Investigating the development of number sense in a 6th grade class in Taiwan

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This research study reports snapshots of a 6th grade Taiwanese class depicting how a teacher investigated and promoted his students' development of number sense. It illustrates students' tendency to rely on written algorithms and reveals some misconceptions that may exist among students that are generally proficient in written computation. It demonstrates an effort to integrate number sense activities into the mathematics class in ways that encourage exploration, discussion, thinking, and reasoning.

Number sense, like common sense, is valuable and plays a key role in elementary mathematics education. For example, *Everybody Counts* (National Research Council, 1989) indicates that the major objective of elementary school mathematics education should be to emphasize the development of number sense. Furthermore, the Number and Operations Standard in *the Principles and Standards for School Mathematics* (NCTM, 2000) states that "Central to this Standard is the development of number sense." (p. 32).

Due to its importance, number sense has engendered much research and investigation among mathematics educators, researchers, and cognitive psychologists. This activity has produced descriptions of characteristics of number sense (Howden, 1989; NCTM, 1989; Reys, et al., 1991; Reys, et al., 1999; Sowder, 1992a; Sowder, 1992b), and theoretically examined the number sense from psychological perspectives (Greeno, 1991; Resnick, 1989; Sowder and Schappelle, 1989). It also stimulated efforts to design instructional activities that encourage the development

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of number sense (Burton, 1993; Markovits and Sowder, 1994; McIntosh, et al., 1997a; McIntosh, et. al., 1997b; Reys, et al., 1991). This body of work has produced several generally agreed-upon components of number sense, including the following: (1) recognizing the relative magnitude of numbers; (2) being able to use benchmarks; (3) knowing the relative effect of operations on number; (4) developing a range of computation strategies, such as estimation, to solve number problems; and (5) deciding the reasonableness of results involving numbers and operations on them.

"Students who have experienced traditional instruction do not exhibit number sense in many numerical situations" (Markovits and Sowder, 1994, p. 5). This situation is more serious in Taiwan. The research evidence has consistently confirmed that Taiwanese students have high performance on mathematics achievement, including written computation performance (Beaton, et al., 1997; Gonzales, et. al., 2000; Martin, Mullis and Chrostowski, 2004; Stevenson, Chen and Lee, 1993; Stevenson, Lee and Stigler, 1986; Stigler, Lee and Stevenson, 1991). However, the study of Reys and Yang (1998) demonstrated that the work of Taiwanese students skilled in written computation does not necessarily reflect good number sense.

Number sense is a complex process involving many different characteristics of numbers, operations, and their relationships (McIntosh, Reys and Reys, 1992). "Number sense develops over time. The development is best if the focus is consistent, day by day, and occurs frequently within each mathematics lesson." (Thornton and Tucker, 1989, p. 21). Providing a class with well-designed activities and establishing a classroom environment that encourages exploration, discussion, thinking, and reasoning is the best way for students to develop number sense.

Other research projects (Cobb, et al, 1991; Treffers, 1991; Warrington and Kamii, 1998) demonstrated the value of instructional activities that focus on students making sense of their computation rather than on the development of high levels of proficiency with standard written algorithms.

This study

This study illustrates how one teacher integrated activities into his mathematics classes to promote the development of number sense. More specifically it discusses two activities used by one Taiwanese 6th grade teacher and captures some of the dynamics within the classroom environment and their impact.

Setting. The classroom teacher, Mr. Hsu, is an experienced mathematics teacher in Taiwan and has a strong interest in number sense. His sixth grade class has 29 students (16 boys and 13 girls). Mr. Hsu values number sense and realizes that the emphasis of the national curriculum on developing computational proficiency does not necessarily develop the understanding and meaningful learning of numbers that characterizes number sense. The instructional activities illustrated here are selected from a semester-long series of activities that were designed to focus on specific components of number sense (cited earlier) and promote their development. (Copies of all of these activities are available from the Taiwanese author).

Procedures. This study monitored the instructional activities and the resulting dialogues as they occurred in the classroom. The researcher recorded whole-class discussions and communications among students and teacher. The recorded data were transcribed and excerpts used to illustrate the student-teacher and student-student dialogue that occurred in the classes.

The episodes

Introduction. Taiwanese students are driven by computational algorithms. So when the students were asked to estimate without doing an exact computation, this represented a new and challenging experience. Benchmarks, or personal referents that provide a basis for making comparisons or estimates, serve as a powerful tool in working with fractions. Thus recognizing when fractions are near the benchmarks of 0, 1 or one-half greatly facilitates comparisons among fractions or estimation with fractions. Yet benchmarks are not emphasized in the Taiwanese mathematics curriculum. These two episodes show how a teacher posed a question and used it to help students develop number sense.

