# Alternatives to Standard Algorithms

## A Study of Three Pupils during Three and a Half Years

Rolf Hedrén

*In this article I will discuss a research project about the use of alternatives to the standard algorithms for the four arithmetic operations and especially the work of three girls who took part in the project. The girls were not taught any standard algorithms during their first five years at school. They were encouraged to use their own written methods, including drawings, for all kinds of computations that they could not do mentally. The three girls often worked together in a group. The girls were taught the standard algorithms during their sixth year at school.*

*The results I got were mainly that the girls could manage to find their own methods, often on their own but sometimes with the help of peers or teachers. I also found that the methods that they used, were mostly less effective than the standard algorithms, but that they were more like those used for effective mental arithmetic and computational estimation. The girls acquired good number sense and good ability in mental computation, and they preferred their own methods, even after they had been taught the standard algorithms.*

## Introduction

In many countries the role of the standard algorithms is discussed today. Is it really necessary to teach pupils in elementary school to compute 503 - 287 with the following algorithm or something like it?

$$\begin{array}{r} \cancel{10}10 \\ \hline \cancel{5}03 \\ -\,287 \\ \hline 216 \end{array}$$

Perhaps it is better to ask them to find their own ways to do the computing. Some of them may do it in this way: 287 plus 13 is 300, 300 plus 200 is 500, and 500 plus 3 is 503. The answer is thus 13 plus 200 plus 3, which is 216. Will this method and other similar ones be sufficient in the pupils' future lives?

---

*Rolf Hedrén is professor emeritus and a teacher educator in mathematics education in Högskolan Dalarna, Sweden.*

I carried out a research project in one class to try to get an answer to the above question. The pupils of this class were not introduced to the standard algorithms during their first five school years, i. e. when they were up to 12 years old. Instead, they were encouraged to find their own computational methods, sometimes on their own and sometimes working together in small groups. Only whole numbers were involved in the computing.

One of the reasons for this research is, of course, that many computations are carried out with the help of calculators and computers today. This reduces the need for paper and pencil procedures, but, on the other hand, it might increase the demand for man's ability to do computational estimation and, thus, mental arithmetic. The technical aids do computations very fast, but at the same time, they also make it possible to make miscalculations just as fast. The calculator or the computer will not check our calculations; we will have to do that ourselves.

To be able to do mental arithmetic and/or computational estimation our pupils will need number sense, an ability to operate with numbers in a skilful way. We might ask if our pupils acquire better number sense when they are allowed to find their own ways of computing than when they are taught the standard algorithms.

In this article I will only take up a part of the research project. I will show how three girls managed to do mental and written computations with whole numbers in the four arithmetic operations during observations and clinical interviews. I will also discuss the girls' opinions of finding their own methods instead of being taught standard algorithms. For a fuller account of the project I have to refer to a report, which was published in Swedish (Hedrén, 2000).

It should be mentioned from the very beginning that by *«mental arithmetic»* or *«mental computation»* I mean computing that is done entirely in a person's head so that only the original exercise and the answer are recorded. When a person uses some kind of drawing or records intermediate results (support notes), I will use the term *«written computation»* (or *«alternative written computation»* to distinguish it from standard algorithms). Although people tend to use the same kind of strategies in alternative written computation as in mental computation, I find the above distinction essential.

## Theoretical Framework

Many researchers and many teachers, as well, see constructivism, especially social constructivism, as the guiding rule of their work. I see this theory of learning as one of the reasons for my research as well. I would like to summarise it in the following three principles:

1. The learner actively builds up her/his own knowledge.
2. The learner's previous experience plays a vital role during that construction.
3. The learner's interaction and dialogue with others is crucial for her knowledge construction.

Presupposing these principles, it is difficult to see how mere teaching of standard algorithms can help our pupils to construct their own meaning of numbers and operations with numbers. It seems more appropriate to allow pupils to find their own computational methods, alone or in cooperation with others.

It should also be mentioned that I made social constructivism my guiding rule not only when designing the project but also when analysing and discussing its results.

