrightarrow{BC}$. They did prove this, but in the continuing discussion the students stated that the parallelogram could have been a rectangle, which was not a parallelogram according to them. To prove that the parallelogram *ABCD* was not a rectangle, they decided that they had to show that the vectors at the vertex A were not perpendicular. The following sequence actuates the group's problem-solving process, which is continued in sequences 3 and 4. They really scrutinise the tasks and the discussion of their negotiated problem starts off like this:

41 Pål: = You can eh::(...) It it is only to employ the usual formula, eh::: If eh:: that is parallel to that, then they are =
42 Kai: Parallel?
43 Pål: = perpendicular, If (.) if AB-vector and(.) for instance BC-vector are perpendicular, =
44 Tor: Yes
45 Pål: = then AB multiplied by BC(.) equals zero

The problem the group faces here is to decide whether two vectors are orthogonal. Pål's solution to this challenge is to "employ the usual formula", as we, from the description, understand is the dot product formula (41). We observe that Pål uses inaccurate mathematical terminology in his argumentation (41, 45). This sequence differs from the first one, because Pål here actually corrects his own way of speaking (43), but that does not happen until Kai has made an objection (42). Pål here makes a statement that does not request or invite an utterance from the others, but still a response is made. Kai obviously reacts to Pål's words as we can see from the repetition of the word "parallel" (42). His response then is minimal but adequate and focally tied to Pål's utterance. The dialogue does not show whether Kai really understands what Pål means in spite of the wrong word being used, but the response from Kai consequently makes Pål correct himself, and he uses the mathematically correct expression in his continued utterance (43). Pål refers to the task and argues how to evaluate their claim (43, 45). Tor most certainly agrees to that (44) even though Pål's utterance is non-soliciting. He contributes to this dialogue, but it seems difficult to decide the functionality of this utterance in the sequence (44). At the moment of Tor's utterance, Pål still has not finished his reasoning. On that basis we interpret Tor's "Yes" to be a response to his understanding of Pål's argumentation. In addition, this utterance makes Pål carry on his statement. In this way the utterance becomes both a response and an initiative.

This sequence exemplifies the argumentation in relation to the dot product: if two vectors are perpendicular, the dot product equals zero. From Pål's choice of words we see that he refers to the implication $\vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0$. In this particular situation the equivalent contra-positive argument would have been mathematically preferable: If the dot product does not equal zero, the vectors are not orthogonal, $\neg(\vec{a} \cdot \vec{b} = 0) \Rightarrow \neg(\vec{a} \perp \vec{b})$. Nevertheless he describes how to proceed to complete the task to show that the angle at the vertex A in the parallelogram is not 90 degrees.

Actually Pål here considers the angle at the vertex B. Whether Pål is aware of that we cannot tell from the transcript. The group decided to investigate the vertex A, and in that case Pål's choice of vectors is wrong. Still it is possible to claim that this does not matter. The students know that ABCD is a parallelogram, and then it is not important which vertex you investigate. Nevertheless Pål begins his argumentation, but in the middle of the reasoning Kai contributes to the discussion. His utterance might be interpreted as if he is not following what Pål is saying. It seems as if Pål takes Kai's utterance as information because of the changed language and the elaborated argument. Pål considers Kai's difficulties with following the reasoning and hence changes the argumentation.

This excerpt is another example of the considerable level of coordinated thinking within this group. It is Pål's utterance (41) that contributes to this view. The words ”usual”, ”that” and ”that” tell about a highly coordinated discursive practice. When Pål says ”usual” he assumes that the other group members understand what he is talking about. It seems as if he assumes that his characterisation of the formula is distributed among them. The words ”that” and ”that” contribute to the analysis in the same direction. Pål mediates his understanding to the group members in a vague way, but the students seem to think together and understand each other.

The excerpt below is another example of a dialogue including inaccurate mathematical terminology, and here too examples of words indicating a thinking-together mode are identifiable. The sequence brings another aspect to the problem-solving process. The level of discussion and argumentation is more elaborated than in the two previous excerpts, but confusion emerges among the group members.