Episode 1: Benchmarks, estimation, and relative effect of operations.

The question:

Without calculating the exact answer, find the best estimate:

21/32 x 7/16

A. Larger than 1/2 B. Less than 1/2
C. Equal to 1/2 D. Without calculating can't decide

Mr. Hsu posed the question, then asked each small group to decide their answer and be ready to explain their reasons. He monitored the groups by listening to their discussions and checking on their progress. He knew this was a tough question and found no groups could produce a correct answer. Here are excerpts of the small group interaction:

First group:

Student A:	Greater than 1/2.
Mr. Hsu:	Can you tell us your reasons?
Student A:	Half of 32/32 is 16/32, so 21/32 is a lot over half, and the half of 16/16 is 8/16 so 7/16 is a little less than 1/2 (The distance between 1/2 and 7/16 is 1/16, a small number). Therefore, the multiplication is probably over 1/2.
Mr. Hsu:	Does anyone have a question?
Student B:	How do you know the multiplication result of 21/32 and 7/16 is greater than 1/2?
Mr. Hsu:	Good question. Please justify your answer.
Student A:	21/32 is over 16/32, and 7/16 is a little less than 8/16. Therefore, I think the answer is greater than 1/2.

Second group:

Student C:	We think the answer is less than 1/2.
Mr. Hsu:	Why is that?
Student C:	Because half of 32/32 is 16/32, and the difference of 16/32 and 21/32 is 5/32. Half of 16/16 is 8/16, and the difference of 8/16 and 7/16 is 1/16. Therefore, we thought the answer is less than 1/2?
Student D:	I don't understand what you said? How does that help you know the answer is less than 1/2?
Student C:	[Long pause] I can't explain it any other way.

None of the students in either group were able to make a convincing argument. Rather than settle the debate, Mr. Hsu decided to listen to the remaining groups' answers and explanations.

Third group:

Student E:	Less than 1/2.
Mr. Hsu:	Why is that?
Student E:	Because the benchmark of 21/32 is 20/32 and the benchmark of 7/16 is 5/16. 21/32 is greater than 1/2 and 7/16 is

less than 1/2. Hence, the answer is less than 1/2. [Many students were surprised how this group selected the 20/32 as benchmark of 21/32]

Student F: Why did you select 20/32 and 5/16 as benchmarks?

Student E: Because ... [Paused for a long time, but this student (and in fact no one in the group) was unable to give a reasonable explanation.]

Fourth group:

Student G: Less than 1/2.

Mr. Hsu: Please tell us why?

Student G: Because the denominators of the fractions 21/32 and 7/16 are greater than the numerators, so the multiplication is less than 1/2.

Student H: How do you know that if the denominators of the 21/32 and 7/16 are greater than numerators, then the answer is less than 1/2?

Student G: Because a real fraction times a real fraction will make the result smaller. Therefore, the answer should be less than 1/2.

This response seemed connected to a multiplication algorithm rather than any thinking related to number sense. This conclusion was reinforced by the fact that several people nodded in agreement but were unable to add anything that was not related to the multiplication algorithm. Mr. Hsu then listened to the last group's explanations.

Fifth and final group:

Student 1: Less than 1/2.

Mr. Hsu: Please tell us why?

Student 1: Because the product of these two real fractions is impossible to be over 21/32, hence the answer is less than 1/2.

Student J: How did you know a real fraction times a real fraction is less than 21/32?

Mr. Hsu: Good question! How did you know?

Student 1: Because a real fraction times a real fraction will make the result smaller. 21/32 is a real fraction, 7/16 is also a real fraction. Then 21/32 x 7/16 should be less than 21/32.

Student K: How did you know the result of multiplication is less than 1/2?

This group recognized 21/32 and 7/16 as proper fractions [i.e., fractions where the numerator is less than the denominator] and concluded their multiplication should not exceed 21/32 (the multiplicand). Yet no member of the group could explain why the result is less than 1/2. At this time, Mr. Hsu intervened and asked some specific questions that he thought would promote their thinking and further development.

