Open ended problem solving in geometry revisited

Hans Erik Borgersen

This article is a successor to Borgersen's (1994) study of a project on cooperative problem solving in small groups, organized as part of a geometry course at the college level. It focuses on the effects of a gradual change of the project. This report studies the transition of the project from 1994 to 1996 (Borgersen 1995, 1997, 1999) as the course changed from being a medium size course for about thirty students from the same study program to become a large course for nearly one hundred students from three different study programs.

1 Introduction

Two major challenges in mathematics teaching at all levels appear to be 1) to have students experience a true picture of doing mathematics and 2) to create learning environments of acceptance, safety and trust. Borgersen's (1994) study of a project on cooperative open ended problem solving in geometry, verifies one way of meeting these challenges in small and medium size classes at the college level.

> Open ended problem solving makes it possible for students to experience the whole process of doing mathematics. Elementary geometry is a rich resource for finding problems to do so. It is possible to create learning environments where students cooperate and give each other support to go through the process together. It is possible to do this within an ordinary first year college geometry course.
> (Borgersen 1994, p. 32)

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The focus of the 1994 study was why the project was run, how it was organized, what the results were, and how it developed from 1986 to 1993. These years the classes consisted of about 30 students and had mainly mathematics students in their second (or fourth) semester of university studies, and the students came from the same study program (Mathematics and Sciences).

The project reported on in the 1994 study is still part of our current geometry course. But the course has changed to become a large course of students from different study programs. This article studies the transition of the project from 1994 to 1996 (Borgersen 1995, 1997, 1999). We do not repeat our arguments for why the project was run, and only briefly describe how it was organized. For a more detailed discussion of theory and method used, we refer to Borgersen (1994).

We raise the following questions:

- Is it possible for one teacher to manage the project with a large class of students from different study programs? If so, can all the qualities of the project be maintained? Particularly, we question how to maintain the processing in plenum and the students' presentations of their solution in class.
- Which problems were given? How did the problems work?
- How will the students' process writing, proof writing, and writing on generalization and evaluation develop?
- The students form groups on their own, and we investigate how this was done related to study program and gender.
- Before the project started in 1995 and 1996, the students were given a short introduction to the computer program Cabri. Based on this introduction, are the group reports typed and the figures drawn by Cabri or similar computer programs?

Finally, what do the groups' evaluations tell the researcher about how the students have experienced the project in general, the cooperation in small groups, the given problems, and the process of problem solving? What mathematics have they learned from it? And what do the evaluations say about the metacognitive and affective sides of the students' problem solving process?

2 Background

The project studied in this article is part of a geometry course, and the following section gives a short description of the content, theory, method and context of the study.

2.1 Geometry

The geometry course has three main parts:

1 The history of geometry with emphasis on Euclid's "Elements", classical Euclidean geometry.

2 Plane real projective geometry.

3 Modern Euclidean geometry (transformation geometry) with applications on symmetry.

After 2–3 weeks of recalling and reviewing classical geometry, the students start their own projects in small groups, and they work without assistance in parallel with the ordinary teaching in class.

We call the activity, starting with an elementary problem in geometry and going through the whole process of doing mathematics, *open ended problem solving in geometry*.

The geometry problems we have used in the projects are collected from different sources (journals, books, and generous colleagues) over the years. The groupwork is not graded, but the groups are given written feedback on their reports. During the written exam at the end of the course, the students are tested individually, and one part of the exam is on problems similar to the type given in the project.

2.2 Theory

The main philosophy of the project described in Borgersen (1994) has been a basis for several studies focusing on problem solving and cooperation in small groups at different grade levels (Bjuland 1997, 1998, 1999, 2002; Mydland 1998; Eriksen 1999; Damsgaard 1999; Carlsen 2002; Wrånes 2003). The method used in these studies is based mainly on micro-analysis of the dialogue in small groups. This dialogical approach was introduced and developed by Cestari (1997, 1998, 2000). In a multigrade school, Wrånes (2003) studied problem solving strategies used in a sixth and seventh grade class solving problems in geometry. Mydland (1998) studied students' responsibility for their own learning by focusing on problem solving in geometry in small groups in a ninth grade class. Eriksen (1999) and Damsgaard (1999) studied open problems and communication in small groups, in a freshmen senior high school class. Carlsen (2002) studied students' sharing of responsibility and development of students' understanding of a mathematical concept (scalar product) through the dialogue in a small group of high ability final year high school students. Bjuland (1997, 1998, 1999, 2002) studied problem solving processes in geometry as part of teacher education when pre-service teacher students cooperated in small groups. The study was done in the first semester at a teacher training college (for elementary teachers). In all these studies the main focus was on few selected groups, 1 – 3 groups, each of 4 – 5 students.

These projects, and the project studied in this article, are all based on the philosophy expressed in American research literature on problem solving (Polya 1945; Stanic and Kilpatrick 1989; Silver 1985; Schoenfeld 1985, 1992; Lester 1994), and in parts of the American research literature on cooperation in small groups (Johnson & Johnson1990; Davidson 1990; Weissglass 1993). A discussion of this research is given in Borgersen (1994), in which our problem solving model is related to Polya's four-stage model through an example (Best place on Stadium). Our use of small group work is developed from our everyday teaching of mathematics, which is a blend of lecturing and problem solving in small groups (Dahl 1995) and which is one of several examples of large-scale implementations of cooperative learning in undergraduate mathematics (Davidson et al. 2001).

2.3 Method

The students work in their project in groups of about 5, which they form on their own. In the first meeting they are supposed to make a schedule for at least one meeting a week. The project lasts for 4 weeks. The teacher of the course has the role as instructor of the project as well as researcher.

The students are supposed to write group reports. They are expected to write down the schedule for their meetings and their way of organizing the group work (*log*). They are expected to write about their problem solving process (*process writing*) and to write down and provide an argument for their solutions (*proof writing*). They are also expected to evaluate their own group work and the project in general (*self-evaluation*).

In preparing for the project, the students are introduced to Borgersen's (1994) problem solving model and given recommendation for working together in small groups similar to Johnson & Johnson's (1990) basic elements for cooperative learning. They are also given a short introduction to the computer program Cabri as a tool for geometry work. But the students make their own decision on how they will organize their meetings, how they will cooperate, and how they will present their final reports.