## Previous Discussion and Research

In the introductory article of the proceedings of a working group: «A Curriculum from Scratch (Zero-Based)» at the eighth international conference in mathematics education, Anthony Ralston, the organiser of the group, writes

> *So, if we were inventing primary school arithmetic today, should there be any p-p-a (paper-and-pencil arithmetic) at all? The answer to this question could be, no, only if each of the following questions has an affirmative answer:*
>
> 1. *Can you teach children all they need to* understand *about arithmetic without p-p-a?*
>
> 2. *From the standpoint of efficiency only, are all arithmetic calculations better or, anyhow, as well performed either mentally or by calculator than with p-p-a?* *(Ralston, 1997, p.4.)*

By p-p-a (pencil-and-paper arithmetic) the author means computing carried out with traditional standard algorithms. However, he also writes that paper and pencil (or ballpoint pen) are indispensable tools for recording intermediate results or for drawing pictures. As he sees mental arithmetic and use of the calculator as the only alternatives to p-p-a, his definition of mental arithmetic also includes the possibility to make support notes and drawings.

According to the author, a curriculum from scratch should, contain other essential components than arithmetic, although arithmetic should remain the most important portion of it. «But it is understanding of the operations of arithmetic ... not facility with arithmetic computation which is crucial to the further study of mathematics.» (Ibid p. 6.)

In this connection I cannot avoid citing what Plunkett (1979) wrote on standard algorithms as far back as the end of the 1970:s.

> (The algorithms) are *analytic*. They require the numbers to be broken up, into tens and units digits, and the digits dealt with separately.
>
> They are not *easily internalised*. They do not correspond to the ways in which people tend to think about numbers.
>
> They encourage *cognitive passivity* or suspended understanding. One is unlikely to exercise any choice over method and while the calculation is being carried out, one does not think much about why one does it in that way. (Ibid p. 3.)

Besides, the author goes on, they are used very little even by children. They are also often applied unthinkingly to computations like 1000 - 995 or 100 x 26. He points out that, in these cases, it would be better to look upon the numbers holistically, for instance 995 is very near to 1000.

I will also say a few words about number sense. In my country there have recently appeared a lot of articles on this subject in our journal on mathematics education, Nämnaren (Emanuelsson & Emanuelsson, 1997; Reys & Reys, 1995; Reys, Reys & Emanuelsson, 1995; Reys et al 1995a, 1995b). Various authors have emphasised different aspects of number sense, and I will here restrict myself to four of them, which I believe have been especially important in my research. A pupil with good number sense:

1. understands the meanings and magnitudes of numbers;
2. understands that numbers can be represented in different ways;
3. knows the divisibility of numbers;
4. knows how to use the properties of arithmetic operations.

I will discuss some applications of each of these aspects that are appropriate in connection with alternative written computation and mental arithmetic.

1. The understanding of place value is a part of this aspect, both in whole numbers and in decimal numbers. For instance, a pupil should understand that 998 is very near to 1000, and that 0.05 is greater than 0.0375. She should also acquire a good conception of very big and very small numbers, one thousand, one million, one thousandth, one millionth.
2. Above all, the connection between whole numbers, decimal numbers, and fractions belongs to this aspect. A pupil knows that 12.0 is mathematically the same as to 12, 9/3 as 3, and 2/5 as 0.4.
   The ability to partition numbers in different ways belongs to this aspect as well. It is sometimes practical to realise that $8 = 6 + 2 = 5 + 3 = 4 + 4$; $36 = 2 \times 18 = 3 \times 12 = 4 \times 9$; $316 = 320 - 4$.
3. This aspect stresses the advantage of knowing, for instance, that 25 is a divisor of 175, that 4 is a divisor of 16, and that $4 \times 25 = 100$. In such a case I can easily compute $16 \times 175$ as $4 \times 7 \times 4 \times 25 = 28 \times 100$. (In this case I also had to use some of the properties of arithmetic operations).
4. This aspect contains the ability to transform arithmetic expression with the help of among others the commutative, associative, or distributive properties of arithmetic operations. A pupil can look upon $27 + 8$ as $27 + (3 + 5) = (27 + 3) + 5$. The multiplication $6 \times 83$ can be computed, either mentally or with support notes, as $6 \times 80 + 6 \times 3$. (The aspects mentioned in points 1 and 2 have been used as well.) The computing of $25 \times 7 \times 4$ will be easier, if the order of the factors 7 and 4 is reversed.

I will say a few words about research in mental arithmetic and then in written computation with support notes.

Klein, Beishuizen, and Treffers (1998) use a way to facilitate mental arithmetic with the help of what they call «the empty number line». The authors encourage their pupils to draw a line on a paper and mark appropriate numbers on it. If, for instance, a pupil is supposed to compute 74 - 36, she marks the number 74 on the line, draws an arrow to the left to indicate that she subtracts 30, marks the number 44, draws another arrow to the left indicating a subtraction of 6, and finally marks the number 38. She has subtracted 36 from 74 and reached the number 38. The answer of the exercise will thus be 38.