Mr. Hsu:	Some of you said 21/32 is over 1/2 and 7/16 is less than 1/2. You also said the product of two real fractions (21/32 x 7/16) is impossible to be over 21/32 is a good point. If I changed the 21/32 x 7/16 to 7/16 x 21/32, what would you get?
Students:	[Many students answered at the same time] It's amazing! That helped when you turned it around.
Mr. Hsu:	Many of you had good ideas, but you limited your thinking. When you solve problems, you should try to think from different perspectives. Now, who can tell me your answer?
Student 4:	[Many students raise their hands excitedly] 7/16 is less than 1/2, if multiplied by 21/32 a real fraction, then the answer is less than 1/2.
Student 2:	7/16 is less than 1/2, and 21/32 is a real fraction and less than 1, then 7/16 x 1 is equal to 7/16 which is less than 1/2. Therefore, 7/16 x 21/32 (less than 1) must be less than 1/2.

Comments. This discussion resulted in students recognizing an easy computation (7/16 x 1) as being less than one-half and using it as a benchmark to reason that 7/16 times any fraction less than one will result in a product less than one-half. This is the kind of thinking and reasoning with numbers that characterizes good number sense.

This problem is different from the textbooks used in Taiwan, because the textbooks focus on standard written computations. Mr. Hsu knew this would be a big challenge to his students, but he thought it would be good to encourage thinking and discussion. In this lesson, although some groups provided a correct answer, their explanations were unclear. Nevertheless Mr. Hsu was successful in encouraging his students to think and to engage in discussions and thinking that promote number sense.

Episode 2: Understanding basic number meanings and number magnitude.

The question:

> Which fraction 16/17 or 18/19 is larger?

This question was used to jump-start a lesson focusing on comparing two fractions. Mr. Hsu knew students tended to use written methods as is taught in Taiwanese mathematics textbooks when comparing two fractions. He also knew these written methods often fostered misconceptions. After the groups had discussed the question and formulated answers, Mr. Hsu then asked students from different groups to share their results.

One group:

Student A: We think it's very easy. Because we found the common denominators, 16/17 = (16 x 19)/(17 x 19) = 304/323, and 18/19 = (18 x 17)/(19 x 17) = 306/323, so 18/19 is larger.
Mr. Hsu: Well! Can you tell us why?
Student A: [Pause for a few seconds] This is the way we do it in mathematics class. We don't know how to explain it.

While students were comfortable applying standard written algorithms when solving numerical problems, Mr. Hsu encouraged them to solve problems in ways that made sense to them.

Second group:

Student B: I have two cakes with the same size.
Mr. Hsu: That's a good start because you need to have the same size unit when comparing two fractions.
Student B: One cake was cut into 17 pieces of the same size and the other cake was cut into 19 same size pieces. Since each piece of 17 is larger than each piece of 19, we think 16/17 is larger.
Student C: Mr. Hsu, I don't understand what he said?

No one in the group was able to explain why 16/17 was larger with this argument. No student realized the inverse relationship between the number of pieces and the size of the fraction.

Third group:

Student D: We believe they are equal.
Mr. Hsu: Can you tell us your reasons?
Student D: Since 16 is near 17, and 18 is also near 19. If you add I to the numerators of both fractions the result is 1. Therefore, we think they are equal.
Student E: Why did you add 1 to the numerators of both fractions?
Student D: Because [long pause followed by] I don't really know.

A very good question was raised and although many students agreed with the method (adding one to the numerators), no one could explain why this approach made sense.

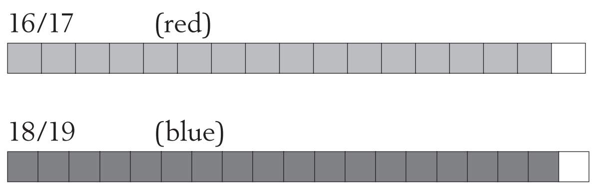

Figure 1.

Fourth group:

Student F: 18/19 is larger.
Mr. Hsu: Tell us why?
Student F: You can see the graph in here [see Figure 1]. The 16/17 means this red area and 18/19 is here. Since this blue area [He pointed to 18/19 in Figure 1] is larger than the red area (16/17). Therefore, 18/19 is larger.

This graph to compare the fractions seemed to make sense to the students in this group.

Fifth group:

Student G: Our answer is 18/19.
Mr. Hsu: Please tell us why?
Student G: We used the pictures in here to explain our answer. You see, because they all left a part, you can see the blanks (He showed the blank area: 1/17 and 1/19 in Figure 2). We only needed to consider 1/17 and 1/19, since 1/19 is smaller than

> 1/17, as we learned from earlier discussions. Therefore, the dark area (18/19) is larger.