In this study we use the same macro-analysis as described in Borgersen (1994), which means that the analysis is based on what the students have written in their group reports. When we refer to what the students say, it is to what the students write about what they said or meant or experienced as they wrote the report collectively. Some of our comments are also given on the basis of our general impression by reading the reports. So, our analysis in this article is not based on transcripts of the group discourse (micro-analysis). It is based on what the students have written about their problem solving process, their solutions, and their evaluation of the group work.

In the following sections we present the problems given in 1994, 1995 and 1996. Comments are made on whether the problems were appropriate and on the final group reports. We give short characteristics by placing a code on each of the reports, to indicate if it includes a log (L), analysis and drawings (A), process writing (P), proof writing (C), generalization or asking new questions (G), and self evaluation (E). Small written letters have the same meaning as capital letters, and mean ”a little of”. A capital letter, such as P, is also interpreted as ”the report contains satisfactory process writing”.

3 The 1994 project

Compared to previous projects the number of students increased substantially in 1994 to a population of 51 students (Borgersen 1995). They were still mainly students in the Mathematics and Science program, but some students in the Teacher Education program had also been added. Our presentation here makes it easy to compare with the 1993 project (Borgersen 1994).

3.1 The problems

Fifty-one students in 13 groups cooperated on Problem 1 or Problem 2 and at least one of the remaining problems.

Problem 1

Given a triangle ΔABC and a point P ($P \neq A,B,C$). The perpendiculars on l(P,A) through A, on l(P,B) through B, and on l(P,C) through C make a triangle $\Delta A'B'C'$ such that each triple of points A, B', C'; A', B, C'; A', B', C are collinear. We call $\Delta A'B'C'$ the Napoleonic of ΔABC with respect to P.

a Prove that if P is the orthocenter of ΔABC, then P is the circumcenter of $\Delta A'B'C'$.

b Investigate the Napoleonic when i) P is the circumcenter of ΔABC, and ii) P is the incenter of ΔABC. Characterize P in relation to the Napoleonic in each case.

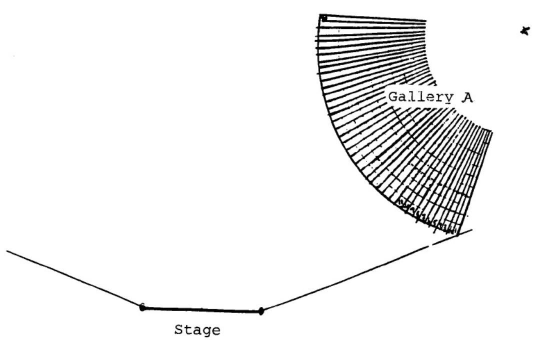
c Investigate the process of repeated constructions of Napoleonics with respect to P, when P is one of the points above (orthocenter, circumcenter, incenter).

d Generalize and formulate new problems.

Problem 2
Given a tetrahedron. Investigate the triples of angles subtended by the edges of a face at the point of contact of this face with the insphere.

Problem 3
Construct all circles tangent to three elements when the elements are points or lines (i.e. three lines, two lines and one point, one line and two points, three points).

Problem 4
The figure below is a sketch of the stage and a gallery in a theater. The benches on the gallery have the form of arcs of concentric circles. You have ticket to a front seat on the gallery. Where do you sit to see the stage under the largest angle? Give a mathematical formulation of the problem. Describe and construct a solution.



Problem 5
Construct a triangle ΔABC given c, $\angle C$, and i) the altitude from A, h_A; ii) a + b; and iii) the inradius, r.

3.2 The group reports and results from the 1994 project

Table 1. *The group reports and results from the 1994 project*

Group	# of students	Problem chosen					# of problems	Characteristics of the final group reports[1]					
		1	2	3	4	5							
1[2]	3	x			x		2	L	A	P	C	g	E
2	5	x		x			2	L	A	P	C	G	E
3	5		x		x		2	L	A	P			E
4	5	x			x		2	1	A	P	c	g	E
5	6	x				x	2	L	A	P	C	G	E
6	5	x				x	2	L	A	P	C	g	E
7	6	x				x	2	L	A	P	C	G	E
8	6	x		x			2	L	A	P	C	G	E
9	6	x		x			2	L	A	P	C	G	E
10[3]	1	x			x		2		A	p	c		
11[3]	1	x		x			2	L	A	p	C	G	E
12[3]	1	x				x	2	1	A	P	C	G	E
13[3]	1	x				x	2	1	A	P	C	G	E
Σ	51	12	1	4	4	5	26						

1 log (L), analysis and drawings (A), process writing (P), proof writing (C), generalization or asking new questions (G), evaluation of the group work (E). Small written letters have the same meaning with the appendix ”a little of”.

2 Group 1 had only three members because they started the project out of order.

3 Four students worked on their own due to practical reasons. They were organized in groups 10, 11, 12 and 13. The students in groups 12 and 13 were students in a distance education program.

Comments

All groups except one chose Problem 1. It worked very well for open ended problem solving, as did Problems 3, 4 and 5. Problem 2 did not work well at all. The students didn’t get started even though they built a tetrahedron with an insphere. Some groups didn’t choose Problem 2 because ”It is difficult to draw figures in a 3-dimensional problem” or ”The formulation of the problem is very short and without any limits”. It would be interesting to give Problem 2 as a choice again making a more detailed formulation like in Problem 1. No group or individual handed in more problems than required.

All groups proved Problem 1a and formulated correct hypotheses in Problem 1b. Only five groups proved at least one of their hypotheses. In Problem 1c several groups read "repeated construction" as constructing the Napoleonic given different types of triangles. Only four groups discovered the beautiful periodicity of P being orthocenter – circumcenter – incenter – orthocenter respectively, by repeated construction of Napoleonics. The main generalizations in Problem 1d were of the following type: i) Vary the position of P, ii) Formulate similar problems for other polygons, iii) Question a periodicity for an arbitrary point P by the process of repeated construction of Napoleonics, iv) Formulate similar problems in 3-space.

Four and five groups respectively proved Problem 3 and Problem 5 in different ways. Four groups had chosen Problem 4 and formulated correct hypotheses without proof. By the processing at the end of the project one individual came up with a proof of Problem 4. More details are given in section 3.5.

The progress on process writing and evaluation reported in 1993 (Borgersen 1994) was stabilized or improved a little in 1994. The proof writing showed a small decline. However, there was remarkable progress in the students' attempts to generalize or formulate new problems. Half the groups typed their manuscript, and two groups out of thirteen made drawings by use of a computer program.