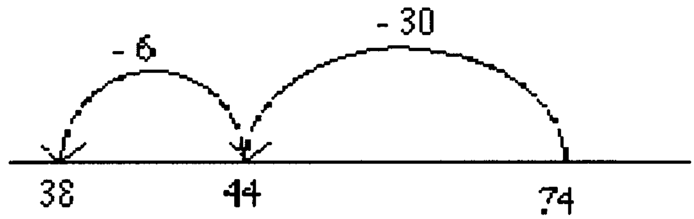


*Fig. 1. The use of the empty number line for the subtraction 74 - 36. (Klein, Beishuizen & Treffers, 1998, p. 450, my interpretation.)*

With the strict definition of mental arithmetic that I gave in the introduction, this is not to be regarded as a solution that is achieved mentally. However, the method given is, according to the authors, but a step on the way to effective strategies for mental computation.

Beishuizen, van Putten, and van Mulken (1997) discuss different strategies in mental arithmetic and their importance for pupils' possibilities to solve more complex exercises. Especially, they take up two methods that pupils often use in addition, which they call 1010 (split strategy) and N10 (jump strategy) respectively. A pupil uses 1010, if she computes 46 + 23 as 40 + 20 = 60; 6 + 3 = 9; 60 + 9 = 69. On the other hand, she uses N10, if she does the same sum in this way: 46 + 20 = 66; 66 + 3 = 69. The corresponding strategies may also be used in subtraction.

The authors point out that the N10 method is more difficult to learn, as the pupils have to build up a mental number line with marks like 16, 26, 36 both forwards and backwards. In the long run, however, N10 proves to be the more effective method. In subtraction, above all, the 1010 method may cause many mistakes, for instance the idea of the «commutative property» of subtraction. Too many pupils solve 65 - 38, for instance, as 60 - 30 = 30; 8 - 5 = 3 and give the answer 33. Especially underachievers using the 1010 method can look upon 65 as 60 + 5, but, in spite of that, they have difficulties to look upon the numbers holistically. To be able to do the sum 65 - 38, the pupil has to partition the numbers 65 and 38 in other ways than as 60 + 5 and 30 + 8.

In the CAN-project (Calculator Aware Number) in Britain (Duffin, 1996) the children, besides using their own methods for written computation, always had a calculator available, which they could use whenever they liked. Exploration and investigation of «how numbers work» was always encouraged, and the importance of mental arithmetic stressed.

One of the reported advantages of the CAN-project was that the teachers' style became less interventionist. The teachers began «to see the need to listen to and observe children's behaviour in order to understand the ways in which they learn». (Shuard et al, 1991, p. 56.) The teachers also recognised that the calculator «was a resource for generating mathematics; it could be used to introduce and develop mathematical ideas and processes». (Ibid p. 57.)

Kamii (1985, 1989, 1994) worked together with the children's class teachers in grades 1 - 3 in a similar way in the U. S. She did not teach the traditional algorithms but encouraged the children to invent their own methods for the four arithmetic operations. She also devoted much time to different kinds of mathematical games.

According to Kamii et al (1993/94): *«many of the children who use the algorithm unlearn place value ...»* *(p. 202.)*
They give

$$\begin{array}{r} 987 \\ \underline{+654} \end{array}$$

as an example and compare pupils, who use their own methods with those using the algorithm. The former start with the hundreds and say: « 'Nine hundred and six hundred is one thousand five hundred. Eighty and fifty is a hundred and thirty; so that's one thousand six hundred and thirty ...' «. The latter «unlearn place value by saying, for example, 'Seven and four is eleven. Put one down and one up. One and eight and five is fourteen. Put four down and one up. ...' «. They state that children, when working with algorithms, have a tendency to think about the digits of every column as units, and therefore the algorithm rather weakens than reinforces their understanding of place value. (Ibid p. 202.)