3 *Attempts to resolve the confusion*

Further on in the discussion, still dealing with the task mentioned B 173, the students try to evaluate which vectors to consider when calculating the angle at the vertex A of the quadrilateral. The group members seem to be struggling with the question of which vectors should be dotted to find the intermediate angle. The discussion swings back and forth and the confusion seems to be considerable. After a while Tim and Pål agree, because of Jon's contribution, that they have to begin with two vectors with a common starting point. In the transcript the students argue with the two formulae or definitions

$$(^*) \qquad \vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos(\vec{a},\vec{b})$$

$$(^{**}) \quad \vec{a}\cdot\vec{b} = [a_1, a_2, a_3]\cdot[b_1, b_2, b_3] = a_1b_1 + a_2b_2 + a_3b_3$$

71 Tim: You just take the length of AB multiplied by, it is the good old formula. =
72 Pål: Yes
73 Tim: = The length of AB multiplied by BC. =
74 Pål: No, not BC
75 Tim: I took vector BC. That's right isn't it
76 Pål: No, we are supposed to find the angle. No::, not that
77 Jon: Yes you just can take that
78 Pål: We are supposed to- we just employ
79 Tim: Yes, that was two nearby each other
80 Pål: Yes, it is DA multiplied by
81 Jon: AB multiplied by AD
82 Pål: AB BC for instance. No it doesn't become that. AB
83 Tim: AB AD
84 Pål: AB AD. =
85 Jon: Yes yes yes
86 Pål: = The length of the AD-vector
87 ((Some mumble and discussion about how to calculate the AD-vector))
88 Kai: I hope we now used the dot product?
89 ((Pål and Tim discuss, and Pål has found that the length of the AD-vector is the square root of 14))
90 Jon: ((:-D))Isn't it ((:-D))just to do like this, multiply and add and find that it's not zero, ergo it isn't 90?((degrees))

Tim opens with an argument for using a formula (71). His choice of words may give the impression that he thinks this is easy. The expressions "you just take" and "the good old formula" point in this direction. It seems as if Pål agrees (72). Pål responds to Tim's utterance because the utterance invites a response. Tim maintains that the length of $\overrightarrow{AB}$ has to be multiplied by the length of $\overrightarrow{BC}$ (73), but Pål does not follow this thought (74). He states that it is not $\overrightarrow{BC}$ that has to be multiplied by $\overrightarrow{AB}$, because they are going to calculate the angle at the vertex A. This disagreement then becomes the object of discussion (75-76). It apparently seems as if Jon is of a different opinion. He says that in this case they could just have used $\overrightarrow{BC}$ (77). The utterance is therefore both locally and focally tied to the previous discussion, but non-soliciting.

Pål amplifies why he is of a different opinion (78) and, without being concrete, he tries to explain how to calculate the angle between two vectors. Nevertheless it appears that Tim understands what Pål is trying to say. It seems as if Tim agrees that he cannot apply $\overrightarrow{BC}$ (79). This expression we interpret as an attempt to say that, in order to calculate the angle between two vectors, they have to have the same origin. Again we identify a faulty use of words, but it appears that Pål understands Tim (80). In any case he agrees with what Tim is saying, and at this moment he announces an alternative solution: we should start with $\overrightarrow{DA}$. We also observe that they apply the words in connection with descriptions of vectors in an inaccurate way. Pål says DA instead of the preferred $\overrightarrow{DA}$. Moreover, they once again use "multiplied by" instead of "dotted by". From a dialogical point of view, Tim's and Pål's utterances (71-80) are locally and focally tied to each other and invite a following contribution. The utterances are minimal but adequate. It appears that Jon has followed the discussion between Tim and Pål and on this basis he makes his utterance. Jon claims that it is the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ which have to be multiplied (81). No one in the group reacts to Jon's words, but that has been the case on several occasions. It is possible to consider Jon's utterance as a response to the preceding discussion between Tim and Pål (78-80). The fact that Jon gives an alternative solution like this initiates the continued debate. Pål gives his solution and his suggestion is to use different vectors compared to what Jon suggested (82). On this basis it may seem as if Pål considers Jon's utterance as inadequate. In any case Jon's statement does not immediately relate to Pål's chain of thought. But as the dialogue goes on they agree that they have to start with $\overrightarrow{AB}$ and $\overrightarrow{AD}$ (83-87).