3.3 Processing

Once a week throughout the project period we did *processing in plenum*; i.e., the groups shared their experiences and evaluated their work in full class. (A representative from each group gave a short report on the group's work that week, and a discussion followed.) The processing developed very similarly to that in 1993 (Borgersen 1994), but by the third week the students were more outspoken about their frustration in the creative process. One group that had cooperated closely had run out of ideas. In another group that worked together and individually the members had not been able to communicate their individual work back to the group. A third group told about their ongoing struggle to find a proof. However, finally two different proofs of the same hypothesis appeared in this group.

3.4 Evaluation – a summary

In this section we make a short summary of the project evaluations given by the groups in the final reports. The students used words as "instructive",

”challenging”, ”useful” and ”fun” to describe their experiences when working on the project in small groups. The positive experiences dominate the negative ones. As one group said, ”We have had good and bad moments, but the total outcome of our work is positive.” Several groups emphasized the importance of attendance and active participation. Absentees created frustration in the groups. The students learned that in problem solving ”it's useful to put a problem aside for a while and to try to look at it with new ideas and from another angle next time”, or that ”problem solving is a maturing process”. The students learned that in geometry ”there are several roads leading to Rome”. The students expressed their feelings, for example: ”We enjoyed the process of proving and finding holes in each other's arguments.” ”We are proud and satisfied with our work.” ”We worked very well together, but we were frustrated by two members missing.” ”After lots of desperate attempts our joy was enormous when we finally saw a light in the darkness.” And one group underscored the importance of good humor: ”We had lots of laughter to handle our frustrations on Problem 4.” Several groups point to proving as the most difficult part of the open ended problem solving process.

3.5 Final presentation in plenum

The project was finished by two sessions in plenum where the instructor commented on the group reports and the group evaluation in general terms, and the students presented their solutions to some of the problems. (The groups appointed members to do this after having been asked in advance to prepare such a presentation.) As an example we will take a closer look at some groups' solutions to Problem 3 and Problem 4.

As mentioned above, four groups had chosen Problem 4 and formulated correct hypotheses without a complete proof; i.e., they were not able to construct a circle through two given points tangent to a given circle. They had tried hard and got frustrated. Group 5 said: ”We thought this problem would be easy to solve, and we quickly found a hypothesis. (...) It has been fun to work on this problem even if the result is poor. But most of us lost courage on it at the end. We were simply sick and tired of the problem. The problem is unsolved and that's that for the near future.” At the end of the students' presentation in plenum one student from group 5 eagerly raised his hand and told he had just found a construction of the circle C in Problem 4! In this section we quote his solution and some of the students' presentations that led to this breakthrough.

Solutions of Problem 3

Construct all circles tangent to three elements when the elements are points or lines.

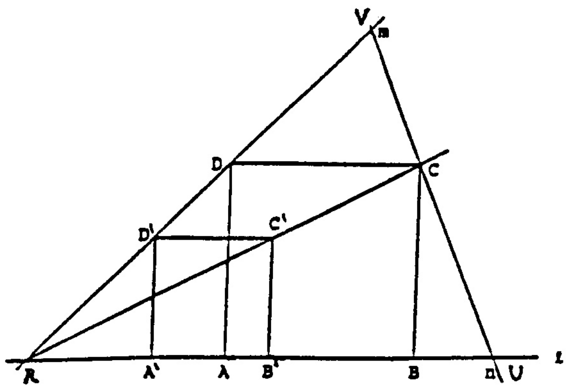
The students presented the dual cases, three elements being

i) two lines and one point, and

ii) two points and one line.

Problem 3 i) (with two non parallel lines) solved by group 8:

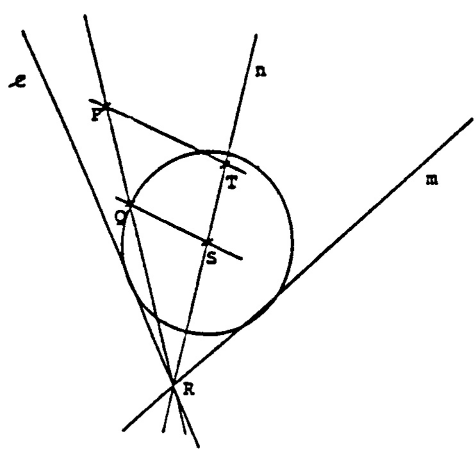
Given two non parallel lines l and m and a point P. We struggled a lot with this problem. The following hint helped us to find a solution: "How to inscribe a square in a triangle having one sideline in common?"



Construct a square A'B'C'D' as shown in the figure. Then the line RC' intersects UV in C, and ABCD is a solution.

Our solution of the problem is as follows:

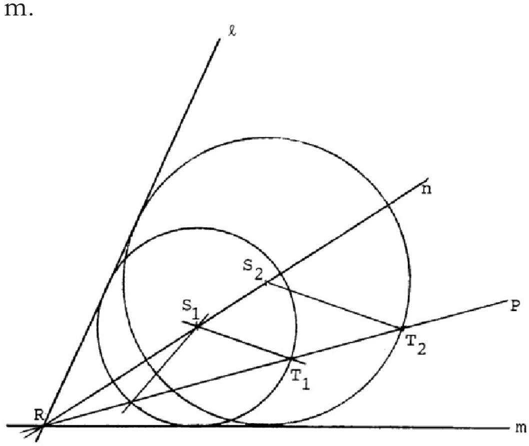
Construct the angle bisector n (see figure). Construct an arbitrary circle, C', tangent to l and m with its center, S, on n. The line RP intersects C' in a point Q. The parallel to QS through P intersects n in T, which is the center of the unknown circle C.



Comments

The group was not explicit in explaining that the circle C is the image of the circle C' by a homothety with center R (as ABCD is the image of A'B'C'D' by a homothety with center R in the hint above). The group missed a second solution because they only used one of the two intersecting points between the line RP and the circle C'. Group 9 used the same method giving a complete solution. They also got some help from the instructor. Both groups used a similar method to solve Problem 3 ii). Group 2 gave a similar proof of Problem 3 i) and 3 ii). Their presentation became important for the solution of Problem 4, as described below.

Problem 3 i) solved by group 2:
Let C_1 and C_2 be two arbitrary circles tangent to l and m. Their centers S_1 and S_2 are on the angle bisector n. The line RP intersects C_1 and C_2 in two points T_1 and T_2 such that $\angle RT_1S_1 = \angle RT_2S_2$. Therefore the point S on n such that $\angle RPS = \angle RT_1S_1$ is the center of a circle through P tangent to l and m.