It is also interesting to follow the research carried out by Murray, Olivier, and Human in South Africa (e. g. Murray, Olivier & Human, 1994; Vermeulen Olivier & Human, 1996). Like the researchers mentioned above, they had their pupils invent their own strategies for computing, and above all they discussed strategies used for multiplication and division. In a summary of the results of their problem-centred learning they state among other things:

> *... students operate at the levels at which* they *feel comfortable. When a student transforms the given task into other equivalent tasks, these equivalent tasks are chosen because the particular student finds these tasks more convenient to execute.* *(Murray, Olivier & Human, 1994, p. 405.)*

Narode, Board and Davenport (1993) concentrated on the negative role of algorithms for the children's understanding of numbers. In their research with first, second and third graders the researchers found out that after the children had been taught the traditional algorithms for addition and subtraction, they discarded their own invented methods, which they had used quite successfully before the instruction. The children also tried to use traditional algorithms in mental arithmetic; they gave many examples of misconceptions concerning place value, and they were all too willing to accept unreasonable results achieved by the wrong application of the traditional algorithms.

It ought to be stated, however, that there are also researchers with a quite different opinion. Kilpatrick writes:

> *A neglected yet critical item both in implementing the NCTM standards and in gaining a better grasp of the role skill development plays in learning mathematics concerns the folk wisdom in today's school practice. Why is it that so many intelligent, well-trained, well-intentioned teachers put such a premium on developing students' skill in the routines of arithmetic and algebra despite decades of advice to the contrary from so-called experts? What is it the teachers know that the others do not?* *(Kilpatrick, 1988.)*

Bauer (1998) is also critical of letting the pupils use their own computational methods, which are often called «halbschriftliches Rechnen» (half-written computing) in German. He points out that these methods might, in everyday school practice, often fall into the decay of normalisation and automatisation, i. e. the development of new algorithms. These algorithms will, however, be much less effective than the traditional algorithms, and therefore there will be no reason to abandon the latter.

## Purpose and Questions

As I have mentioned earlier there were mainly three reasons for starting the whole project:

1. The existence of calculators and computers to make computing faster, simpler, and more reliable.
2. An increasing demand for a citizen's number sense, which, in my opinion, is closely related to her/his skill in estimation.
3. Social constructivism as a theory of learning.

Research has shown that the pupils' own methods for computing in the four arithmetic operations are more like effective methods for mental computation and estimation than standard algorithms are. It has also been shown that pupils will acquire better number sense by inventing their own methods for computing than by following given rules.

I, therefore, wanted to investigate what effects teaching, where the pupils were not taught the traditional standard algorithms during their first five years at school (ages 7 - 11), might have in a Swedish classroom. As I have already mentioned, a lot of research in this area has already been carried out. I wanted, however, to follow the process in one class very thoroughly and for a long period of time. The research questions, which will be discussed here, are:

1. How is the pupils' number sense affected?
2. How is the pupils' ability to do mental computation affected?

## Method

I followed one class from their spring semester in year 2 up to and including their spring semester in year 5. The reason that I started in year 2 was that in Sweden we generally start teaching the standard algorithms during that year at school. Due to some pressure from parents, the pupils who wanted were given the opportunity to learn standard algorithms during year 6. I therefore met a few of the pupils at the end of this year for clinical interviews and to ask some other questions.

To give a better picture of the experimental design, I summarise it in the following way:

1. The pupils were encouraged to use their own written methods (including the use of drawings) for all kinds of computations that they could not do mentally. No special methods were taught. The pupils' methods were often discussed in groups or in the whole class. The pupils' parents were invited to help their children find alternative methods and asked not to teach them standard algorithms.
2. Mental computation and estimation were encouraged and practised.
3. The pupils had calculators in their desks. They were used for number experiments, for control of computations and for more complicated computations.
4. Due to lack of suitable textbooks, we used a traditional book during years 2 and 3, but the pupils calculated all exercises intended for computing with standard algorithms in their own ways. During years 4 and 5, we could use a textbook that was more adapted to the use of non-traditional methods (Rockström, 1996, 1997). The class teacher taught all lessons. The textbooks used covered all other items of the syllabus, and the teacher taught these items in the way he had done before.

I want to add that the calculators did not play a major role in this experiment. However, I chose to let the pupils have them in their desks and use them when appropriate, because they were one of the reasons for the realisation of the project.

The research and evaluation methods, which will be discussed in this article, were qualitative:

- Clinical interviews
- Observations
- Copies of the pupils' computations during observations
- Ordinary interviews with pupils

The interviews and observations were tape-recorded and transcribed. I supplemented the recording with simple notes in case an important part of the recording would fail.