At this point in the sequence Kai's only utterance occurs, and it concerns his hope of using the dot product in this case (88). This utterance may be interpreted as a question Kai asks to confirm that he has understood some application of the dot product. In spite of soliciting a following contribution, Kai's statement is not responded to. However, whether this utterance is considered as inadequate, we cannot actually tell.

Jon closes the sequence with the wish to get an answer to a question (90). From the choice of words we understand that Jon has understood how to calculate a dot product. He wishes by means of his request to have his understanding confirmed. This utterance has about the same content of meaning as Pål's utterance (41, 43, 45) in sequence 2, but the specification is different. When Jon here talks about multiplication and addition he concretises at a different level. The word "multiply" we interpret as a description of the process connected to multiplication of coordinates, and the word "add" in this setting becomes the addition of the three coordinate products afterwards. If this adds up to something different from zero, the angle is not 90° and then the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are not perpendicular. Even though the utterance is formulated as a question, it is our opinion that Jon is quite confident that his understanding is accurate.

This sequence exemplifies a new aspect of the group's development towards a deeper understanding of the dot product. That you have to use vectors with the same origin to find the intermediate angle is an important part of the dot product and its applications. To decide whether an angle is right or not, you do not have to consider the direction of the vectors. This was the case in the problem at hand. Tim's suggestion that they calculate the angle at the vertex A using $\vec{AB}$ and $\vec{BC}$ would have led to a solution. At this point the group already has shown that $\vec{AD}$ and $\vec{BC}$ are equal, and then it is arbitrary to employ $\vec{AB}$ and $\vec{BC}$ or $\vec{AB}$ and $\vec{AD}$ to determine the angle at the vertex A. If this is the meaning of Tim's utterance (75), Pål does not catch or understand it. In this light it is also possible to interpret Jon's utterance (77) in a new way. This statement then becomes a focally tied response and a confirmation of Tim's idea.

This sequence is another example of what we can call the group's mode of thinking together. They do not just talk and discuss but really coordinate their thinking while solving the problem. The words in (71) say something about this highly coordinated thinking. Tim uses the expression "the good old formula". By doing this he appeals to his fellow students' thinking and understanding of formulae or definitions related to the dot product. Pål does not question this expression, but he confirms his understanding of Tim's expression by saying "yes". Tim and Pål are here closely related in their thinking and that moves the problem-solving process further on.

It is also worth noticing that Jon's utterance at the end of the sequence (90) concerns formula (**) and not formula (*) as was the case in the beginning of the sequence. The strategy to use (*) would not lead to a solution because the cosine of the angle makes a factor, and it is this angle they actually are going to determine. Jon here uses the implication $\vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0$, and he employs a contra-positive argument when he mentions the equivalent formulation $\neg(\vec{a} \cdot \vec{b} = 0) \Rightarrow \neg(\vec{a} \perp \vec{b})$. By discussing and explaining each other how to proceed, the group members are internalising and appropriating the dot product.

The last sequence is another example of how the students discuss in order to solve the mentioned task B 173, and how they apply and connect the dot product to this. The excerpt that is analysed below continues the previous discussion in sequence 3, but the transcript in sequence 4 shows a further elaborated discussion.

4 *Application of the dot product*

This sequence closes the part of group lesson 7 where the group discuss the dot product and its applications. The argumentation in this sequence seems interesting in a developmental perspective. It is almost just Jon and Tor who are actively engaged in the discussion, but all of the group members are represented. This sequence is closely connected to the sequence above, and the context is that Jon stands at the black board and explains what he tried to argue orally in sequence 3.