Comments
The angle equality was not proved, but the two solution circles were found.

Solutions of Problem 4
"Where to sit on a front gallery seat to see the stage under the largest angle?"

In their reports, the groups gave similar descriptions and arguments, as follows:

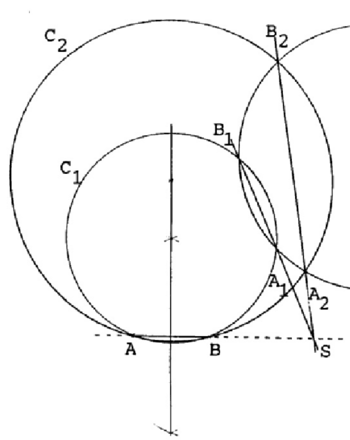
Hypothesis: The point of tangency between the front gallery circle and a circle through the endpoints of the stage (A and B) gives the place where you see the stage under the largest angle.

Proof: Let C be the circle through A and B tangent to the front gallery circle. A point inside C gives a larger angle, but is outside the gallery.

A point outside C and on the gallery gives a smaller angle. This proves that the point of tangency is a unique solution. The following construction of the circle C was discovered by an individual member of group 5 at the end of the final presentation in plenum.

A solution by analogy: The problem is to construct circle C, through two given points A and B, that is tangent to a given circle G (gallery circle).

Choose two circles C_1 and C_2 through A and B intersecting G in A_1, B_1 and A_2, B_2 respectively. The lines A_1B_1 and A_2B_2 intersect in a point S. Construct the tangent (s) from S to G, which touches G in T (on the gallery side). Then C is the circle through A, B and T.



Comments

The construction, which is correct, was given without a proof. We asked the student to explain how he got the idea. He said it was the presentation by group 2 of Problem 3 i) where they had chosen two auxiliary circles C_1 and C_2. It was not the method they used but the move to choose an auxiliary quantity that gave him the idea. Together we found an argument which convinced us that the construction is correct: The point S is a point of common power with respect to the circle G and all circles through A and B. S is on the line AB and could also be found by intersecting the line AB and, for example, the line A_1B_1. The common power is $|SA|\cdot|SB|=|ST|^2$.

4 The 1995 and 1996 projects

In 1995 and 1996 the number of students nearly doubled from that in 1994. The 1995 project had 97 students in 22 groups (Borgersen 1997), and the 1996 project had 80 students in 18 groups (Borgersen 1999). Mainly teacher students being recommended to take the course caused the increase from 1994. The projects will be presented simultaneously, organized in the same manner as for the 1994 project above. In addition there will be sections on students and grouping related to their study program, gender, and computer usage. No section on the students' presentation of solutions is included.

4.1 The students and the groups

Let us have a closer look at the number of students and groups in the 1995 and 1996 projects with respect to study program and gender. We have used the following abbreviations for the study programs:

I Computer Science and Mathematics Methods
3-year program (the geometry course is usually taken in the 4^{th} semester).

M Mathematics and Sciences
1–4 year program (the geometry course is usually taken in the 2^{nd} or 4^{th} semester).

T Teacher Education
1-year program (the geometry course is taken in the 2^{nd} semester).
4-year program (the geometry course is taken in the 6^{th} semester).

Students in these study programs are called I-students, M-students, and T-students respectively. We denote groups with only I-students, M-students, or T-students as I-groups, M-groups, or T-groups, respectively. MIX-groups are groups with students from at least two different study programs. We denote groups with only female students as female groups, and groups with only male students as male groups. Groups with both female and male students are denoted mix groups (groups of mixed gender). Groups with more than one individual are called ordinary groups.

Table 2. *The number of female/male students in the different study programs in the 1995 and 1996 projects*

	I		M		T		Total	
	1995	1996	1995	1996	1995	1996	1995	1996
female	1	4	15	11	27	15	43	30
male	9	7	32	23	13	20	54	50
Total	10	11	47	34	40	35	97	80

Comparing the 1995 and 1996 projects, the number of students was somewhat reduced. The distribution of students in the three study programs was about the same. The M-students and the T-students made up most of the students in the projects with a majority of M-students in 1995 and a slight majority of T-students in 1996. Both years there were more male students than female students. In 1996 the male students had become a majority in all study programs, also in Teacher Education.

Table 3. *The number of ordinary female, male, and mix groups with respect to the I-, M-, T-, and MIX-groups*

	I		M		T		MIX		Total	
	1995	1996	1995	1996	1995	1996	1995	1996	1995	1996
Female	0	0	1	0	2	0	1	0	4	0
Male	0	0	2	1	0	0	2	2	4	3
Mix	0	0	2	2	4	4	3	6	9	12
Total	0	0	5	3	6	4	6	8	17	15

Comparing the 1995 and 1996 projects, the distribution of ordinary groups has changed towards more MIX-groups and groups of mixed gender. In 1995 a clear majority of the groups only had students from the same study program, and a slight majority of the groups were of mixed gender. In 1996 a slight majority of the groups had students from different study programs, and a clear majority of the groups were of mixed gender.

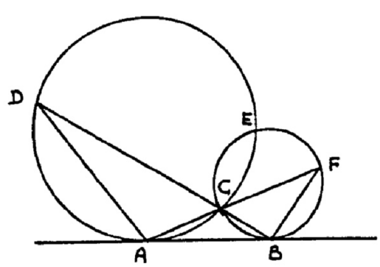
4.2 The 1995 project

The problems

Ninety-seven students in 22 groups cooperated on Problem 1 or Problem 2 and at least one of the remaining problems.

Problem 1

In the given figure two circles intersect in C and E. The circles have a common tangent, which touches the circles in A and B. The lines BC and AC intersect the circles in D and F respectively.



a Assume that the triangle ΔABC is given such that $\angle CAB = 30°$ and $\angle CBA = 45°$.Construct the points D, E, F. Measure the angles in ΔACD and ΔCBF. Are D,E,F collinear? Investigate this question for other angle measures, such as: (15, 30), (15, 45), (30, 60).

b Assume that $\angle CAB = u°$ and $\angle CBA = v°$. Try to find a necessary and sufficient condition for D, E, F to be collinear.

c Given two circles with radii R and r. Is it possible to place the circles so that in a figure like the one above, the points D, E, F are collinear?

Problem 2

Given an angle with vertex A and legs m and n.

Let P be an arbitrary point interior to the angle. A line l through P meets m in B and n in C. Try to find a position for l that makes the area of ΔABC minimal.