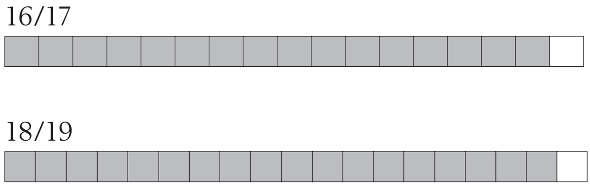

Figure 2.

Comments. The last group used a different graph to illustrate the fractions and to explain their reasoning. They were able to switch back and forth between the fraction (18/19) and its complement (1/19). They recognized that the sum of these fractions was one, and were able to move fluently in exploring the relationships of complements and how this relationship helps solve the problem.

Mr. Hsu asked his students to keep a mathematical diary and record thoughts from the daily lessons. Their reflections provide additional insight into what the students were learning and their level of understanding. An examination of their diaries demonstrates that this lesson was helpful. For example.

One student said:

> "We used to compare fractional size by finding the common denominators and expanding the fractions. After this class activity, I know other ways to compare fractions with different numerators and denominators. "

Another student wrote:

> "After I heard Santy's explanations, I thought her opinions made sense. She used the left parts (1/17 and 1/19) to compare 16/17 and 18/19. Since 1/19 is smaller, then the fraction 18/19 is larger. This is easy and makes sense. This gave me a different way to think. For example, in comparing 1/2 and 2/3. We can consider two cakes with same sizes. One was cut into two pieces, picking up one piece then the remainder is 1/2. The other cake was cut into three pieces, picking two piece then 1/3 is left. Since 1/3 is less than 1/2, therefore, 2/3 is larger. I used to compare fractions by finding the common denominators, now I know other ways that I can use and do it in my head."

Discussion

These episodes illustrate different ways of student thinking. In the process of promoting and developing number sense, the teacher demonstrated the following tenets associated with quality teaching:

The teacher p1ays a key role in creating a good learning environment that encourages exploration, discussion, communication, and reasoning. Mr. Hsu knew his role was not only to pose challenging questions and encourage discussions, but to provide opportunities for students to communicate and share their thinking and reasons with classmates. Rather than rely on standard written computations, Mr. Hsu encouraged his students to solve the problem with different strategies along with meaningful explanations. His actions reflect the statement in the *PSSM Teaching Principle* "Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well" (NCTM, 2000, p. 16).

The teacher needs to know his role in the class. Mr. Hsu avoided the tendency to 'work' the problems for the students or 'tell' them the answers. He understood his role was to pose challenging and worthwhile mathematical problems and then lead students to learn mathematics with understanding. Even though correct explanations were not always forthcoming, he patiently listened to their answers, and then asked probing questions that challenged them to explain their thinking.

The teacher chooses what to teach and how to teach it. Textbooks provide structure for the mathematics content. Yet the teacher decides what important mathematical ideas need to be emphasized. Mr. Hsu showed strong self-confidence by focusing on activities that were different from the national mathematics curriculum in Taiwan. He knew his students could find exact answers by paper-and-pencil computation, yet he realized that they were not necessarily developing number sense. He knew "conceptual understanding is an important component of proficiency" and "learning with understanding is essential" (NCTM, 2000, p. 20-2 1).

The teacher needs to have a strong background and deep understanding in mathematical knowledge. Mr. Hsu not only knew what is a worthwhile mathematical task for students to learn, but also knew how to lead students to meaningful learning and understanding. He started with worthwhile mathematical tasks, then challenged students to answer the questions and defend their thinking. He reflected the trait that teachers should "know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks" (NCTM, 2000, p. 17).

The message from this study is that the teacher determines what is important and focuses on its development. Mr. Hsu is a mathematics teacher in Taiwan and recognizes that computational proficiency is an expectation for his students. However, he knows high levels of proficiency in written computation is not necessarily accompanied by understanding of the written procedures and the consequences of using them. Therefore this teacher has made a conscious effort to challenge his students to not only develop proficiency with standard written algorithms but also to explore computational alternatives and explain their thinking along the way.

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Sammanfattning

Denna studie ger en bild av hur en taiwanesisk lärare undersökte och stärkte sina elevers taluppfattning. Eleverna gick i årskurs 6 och man sysslade med bråk. Resultaten illustrerar elevernas benägenhet att förlita sig på nedskrivna algoritmer och avslöjar några missuppfattningar som kan förekomma bland elever som är skickliga på sådana. Studien presenterar ett försök att i klassens arbete integrera aktiviteter som stöder taluppfattningen genom att uppmuntra till utforskande, diskussion, tänkande och resonemang.