102 Pål: That is we actually have we actually have, we don't have to find the absolute value to multiply things together. We can just use the <u>other</u> formula
103 Jon: ((Jon writes on the black board AB · AD = 0)) If that's equal to zero then they are perpendicular, that is they are 90 degrees. =
104 Tor: Yes, that is actually right
105 Jon: = You're following right? Well then you just write(.) the coordinates to <u>that</u>, =
106 Tor: Yes A
107 Jon: = AB, the coordinates to AB, =
108 Tor: Yes yes, just yes
109 Jon: = and then multiply them and then add them the three coordinates, =
110 Tim: Okay
111 Jon: = and then you get like that is 28, I got. And then AB multiplied by AD actually is 28 and then they aren't 90 degrees
112 Tor: Totally correct
113 Ann: Yes
114 Kai: You're a genius

The sequence starts with Pål's non-soliciting utterance that they just can "use the other formula" (102). From the context it is fairly clear that he means formula (**). He argues about the unnecessary struggle to calculate the absolute value to multiply things together. We interpret this utterance as an attempt to say that it is not necessary to find the length of the vectors to calculate the dot product between them. We notice that the word "other" is emphasised, and it is possible to interpret this as if he wants to distinguish between the two formulae. Maybe Pål has difficulties in catching the two formulae as two different approaches to calculating the dot product.

As mentioned, Jon stands at the black board and actually teaches the whole group, but immediately it seems as if just Tor and Jon are communicating. Jon writes the expression $\vec{AB} \cdot \vec{AD} = 0$ on the black board (but without the arrows) and his arguments are based on that expression (103). He maintains that if the dot product equals zero, then $\vec{AB}$ and $\vec{AD}$ are perpendicular. Jon here draws a logical correct conclusion, and the argumentation has the following pattern: it starts with the implication

$\vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$, but throughout the utterance the focus changes to the implication $\neg(\vec{a} \cdot \vec{b} = 0) \Rightarrow \neg(\vec{a} \perp \vec{b})$ that is equivalent to $\vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0$. In this way Jon shows that he has understood that the formulations $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \perp \vec{b}$ are equivalent. It seems difficult to decide whether this utterance is a response to Pål's utterance. Thematically they consider the same thing, but Jon uses different and more precise words. Jon's expanded response contains an aspect of initiative because he establishes premises for when it is possible to conclude that two vectors are orthogonal. Tor considers this initiative and obviously agrees with the given argumentation (104). Whether it was Jon's intention only to give Tor the explanation we do not know. But it seems, from the choice of words, as if Jon in the beginning of his explanation is addressing the whole group. The utterance (103) has a lack of preferences that tells us that Jon only gives Tor an explanation, but that is probably because it was Tor that responded to the utterance before (104). Nevertheless, Jon continues to concretise and to explain how he practically proceeded to calculate the dot product (105, 107, 109, 111). Tor (106, 108) and Tim (110) both interrupt his reasoning. It seems as if Tor fully understands what Jon is trying to explain to him. Tim's utterance, on the other hand, may be indicating a different understanding (110). The word "okay" may be interpreted as an expression of acceptance of Jon's explanation, but that Tim still is not fully confident with what Jon is saying.

In the utterance (109, 111) Jon describes how to calculate the dot product. First you multiply and then you add. This announcement may be described as faulty. It is another example of how the students use inaccurate mathematical language or terminology. You do not add three coordinates, but three products of coordinates. Again it does not seem to be a problem for neither Jon nor the listeners. They seem to understand what Jon is trying to explain to them. Perhaps the reserved utterance by Tim in (110) is a consequence of the inaccurate language, but that is not possible to determine from the dialogue. Jon talks about the dot product between $\overrightarrow{AB}$ and $\overrightarrow{AD}$, $[5, 6, 4] \cdot [2, 1, 3] = 5 \cdot 2 + 6 \cdot 1 + 4 \cdot 3 = 28$, and concludes that the vectors are not orthogonal. Once again we observe the inaccurate use of words, "multiply" instead of "dot", but this does not seem problematic for the students. Jon draws at least the correct conclusion, and several in the group agree with this conclusion. Both Tor (112) and Ann (113) agree that $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are not perpendicular. This is the only place in these sequences Ann actually expresses herself. It seems as if Ann has followed Jon's argumentation since he started at the black board. We can only speculate why she has not said anything before, but it seems as if she has concentrated on what Jon has actually said. This also confirms that Jon really has more listeners than Tor, although he only addressed him verbally. Kai finishes this sequence by praising Jon and his explanation, showing that he values Jon's explanation. In addition the utterance shows that he has followed the argumentation from the sideline. The last part of the sequence (112-114) is responding Jon and his utterances, and the content shows that Jon has made himself understandable. The sequentiality in the dialogue seems to be coordinated, and the utterances are locally and focally tied to each other.