Problem 3

This winter we had some nice clear days with an extremely sharp horizon. From the sixth floor in the main building of the university we have a marvelous view to a lighthouse called Grønningen. About one half of the lighthouse is visible above the horizon. Inspired by the view from his office my colleague Knut Øyma had an idea to calculate the diameter of the earth. He used the following data.

The position of Knut Øyma's binoculars was 32.3 meters above sea level. The top of Grønningen lighthouse is 18.7 meters above sea level. The distance from the office to Grønningen is 9820 meters. With these data Knut Øyma calculated the diameter of the earth with an error of $\pm 10.5\,\%$. How did he make this calculation, and what was his reasoning? Try to find an estimate for the diameter of the earth, given the data above. Discuss the model. (The circumference of the earth is 40 070 km.)

Problem 4

Given a square $\square ABCD$ and an inner point P.

a Prove that if ΔPCD is equilateral, then $\angle ABP = 15°$.

b Is the converse true?

Problem 5

Construct a triangle ΔABC given the length of the side opposite to A, a, the circumradius, R, and i) the altitude from B, h_B; ii) the inradius, r .

Table 4. *The group reports and results from the 1995 project*

group	# of students	problem chosen					# of problems	characteristics of the final group reports				
		1	2	3	4	5						
1	5	x			x	x	3	L	A	P		E
2	4		x		x		2	L	A	p	g	E
3	7	x		x			2	L	A	P		E
4	5	x			x		2	L	A	P		E
5	5	x		x	x		3	L	A	p		E
6	5	x			x		2	L	A	p		E
7	6	x			x		2	L	A	p		E
8	5		x		x	x	3	L	A	P		E
9	6		x		x		2	L	A	P	G	E
10	5		x	x	x		3	L	A	P	g	E
11	6		x		x		2	L	A	P	g	E
12	5	x			x	x	3	L	A	P		E
13	5	x			x		2	L	A	P	G	E
14	6	x		x		x	3	L	A	P		E
15	6	x			x	x	3	L	A	P	g	E
16	5	x			x	x	3	L	A	p		E
17	6		x		x		2	L	A	P		E
18[2]	1		x		x		2	1	A	P	G	E
19[2]	1	x			x	x	3	1	A	P	G	E
20[2]	1	x			x		2	L	A	P		E
21[2]	1	x			x		2	L	A	P		E
22[2]	1	x			x	x	3		A	p	g	E
Σ	97	15	7	4	20	8	54					

1 log (L), analysis and drawings (A), process writing (P), proof writing (C), generalization or asking new questions (G), evaluation of the group work (E). Small written letters have the same meaning with the appendix ”a little of”.

2 Five students worked on their own due to practical reasons. They were organized in groups 18, 19, 20, 21 and 22. The students in groups 18, 19, 21 and 22 were students in a distance education program.

Comments

All the problems worked well for open ended problem solving. About half the groups and individuals handed in more problems than required. In 1994 no group or individual handed in more problems than required.

Fifteen groups chose Problem 1. All groups made nice investigations, tried out different hypotheses, and formulated correct hypotheses. But only one third proved their hypothesis satisfactorily. Some groups clearly had little training in mathematics. Their mathematical language was limited, and they had difficulties with basic logic and with proof. On the other hand, the process writing made it easier to understand their ideas and their way of thinking. Some of these groups wrote very insightful comments about their problem solving process. About half the groups argued for a positive answer to the question in Problem 1c. One group gave a trial and error argument using Cabri, and four groups used a continuity argument. No group used a constructive argument. Only one group formulated new problems related to Problem 1.

All groups formulated correct hypotheses in Problem 2, which was chosen by 7 groups. All groups except one had interesting process writing, and all groups except one provided proofs to support their work. Only one group formulated a new problem (Find the position of l that makes the perimeter of ΔABC minimal).

Among the optional problems most of the groups (15) chose Problem 4. All groups provided proofs but very little process writing. Two groups reformulated 4b so that the converse implication was correct. Two groups formulated new problems. In Problem 3 all the groups gave correct proofs. In Problem 5 all the groups solved 5i, and only one group solved 5ii.

The process writing and the evaluation were stabilized at the 1994 level. The decline in proof writing continued. All groups gave evidence of some proof writing, but only a little more than half of the groups had satisfactory proof writing (i.e., capital C in the table). The progress on generalization reported in 1994 was not maintained. A little less than half of the groups tried to generalize or formulate new problems, of which less than half were satisfactorily written. A little less than two thirds of the groups had typed their reports and a little less than half of the groups had used a computer program to construct their drawings, two thirds of those groups by use of Cabri.

4.3 The 1996 project

The problems

Eighty students in 18 groups cooperated on Problem 1 or Problem 2 and Problem 3.

Problem 1

a Choose a point P in the plane. Construct an equilateral triangle so that P is an interior point and so that the distance from P to the sides of the triangle is 3, 5, and 7 cm respectively.

b Choose an arbitrary equilateral triangle ΔABC. Let P be an interior point. Let d_a, d_b, d_c be the distances from P to the sides of the triangle. (d_a is the distance from P to the side opposite of A, etc.)

   i) Choose different positions for P and measure d_a, d_b, d_c each time. Make a table and look for a pattern. Try to formulate a conjecture.

   ii) Try to prove the conjecture in a).

   iii) Try to generalize the problem above.

Problem 2

A square sheet of paper with vertices A, B, C, D is folded by placing the vertex D at a point D' on BC. By this folding A goes to A'. A'D' intersects AB in E.

a Cut a square paper with the side equal to the shortest side of an A4 sheet of paper. Choose different foldings (placements of D) and measure the lengths of the sides in $\Delta EBD'$ each time. Make a table and look for a pattern. Try to formulate a conjecture.

b Try to prove the conjecture in a).

c Prove that $|A'E|$ is the inradius of $\Delta EBD'$.

Problem 3

Given a right-angled triangle ΔABC ($\angle B = 90°$) and a semicircle Ω with center O and diameter AQ, where Q is a point on AB. The points P ($P \neq A$) and R on Ω are given so that P is on AC and OR is perpendicular to AB.

a Find $\angle APR$ and $\angle QPC$

b Prove that $\angle BQC = \angle BPC$

c Prove that if B, P, R are collinear (are points on a line), then BC and BQ are of equal length.

d Formulate the converse of the theorem in c). Is this formulation a theorem?