Three girls and three boys were picked out for clinical interviews. They were chosen to represent different levels of achievement in mathematics.[1] The three girls formed a special group during the observations. Clinical interviews and interviews with pupils were

[1] Actually, I used the results of tests in year 2 to pick out these pupils. However, it turned out that, during the course of the experiment, their levels of achievement changed, so that the order among the girls was totally reversed.

undertaken at the beginning of the spring semester in year 2 and in the middle of the same semester in years 3, 4, and 5. In the clinical interviews I spoke to one pupil at a time.

In this paper I will concentrate on the activities, procedures, and achievements of the three girls mentioned, when doing mental and written computation, and the cooperation in their group. During my observations, it has been my intention above all to notice and register:

whether the pupils have done all computing mentally or made drawings or notes

what methods the pupils have used

whether and how the pupils have used some aspect(s) of number sense

whether the pupils have cooperated to find strategies or correct possible mistakes

whether the pupils have come across unexpected difficulties

I will also take up results from the clinical interviews and ordinary interviews with the three girls. During the course of the project, the girls were also given exercises in estimation. However, there is no space to discuss the outcome of these in this article, instead I have to refer to my previously mentioned report (Hedrén, 2000).

## Social constructivism as the guiding rule of the project

As I have mentioned before, I tried to follow the ideas of social constructivism when I designed the project, discussed the actual carrying out of the project with the class teacher, and analysed its results. I will therefore give an account of how we tried to realise this as far as the role of the teacher, the parents, the textbook, and even the researcher himself were concerned.

The teacher often led discussions in class where the pupils got an opportunity to show their different solutions of an exercise. However, he also took up common misunderstandings which he had seen when the pupils were working on their own or in groups; e. g. the misuse of the equal sign or the too common belief that an exercise like 34 - 27 could be solved as 30 - 20 = 10; 4 - 7 = 3, giving the answer 13. He might also sometimes give his own suggestions how to solve an exercise, albeit never forcing or even recommending the pupils to use his methods.

The class teacher and I discussed with the parents why we did not want them to teach their children the standard algorithms. We even arranged a special parents' meeting, where we discussed the same kind of exercises that the pupils were given and in the same way as the teacher did together with his pupils. Of course, we could not prescribe in detail how the parents should help their children. We even know that there were a few occasions, where parents could not refrain from teaching their children the algorithms after all, of course, with the best of intentions.

In the textbook there were suggestions how different kinds of exercises could be solved without the standard algorithms, and in division even with a kind of algorithm, which in Sweden is called «short division». In my opinion the textbook should have given the pupils more freedom to invent their own methods, but the one used was in my opinion and for the project in question the best one available in the Swedish market. As far as I could see from the observations and clinical interviews, the suggestions of the textbook did not influence the pupils' solutions to a very high degree.

I as a researcher tried to behave as neutrally as possible. However, I am fully aware that I did influence the course of events by my mere presence, by my asking questions about the pupils' thinking etc. I never gave any suggestions how to solve an exercise. On the other hand I could, for pedagogical reasons, never let a wrong answer pass without taking steps to let the pupil find the correct one. In the first place, I expected that the pupil herself would realise her mistake, in the second that her peers could help her with that. Only when these measures failed, did I ask a question that made the pupil realise that her solution method was not quite adequate.

## Results and Comments

I find it more convenient to follow the girls' procedures and achievement in one arithmetic operation at a time.

In this paper, the three girls are called Ann, Britta, and Cecilia. Especially in the beginning most of the computing was done mentally, but I always asked the girls to explain their reasoning. In this paper, everything that was said has been translated from Swedish into English as literally as possible.

In Sweden, the school year starts in August and goes on until June in the following calendar year.

## *Addition*

The pupils did not meet with many difficulties in this arithmetic operation. However, I will show a few examples of such difficulties but also of clever solutions.

In March in year 2, Britta solved the exercise 8 + 5 in a very smart way. She said 10 plus 5 equals 15, so 9 plus 5 equals 14, and 8 plus five equals 13.