From this sequence we will discuss four aspects. The first aspect concerns the communication that arose first of all between Jon and Tor, but in the rest of the group too. The explanation and argumentation Jon employs in this sequence are different from previous sequences. Jon thoroughly, quite accurately and concretely explains how to apply the dot product. That the explanation hits home we can tell from the response he gets. It seems as if the group members understand what is being said. In addition we find it interesting that Jon in his explanation directly addresses Tor and asks whether Tor understands what he is saying. Jon argues in such a way that he considers Tor's possible difficulties, and that he sincerely wishes to help Tor in his understanding. Jon here mediates his understanding and appropriation of the dot product through his explanation. The second aspect we will emphasise is the acknowledgement Kai expresses in the closing section of this sequence. The expression "You're a genius" shows that Kai really appreciates Jon's help and wishes to give something in return. This statement, although it may be a bit exaggerated, probably increases Jon's self-confidence, and it may encourage Jon to continue explaining parts to the other group members. The third aspect we will highlight is that this utterance also says something about the social relations in the group, that the learning environment and atmosphere are good. By giving each other positive response the feeling of security and trust in the group probably increase. The fourth aspect is what we already have mentioned a couple of times, the group's mode of thinking together. In (102) Pål uses the word "other" which indicates that he assumes that the other students are thinking about the same thing as him. However this exemplifies a weaker mode of thinking together than we have seen in the previous excerpts. Here Pål to some extent explains which formula he means, formula (**), by describing related issues of formula (*). By doing that he insinuates what he is talking about and facilitates his fellow students' ability to contextualise and understand his utterance.

Discussion

In this article we have studied the duality between spoken mathematical language and the problem-solving process. In general we observe several elements of what can be interpreted as inaccurate mathematical formulations, but this never seems to constitute a problem for the group members' problem-solving process nor for their understanding. This aspect relates to and supports what Webb (1991) maintains, that students who collaborate in small groups have the potential to give each other immediate feedback and they use a mutual mathematical language or jargon. Pål and Tim understand each other in spite of inaccurate use of language in sequence 1. In sequence 3 Jon explains using a language the others can understand and relate to. In spite of this inaccurate mathematical terminology, the group of students communicate very well. This illustrates the complexity of talking about mathematics and human understanding. In spite of conceptual ambiguity the students understand each other's arguments and the problem-solving process evolves. This illustrates that humans are able to interpret and understand oral mathematical arguments in spite of incorrect mathematical terminology. This also relates in some sense to Lave and Wenger's (1991) notion of learning as *legitimate peripheral participation*. These mathematics students are metaphorically speaking peripheral, on the border of the mathematics community and newcomers to the realms of mathematics. That their mathematical discussions are characterised by inaccurate terminology is therefore natural and to be expected. This is due to a socio-cultural perspective on language, that the level of precision does not have to be high in oral language for participants to understand each other (Vygotsky, 1975).

The students have a common understanding of what they actually are doing and seem to be quite coordinated in their thinking about the dot product. As mentioned it is possible to interpret several aspects in the excerpts as what we have called the group's mode of thinking together. The students use words and collaborate in such ways that a highly coordinated discursive practice seems to have been developed among the group participants. Some of the words the students use are characterised by not being informative, but because of the group's highly coordinated thinking-together mode the words work very well communicatively. Another possibility is to look upon all the sequences 2, 3 and 4 as examples of what Kieran and Dreyfus (1998) refer to as entering another's universe of thought. In sequence 2 we see that Pål starts with an argument. In the middle of the utterance Kai objects, and it seems as if Pål considers Kai's utterance when he changes his language and concretises the argument. Pål considers and changes the argumentation because of Kai's difficulties in following the reasoning. After listening to the others' discussion in sequence 3, Jon takes part in and understands their way of thinking. He considers and relates this to his own opinion in his explanation. In this way an inhomogeneous interaction (op. cit.) emerges among the group members. This is exemplified in sequence 4 too. Here Jon argues in a considerate way because Tor may have problems in explaining. At least he ensures that Tor follows his explanation. The content of these three sequences also differs in the level of argumentation. The practice of logical implications and the argumentation develop from being unfinished and inaccurate to correct. In sequence 4, after insecurity and spending a lot of time in the sequences before, the argumentation is more accurate. This is what Vygotsky (1978) calls microgenesis, i. e. the students' argumentation evolves from diffuse to more articulated conceptualisation over a short period of time. All in all it seems as if the students use their mathematical language as part of their mathematical thinking.