Table 5. *The group reports and results from the 1996 project*

group	# of students	problem chosen			# of problems	characteristics of the final group reports[1]					
		1	2	3							
1	5		x	x	2	L	A	P	C		E
2	5	x		x	2	L	A	p	C	G	E
3	5	x		x	2	L	A	P	C	G	E
4	3	x		x	2	L	A	p	C	g	E
5	6	x		x	2	L	A	P	C		E
6	6		x	x	2	L	A	P	c		E
7	4	x		x	2	L	A		C		e
8	5	x		x	2	L	A	p	C		E
9	5	x		x	2	L	A	P	C	G	E
10	7	x		x	2	1	A	p	C	G	e
11	5	x		x	2	L	A	P	C		E
12	6	x		x	2	L	A	p	C	g	E
13	5	x		x	2	L	A	P	C	g	E
14	5		x	x	2	L	A	P	C	g	E
15	5	x		x	2	L	A	p	C	g	E
16[2]	1	x		x	2	L	a	P	c	G	E
17[2]	1	x		x	2		A		C	G	
18[2]	1	x		x	2	1	A	P	C	G	E
Σ	80	15	3	18	36						

1 log (L), analysis and drawings (A), process writing (P), proof writing (C), generalization or asking new questions (G), evaluation of the group work (E). Small written letters have the same meaning with the appendix ”a little of”.

2 Three students worked on their own due to practical reasons. They were organized in groups 16, 17 and 18. The students in groups 17 and 18 were students in a distance education program.

Comments

All the problems worked well for open ended problem solving, even though Problem 2 appeared to be more difficult than the other problems. No group or individual handed in more problems than required, which was also the case in 1994, but not in 1995.

Problems 1 and 2 seemed to have inspired the groups to investigate, to try out different hypotheses, and finally to formulate correct hypotheses. All the groups did so.

Thirteen groups proved their hypotheses in Problem 1. They gave nice proofs, and two groups even gave three different proofs. Two groups proved their hypotheses for a special position of the point P (as the orthocenter). For one of these groups the argument (based on area) would have worked for an arbitrary point as well. Ten groups used an area argument. Totally, the groups produced five different proofs for their hypotheses in Problem 1.

Of the three groups that chose Problem 2, one group gave a nice trigonometric proof for their hypothesis in Problem 2 a, and one group almost completed a proof (which can be observed by their very good process writing). The third group did not manage to prove their hypothesis. They made nice drawings and tried to verify Problem 2 c by use of these. The two first groups both proved Problem 2 c. Problem 2 had no specific question on generalization. Only the second group above commented on generalization by stating that the problem is valid for an arbitrary square (and not only for the square sheet of paper given). This group had an interesting formulation of a new problem. As I interpret it, they ask, ”Given an arbitrary right-angled triangle, is it always possible to find a square so that by folding it as in Problem 2 the given triangle is obtained?” They suppose this is not always true, and continue to ask, ”What characterizes such triangles?”

In Problem 1 the students are asked to try generalizing. Three groups that proved their hypotheses by an area argument managed to generalize the problem to a 3-dimensional problem, which they proved, by generalizing the ”area proof” to a ”volume proof”. Three groups generalized to regular polygons in the plane. One group extended the problem by letting P be an arbitrary point in the plane (not necessarily an inner point of the triangle), and claimed that it is correct when P is a vertex. Four groups stated that it is not possible to generalize the problem to arbitrary triangles. Four groups did not try to generalize Problem 1.

Most of the groups gave nice proofs for Problem 3. Two groups based their proofs on the circumcircle of a quadrilateral without proving that it is cyclic. One group tried verifying by use of nice drawings. Problem 3 d created problems for many groups. They did not quite understand the problem.

Although it was less evident than in 1995, a couple of groups clearly had had little training in mathematics. Even if they did not manage to solve the problems, it seems that they had made progress in their doing and understanding of mathematics. Compared with the 1995 project, the main impression is that there had been considerable progress in proof writing, some progress in generalization, and a decline in process writing. The groups that did unsatisfactory ("a little of" or no) process writing did satisfactory proof writing. The two groups that did unsatisfactory proof writing did satisfactory process writing. A little less than half of the groups had typed their reports and a little less than one quarter of the groups had used a computer program to construct their figures, all by use of Cabri. Compared with the results in 1995, there was a clear decline in the number of groups that typed their reports or drew their figures by use of a computer program.

4.4 Processing and final presentation in plenum

In the 1995 project we did weekly processing in plenum, but the processing developed differently than that in 1994. The large number of students (twice as many as in 1994) made it difficult to handle the processing, which degenerated into one-way information and no discussion. Still, the processing in plenum reminded the students to do processing in the groups. An instructor giving feedback based on short, written weekly evaluations from the groups might have done this more effectively. Some sort of processing in plenum seems to be important to maintain, but the way this is done depends on the number of students in the class. In the project evaluation, one group remarked that it was comforting to hear that other groups also had difficulties in their problem solving, but that it was less interesting to hear about the way they organized their group work. Therefore, the time spent on the processing in plenum could have been shorter. Another group wrote that processing in plenum had both positive and negative effects, and pointed to competition between groups as a negative factor. With an almost equally large class in 1996, the processing in plenum was left out, but the students were reminded to do processing in the groups at the end of each group meeting. Our experience so far is that processing in plenum works in classes with as many as 50 students, but it does not work in classes above that number.

As in 1994, the projects in 1995 and 1996 were finished by two sessions in plenum where the instructor commented on the group reports in general terms, and the students presented the solutions to some of the problems. The result of the students' presentations in plenum seems to be related to the number of students in class. In a class of 80–100 students, it is more difficult to keep their interest, and it is more difficult to stimulate reflection than it is in a class of 30 – 50. Both in 1993 (Borgersen 1994) and in 1994 the students' presentations were an important part of the problem solving process. It generated discussions and individual students produced new and interesting solutions. None of that occurred in 1995 or 1996. The students' presentations of their solutions did not have the same positive effect as in earlier projects with fewer students.

4.5 The evaluations by the groups – a summary

The evaluations by the groups were remarkably similar in 1995 and 1996. Here we provide a short summary of the evaluation from the 1996 project with comments when the results are different than in the 1995 project.

From the evaluation by the groups it seems as if they had cooperated well on problem solving. The different subdivisions of groups appeared not to influence the cooperation within the group structure. Some groups reflected in more detail on their cooperation and the outcome of it.