There are other aspects as well connected to the communication in the group. In sequence 1 Pål establishes a learning foundation by admitting his shortcomings. He says that he does not really understand. The opposite situation we discover in sequence 4. The students confirm that they do understand Jon's explanation. They talk about their own understanding. These are positive elements in the dialogue, and according to Webb (1991) it is therefore possible to characterise the group as well-functioning. In addition the students in these two sequences give each other feedback that has the potential to build personal self-confidence. The collaborative learning seems to have positive affective aspects, following what Davidson (1990) reports. Another element to mention from these sequences is the different level of explanation in the utterance by Tim in sequence 1 and the utterance by Jon in sequence 4. Tim only states the answer to the task and does not explain why the calculations have to be done like that, while Jon gives what Webb (1991) calls a content-related explanation. According to Webb, content-related explanations are positively correlated with learning for the explainer, while just giving the answer is not. Learning is a possible outcome of the problem-solving process in the group. When several students inquire into a mathematical problem as they do in the excerpts here, the possibility of learning outcome increases. This group collaboration and investigation relate to both Vygotsky's (1978) notion of zone of proximal development and to Hiebert's (1992) reorganisation of thinking. Nevertheless, whether Jon and Tim really are learning something on the basis of their explanations this dialogical analysis cannot answer.

The working efforts of the group members are considerable, as they are thorough in their approach to understanding the mathematics and spend a great deal of time solving the problems. The problem-solving process does not evolve rapidly but rather iteratively. All the students contribute to the process by expressing their own ideas and questioning others' ideas. The students choose to be thorough and discuss the tasks in elaborated ways. By doing that the students do not manage to work with a broad range of problems, and the problems they do discuss are, in our opinion, actually quite basic. They make this choice in spite of their being high-achieving students. As teachers we would expect them to rush into the really difficult problems, but that is not the case here. The group members choose thoroughness and quality instead of superficiality and quantity.

As already mentioned, conceptual understanding is mediated between participants in social practices resulting from communicative processes (Säljö, 2001). Jon, in sequence 4, mediates his conceptual understanding when communicating an argument consisting of knowledge about the definitions related to the dot product, how to apply these definitions in this particular case, how to formulate a mathematical argument, how to complete the calculations etc. Using the correct words is also part of the conceptual learning process, but we have seen that inaccurate mathematical terminology constitutes limited problems, as long as the understanding related to the words is appropriate.

The above findings to some extent confirm Wiliam's (1986) assertion but contradict Manes' findings (1996). Our students experience a traditional approach to the dot product, focusing on the definitions, and they are struggling in practicing dot-product-related mathematical techniques. However, in the problems cited the students work with geometrical applications of the dot product, and in that sense, following Wiliam (1986), it seems as if they experience the geometrical usefulness of the dot product. According to Manes (1996) students fail to argue in general ways from specific tasks. However, this seems not to be the case here. The students both reason and base their arguments on the definitions and draw on those to complete the tasks. From these findings it seems appropriate to emphasise that the group members have the capacities to extract general properties from the specific cases.