- They experienced that different ways of looking at the problems created interesting discussions. Many different perspectives, ideas and proposals for solving the problems emerged. This contributed to a deeper understanding, and the students learned to think about geometry in a new way.
- They became active participants. They learned to take responsibility, to show interest, and become engaged in the process.
- They felt safe to tell when they did not understand a proposed solution. Group members helped each other and learned from each other. They learned to respect each other's way of thinking.
- They felt that it would have been difficult to solve the problems individually.
- Several groups felt that using the blackboard made it easier to cooperate.

Some additional points from the 1995 project are the following.

- They experienced the importance of starting the project by organizing the total group work and deciding how to cooperate.
- They appreciated working together to solve problems from scratch, and to work on the problems for a long period. It was motivating to have optional problems and problems of differing degrees of difficulty.

- The problems demanded full commitment by all members of the group. They dared to express their ideas and suggestions without being afraid of what the others would say.
- New ideas developed gradually.
- They experienced different ways of solving a problem. By exchanging experiences and knowledge they learned something new together.
- They experienced the importance of good humor.

Several groups were challenged by the problems, which seem to have been suitably demanding and appropriate for discussion and problem solving in small groups. The students experienced that even if the problems appeared difficult, solutions developed during the problem solving process. At the end, all members of the group understood most of the problems. One group said, ”The way they were given, it was necessary to experiment and investigate. None of us found a solution immediately (...) Just this is a central point for a proper problem.” Some groups look at the problems as a means, ”They have introduced us to a new way of thinking.” ”They have given rich and interesting possibilities for mathematical activity.” ”The work also made it possible to refresh important parts of Euclidean geometry.” One 1995 group said, ”The way the problems were given made it easy to get started. We got inspiration to go on even if we ran into difficulties.” To deal with the difficulties they have learned that ”it pays to sleep on the problem and look at it again the next day” or ”it pays not to give up”. Some groups experienced the importance of reading the problems properly and using auxiliary figures.

What did the groups learn about problem solving in geometry? One group said, ”We have seen that trial and error is an essential part of this working process (...) We have also experienced that problem solving activity is a creative and valuable way of obtaining and mastering new knowledge.” Another group interpreted a problem incorrectly – ”That reminded us of the importance of reading the problem properly.” And this group concluded, ”The outcome of such a project is that one often is left with unsolved problems. That was also our experience.” One group experienced that drawing an auxiliary figure is useful, and added,

”When we got stuck on a problem, we went on to another problem. In this way the problem matured in our minds.”

Some groups reflected on their experiences with the project. ”We agreed that this way of working was instructive, demanding and time consuming.” ”Connections and concepts in Euclidean geometry have been the topic of conversation and discussion, which resulted in a deeper understanding. We have experienced joy by discovering a solution and excitement by being so close that we feel we are almost there." "Even if we did not solve all the problems, the process has been inspiring." "It has been fun to find different proofs. Some of us do not like evaluation and this form of group work. Some of us got the great aha-experience." "This type of work was a nice variation." One group reflected on their beliefs about learning mathematics in general, "In our group we agree that it is essential for each individual to participate actively to gain new knowledge in mathematics. It is not enough to get everything from outside, served and chewed. Mathematics should not primarily be an accumulation of rules and formulas to be remembered. It is more a process of investigation, exploration, and generalization. During the group work we have been strengthened in this view of learning mathematics."

Most groups had managed their problem solving process very well. They had organized their work, made adjustments and new choices along the route, and finally written and delivered the report on time. Some groups suggested that more ordinary group hours or lecture hours should be allocated to the project. Most groups had also managed the affective sides of the process. They had experienced ups and downs, but the dominant outcome was that it had been an instructive and enjoyable process.

5 Discussion and conclusions

This article is a successor to Borgersen's (1994) study of a project on co-operative problem solving, organized as part of a geometry course at the college level. It studies the transitions of the project as the course changed from being a medium size course for students from the same study program (Borgersen 1994) to a large course for students from different study programs. In 1993 the project had about 30 students in 10 groups, which increased to 50 students in 13 groups in 1994, mainly from one study program (Mathematics and Sciences). In 1995 the number of students doubled to about 100 in 22 groups and decreased in 1996 to 80 students in 17 groups, both years from three different study programs. (Mathematics and Sciences, Teacher Education, Computer Sciences and Mathematics Methods).

The study confirms the main conclusion in Borgersen (1994) – the co-operative open ended problem solving project in geometry offers one way of meeting two major challenges in mathematics teaching at the college level, namely to have students experience a true picture of doing mathematics and creating learning environments of acceptance, safety and trust. More specifically,

- It is possible to have students experience the whole process of doing mathematics.
- Geometry is a rich resource for finding challenging problems.
- It is possible to create learning environments where students cooperate and give each other support to go through the process together.
- It is possible to do this within an ordinary geometry course in the first year of college.

It is also confirmed that

- Most students take the project seriously and enjoy it even if the project is not graded. It motivates further work in the course and creates a positive learning environment.
- The emphasis on processing and self-evaluation seems to have an effect on the students motivation and understanding of doing mathematics.
- The students appreciate cooperating in small groups and recommend it as an effective method of learning mathematics.
- The students need encouragement and support in the process of doing open ended problem solving.

This result is also consistent with Bjuland's (1997, 1998, 1999, 2002) in-depth study of three small groups of pre-service teachers cooperating on geometry problems (Problem 1 and Problem 3 from our 1996 project). These groups were selected from a class of 105 students.

It has been documented that it is possible for one teacher to manage the project with as many as about 100 students, mainly because the students were organized in small groups and because they took responsibility for their work in the groups. However, it is our experience that with a large class of students, not all the qualities of the project can be maintained.

The study shows that processing in plenum works well with as many as 50 students but not above that number. In 1995 the processing degenerated to one-way information and no communication. In 1996 the processing in plenum was left out. But both years the students were reminded to do processing in the groups at the end of each group meeting. It is difficult to assess the outcome of this, but some groups wrote interesting and insightful reflections on their own problem solving process and on their cooperation in small groups. On the other hand, the decline in process writing may be explained by less focus on processing in class.

The outcome of the students' presentation of their solutions in plenum also seems to be sensitive to the number of students. It is more difficult to keep up the interest of all the students and to stimulate reflection in a large class than in a medium size class. In 1993 (Borgersen 1994) the presentation was an important part of the problem solving process, and individual students produced new and interesting solutions. Surprisingly, this also happened in 1994 with as many as 50 students. But in 1995 and 1996 none of it happened. In classes of 80–100, the students' presentations of their solutions did not have the same positive effect as in the smaller classes.

Most of the problems worked well for open ended problem solving. Problem 2 in 1994 seems to have been too difficult, either because it is a 3-dimensional problem or because the formulation of the problem is very short and concentrated. Otherwise, the problems have been suitably demanding and appropriate for problem solving in small groups. The students have experienced that even if the problems appeared difficult, solutions developed during the problem solving process. The cooperative small group work helped them to get started, and helped them not to give up. And at the end of the term all the students understood most of the problems. They experienced the excitement of discovering a solution and the joy of finding different ways of solving a problem. Having optional problems and problems of diverse difficulty motivated the groups, and they experienced that it would have been difficult to solve the problems individually.

Considering all the projects, the quality of the final reports was variable, but in general, quite high. All group reports had analysis and drawings, and almost all groups had written a log on their meetings and their way of organizing the group work. The progress in process writing and self-evaluation reported in 1993 (Borgersen 1994) stabilized or improved somewhat in 1994 and 1995. All groups had some process writing even if the number showing "a little of process writing" increased in 1995. This trend increased in 1996. Two groups had no process writing at all. As a result, there was a decline in process writing in 1996, which probably started in 1995. We have questioned if this decline was caused by less focus on processing in class. The evaluation writing was still stable in 1996, even though there was a small decline in the number of reports with satisfactory evaluation writing.

The proof writing in 1994 showed a small decline compared to 1993. Several groups pointed at proving as the most difficult part of the open ended problem solving process. The decline in proof writing continued in 1995. Only a little more than half of the groups had satisfactory proof writing, even though all groups had some proof writing. Some groups clearly had little training in mathematics. Their mathematical language was limited, and they had difficulties with basic logic and proof. On the other hand, the process writing made it easier to understand their ideas and their way of thinking. And some of these groups wrote very insightful comments about their problem solving process. The focus on process seems to have made the students aware of their learning and encouraged them to keep trying to solve the problems. In 1996, all the groups had some proof writing, of which almost all was satisfactory. The decline in proof writing in 1994 and 1995 had turned into considerable progress. It is difficult to explain this shift. It could be traced back to the students, the grouping, the problems, or other ”sources”. One could also wonder if the progress in proof writing caused some of the decline in process writing in 1996. And could there be a similar connection between the stability of process writing and decline in proof writing in 1995? Some groups may have used their successful proof writing as a substitute for process writing by arguing: ”Our proofs show how we have been thinking.” On the other hand, detailed process writing may compensate for incomplete proofs or at least ”show how far we came”.

In 1994 there was remarkable progress on the students' attempt to generalize and formulate new problems, compared to that in 1993. All groups except one ordinary group and one individual had some generalization, of which most was satisfactory. This progress on generalization did not continue in 1995. It was more like the result in 1993. A little less than half of the groups tried generalization, a little less than half of which was satisfactory. In 1996 there was an improvement. Two thirds of the groups had some generalization or formulation of new problems in their reports, of which more than half was satisfactory. Some of this variation may be explained by the problems having explicit questions about generalization, which was the case in 1994 and 1996, but not in 1995. Especially since all the groups in 1995 had some process writing and some proof writing, it is likely that more groups would have had some generalization if they had been asked to generalize or formulate new problems.

Considering all the projects, there was a development of the evaluation writing of the groups. The evaluations in 1994 were comparable with those in 1993. And the evaluations in 1995 were remarkably similar to those in 1996. Some were more extensive, detailed and specific, compared to the evaluations in 1994. But the core elements were very much the same in all the projects. The majority of the groups was satisfied with their project work and had experienced the project as interesting, engaging, and instructive, even if they had worked hard and at times felt frustrated. For the three projects studied here, the students expressed their feelings more readily than in previous projects, and they had experienced the importance of good humor in the groups. The evaluations confirm that there are strong emotions connected to open ended problem solving in geometry. It might be that this is even more so in large classes of students where most of the processing happens in the small groups, and very little can be done with the entire class.

Considering the evaluations by the groups in 1995 and 1996, we note the following. It seems that the groups cooperated very well on problem solving. The different ways of forming the groups did not influence the cooperation in such a way that one of those groupings had an advantage over the others. Some groups reflected in more details on their cooperation and the outcome of it,

- They experienced the importance of starting the project by organizing all of the groupwork and deciding how to cooperate.
- They experienced that different ways of looking at the problems created interesting discussions. Many different perspectives, ideas, and proposals for solving the problems emerged and developed gradually. This contributed to a deeper understanding. They learned geometry in a new way.
- They were active participants. They learned to take responsibility and to show interest and engagement. Using the blackboard made it easier to cooperate.
- They felt safe and comfortable to express their ideas and to tell when they did not understand a proposed solution. Group members helped each other and learned from each other. They learned to respect each other's way of thinking.

It is interesting to notice the students' reflections on the importance of all group members being actively involved and willing to help each other, how ideas developed gradually, and how cooperative work contributed to deeper understanding.

The students learned how to cooperate on solving problems from scratch, and how to work on the problems for long periods of time. The project was not meant to be looked upon as an alternative to traditional teaching of mathematics, but as a supplement. And as such, the projects seem to have been ”inspiring” and ”a nice variation”.

Some of the groups' experiences with the problems given and what they learned about cooperative open ended problem solving are discussed above. We have seen that:

- Properly demanding and thoughtfully formulated problems are important for creating interest and inspiration in the students.
- Cooperative small group work helps the students get started on problem solving, and helps them not to give up even if they run into difficulties.
- Explicit questions in the formulation of a problem are important for having the students trying out all steps in problem solving, also generalizing or formulating new problems.
- Cooperating in small groups create learning environments of acceptance, safety and trust, and make it possible for students to be actively involved and take responsibility for their own learning.

It appears as though having students cooperating in small groups focusing on the process of problem solving, promotes reflection on their own learning. It may help them to become aware of their cognitive resources and affective reactions and to develop their metacognitive abilities and social skills.

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Sammendrag

Denne artikkelen er en videreføring av Borgersens (1994) undersøkelse av et problemløsningsprosjekt, basert på samarbeid i smågrupper og organisert som del av et geometrikurs på universitet/høgskolenivå.

Artikkelen fokuserer på effekten av en gradvis endring av prosjektet. Den undersøker prosjektene fra 1994 til 1996 (Borgersen 1995, 1997, 1999), idet geometrikurset endret seg fra å være et middels stort kurs med omlag tretti studenter fra samme studieprogram til å bli et stort kurs med nær ett hundre studenter fra tre ulike studieprogram.