Conclusion

In general we conclude that the mathematical language used is inaccurate, but the precision in the mathematical terminology improves as the problem-solving process evolves. The group members communicate quite briefly, but constantly more concretely and thoroughly about the concept of dot product. The use of definitions and formulae, for instance (*) and (**), and the related understanding, also seem to develop: from insecurity connected to the calculation of the dot product, by way of discussions about which formula to use in concrete cases, to an established, thorough and applicable understanding. It is also possible to conclude that the argumentation connected to the concept develops positively. The level of argumentation in explanations varies from just giving the answer to a question, by way of somewhat hesitating and inaccurate argumentation, to quite exact and thorough formulations. Application as well is an important aspect of a deep conceptual understanding. Both sequence 3 and 4 are examples where the students, after some discussion, apply the dot product in connection with evaluating angles and perpendicularity. These are important applications of the concept, and we see that the group members, after some time, manage this.

Seeing the four sequences as a whole, we conclude that the students' understanding of the dot product improves. In the beginning sequences they are insecure and are fumbling, but as the problem-solving process evolves their arguments and conceptual understanding mature and improve from a mathematical point of view. The students show a highly coordinated thinking-together mode, and although the mathematical language is inaccurate, the problem-solving process evolves. The students collaborate very well communicatively. The selection of sequences shows that something has happened with the students' conceptual understanding and learning connected to the dot product. It is our opinion that the group members' mathematical language, use of definitions, argumentation, and applications of the concept can be taken as evidence of their acquired conceptual knowledge about the dot product. In that sense the approach focuses on important parts of the problem-solving process and elucidates how students develop conceptual understanding.

From the four extracts we also get an impression of how the group work progressed, and it is possible to conclude that the students thoroughly discuss and debate the tasks. In addition we see that the students spend quite a lot of time working on the problems. In the last three sequences they are working with the same problem. This indicates thoroughness and energetic effort, but it also illustrates that high-achieving students go through a common process of learning mathematics. They need practice and thorough discussions with fellow students, and they have difficulties separating their understanding of different concepts and their applications. Furthermore they need, at their own pace, to work with the mathematics over some time. We will also emphasise that these high-achieving students, surprisingly, choose to work with relatively basic problems. They do not rush into the most difficult ones, but struggle through the problems to gain a deep and thorough conceptual understanding of the dot product.

This research has some implications for teaching. It seems unnecessary to stress in great detail the significance of students' oral mathematical language when it comes to conceptual understanding. Students' spoken language does not necessarily reflect their understanding, and they probably understand more than they are capable of articulating. Nevertheless students seem to be able to understand each other in spite of diffuse and limited mathematical language and argumentation. However, more research is needed to inquire into these aspects.

References

Bauersfeld, H. (1980). Hidden dimensions in the so-called reality of a mathematics classroom. *Educational Studies in Mathematics*, 11, 23-41.

Bjuland, R. (1998). Lærerstudenters matematiske tenkning og utvikling i en sosial kontekst. *Nordisk matematikkdidaktikk*, 2, 41-65.

Bjuland, R. (2002). *Problem solving in geometry. Reasoning processes of student teachers working in small groups. A dialogical approach* [Published doctoral dissertation]. University of Bergen.

Carlsen, M. (2002). *Smågruppeundervisning i matematikk. Ei dialogisk tilnærming til samarbeid i ei gruppe av faglig sterke elever i videregående skole* [Unpublished master degree thesis]. Kristiansand: Agder University College.

Cestari, M. L. (1997). *Communication in mathematics classrooms. A dialogical approach* [Unpublished doctoral dissertion]. University of Oslo.

Damon, W. & Phelps, E. (1989). Peer interaction, problem solving, and cognition: multidisciplinary perspectives. *International Journal of Educational Research*, 13, 9-19.

Davidson, N. (1990). *Cooperative learning in mathematics - A handbook for teachers*. New York: Addison-Wesley.

Davidson, N. & Kroll, D. L. (1991). An overview of research on cooperative learning related to mathematics. *Journal for Research in Mathematics Education*, 22, 362-365.

Hiebert, J. (1992). Reflection and communication: cognitive considerations in school mathematics reform. *International Journal of Educational Research*, 17, 439-456.

Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: an introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: the case of mathematics*. London: Lawrence Erlbaum.

Kieran, C. & Dreyfus, T. (1998). Collaborative versus individual problem solving: entering another's universe of thought. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd international conference for the Psychology of Mathematics Education, PME 22, Vol. 3* (pp. 112-119). University of Stellenbosch.

Lave, J. & Wenger, E. (1991). *Situated learning: legitimate peripheral participation*. Cambridge University Press.

Linell, P. & Markovà, I. (1993). Acts in discourse: from monological speech acts to dialogical inter-acts. *Journal for the Theory of Social Behavior*, 23, 173-195.

Manes, M. (1996). Student Learning in linear algebra: the gateways to advance mathematical thinking project. *Proceedings of the fifth Conference on the Teaching of Mathematics*. Retrieved January 13, 2006 from http://archives.math.utk.edu/CTM/5th.html.

Markovà, I. (1990a). Introduction. In I. Markovà & K. Foppa (Eds.), *The dynamics of dialogue* (pp. 1-22). New York: Springer-Verlag.

Markovà, I. (1990b). A three-step process as a unit of analysis in dialogue. In I. Markovà & K. Foppa (Eds.), *The dynamics of dialogue* (pp. 129-146). New York: Springer-Verlag.

Schoultz, J., Säljö, R., & Wyndhamn, J. (2001). Conceptual knowledge in talk and text: what does it take to understand a science question? *Instructional Science*, 29, 213-236.

Springer, L., Stanne, M. E., & Donovan, S. S. (1999). Effects of small-group learning on undergraduates in science, mathematics, engineering, and technology: a meta-analysis. *Review of Educational Research*, 69, 21-51.

Säljö, R. (2001). *Læring i praksis. Et sosiokulturelt perspektiv*. Oslo: Cappelen Akademisk Forlag.

Vygotsky, L. S. (1975). *Thought and language*. Cambridge, Mass: The M. I. T. Press.

Vygotsky, L. S. (1978). *Mind in society. The development of higher psychological processes*. Cambridge, Mass: Harvard University Press.

Webb, N. M. (1991). Task-related verbal interaction and mathematics learning in small groups. *Journal for Research in Mathematics Education*, 22, 366-389.

Wertsch, J. V. (1990). Dialogue and dialogism in a socio-cultural approach to mind. In I. Markovà & K. Foppa (Eds.), *The dynamics of dialogue* (pp. 62-82). New York: Springer-Verlag.

Wiliam, D. (1986). A 'black box' approach to scalar and vector products. *The Mathematical Gazette*, 70, 197-200.

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Note

1 Transcription conventions

(.)	small break
(...)	longer break
=	continued utterances
[[	overlapping utterances
(( ))	non-verbal activity /comments
((:-D))	laughter in voice
:	prolonged sound or letter
( )	inaudible fragments
(guess)	best guess
<u>underline</u>	emphasised words

Martin Carlsen

At the moment Martin Carlsen is a doctoral student in mathematics education at Agder University College. He is in the third of four years of scholarship. His main research interests are in the areas this article also is a part of, conceptual understanding through problem solving in collaborative small groups. The doctoral project is a continuation of the work done in the master thesis in mathematics education, which is the research base for this article.

Martin Carlsen identifies with a socio-cultural perspective on learning and development, and his research, both in this article and in his doctoral project, is thus strongly affected by this theoretical stance.

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Sammendrag

Målet med dette studiet er å undersøke om det gjennom analyser av smågruppedialoger er mulig å belyse utvikling av begrepsforståelse av skalarprodukt. I studiet vil vi fokusere på språk, det vil si på argumentasjonens karakter og utvikling. Artikkelen presenterer en teoretisk bakgrunn for begrepslæring og samarbeidslæring fra et sosiokulturelt perspektiv. Artikkelen fokuserer på fire sekvenser som anskueliggjør hvordan elevene bruker matematisk terminologi og en sterkt koordinert thinking-together mode. Til tross for unøyaktige matematiske formuleringer, utvikler problemløsningsprosessen seg og elevene forstår hverandre. Sekvensene viser også hvordan elevenes argumentasjon utvikler seg og hvordan den endres som følge av lytternes bidrag, og de viser på hvilke måter definisjoner blir forstått, benyttet og anvendt.