On this occasion, Britta did not work together with Ann and Cecilia but with two other girls. They were all asked to solve 49 + 22 in connection with a word problem. Both the other girls got the answer 61. They explained that 40 plus 20 is 60; 9 plus 2 is 11 and then gave the answer 61 by omitting the first digit 1 in 11. Britta, however, gave another explanation:

> *«I thought like this, 11, it is written like this that 9 plus 2, you cannot add that to some six, then it should be something with 70, then you must put 71.»*

I asked her to explain better and got the following explanation:

> *«Well, you cannot go on like this, eh, sixtyten, sixtytwelve or sixtyeleven, like that, one cannot do like that, then you have to start on 70 then 71.»*

Among these girls, I could see no sign of the N10 (jump) strategy mentioned above. In November in year 3 for example, both Ann and Cecilia did the following computation to add 23 + 10: 20 plus 10 is 30; 30 plus 3 is 33. Britta solved the exercise in a similar way: 2 plus 1 is 30; 30 plus 3 is 33.

Similar kinds of solutions were used for more complicated problems. On the same occasion, Ann and Cecilia computed 137 + 148 in a word problem as 100 plus 100 is 200; 40 plus 30 is 70; 8 plus 7 is 15; the answer is 285. Britta used units instead of tens and hundreds and got the correct answer.

Just as in the two exercises just mentioned, Britta very often used units instead of powers of ten. However, she seemed to understand what she was doing. For instance, she solved the exercise 13 + 34 given on the same occasion as 3 plus 1 is 4; 3 plus 4 is 7; the answer is 47.

In January in year 3, the girls were working with an exercise, where they were supposed to use their calculators. They were asked to find two consecutive numbers (good neighbours), whose sum was a given number (i. e. they had to find the value of x and x + 1 in the equation x + (x + 1) = a, where a was given). Britta explained how she found the good neighbours, whose sum is 31:

> *«Because 1 plus 1 is 2, so 10 plus 10 is 20. And then 5 plus 6 is, sure, no, it is 11, sure. And then it will be one ten more. Then it is 30. And then you take this one. And then it is 31.»*

On the same occasion, Cecilia found the good neighbours, whose sum is 51 in the following way:

> *«Well, I thought, 20 plus 20 is 40. And that then, it is not 50, of course. And then I thought that 1 is here, sure. So then you cannot take, what is it called, eh, 20. So then I thought like this, that one could take, eh, 20, 5 plus 26 plus 6 plus 5, it will be 11. And then you take one ten more on 40, and then it will be 50. And then you have 1 left, and it will be 51.»* (See comments below.)

In October in year 4, the girls had to solve the exercise 148 + 74. Cecilia used a somewhat unconventional method. She wrote 148 + 2 = 150 + 50 = 200 + 22 = 222, and explained that 74 minus 52 is 22. Both Ann and Britta got the answer 212, but they found their mistakes while, at my request, Ann explained her computations. This was just one example of how the girls 1) found their mistakes while explaining their solutions, and 2) could help each other to find correct solutions.

In May in year 6, I made a follow-up investigation after the pupils had been taught the standard algorithms. I gave the exercise 129 + 366 + 273. Unfortunately, only Ann and Cecilia could take part in the observation. Ann computed the sum in two ways, with and without a standard algorithm. The algorithm gave the correct answer, while her own method did not. Cecilia only wrote the numbers 600, 150, and 18, and the correct answer 768. She thus abstained from using the algorithm and did most of the computation mentally.

## Comments

It is interesting to see how Britta could use a systematic thinking strategy, when computing 8 + 5 (March in year 2). A corresponding thing occurred when Cecilia computed 148 + 74 (October in year 4). She showed that she could really look at the numbers and find a method that took account of their characteristics: 148 is near to 150, 150 and 200 are both what I would call *«round numbers»*, i. e. numbers easy to do computations with. In my opinion, these are examples of how the girls were using and deepening their number sense.

The issue of regrouping is a stumbling block when working with standard algorithms. Of course, the problem did not disappear when the pupils were working with their own methods. However, in a very concrete way, Britta was able to show that she had grasped the idea when she was working with the exercise 49 + 22 (March in year 2). Both Britta and Cecilia did the same thing when finding the good neighbours whose sum is 31 and 51 respectively, a not too easy task in year 3 (January in year 3). It is a little difficult to understand what Cecilia wants to say. My interpretation is that she understands that the units of the numbers searched have to be pretty big so that they, when added, can form another ten.

I was a little hesitant, when Britta used units instead of tens to solve 13 + 34 (November in year 3). I wondered whether she was thinking of a traditional algorithm. However, this way of computing was repeated later as will be seen, and each time Britta showed a command of the computation. Therefore, my hypothesis is that the use of units instead of tens (and hundreds) is only an example of the «lazy» mathematician's attempt to find a labour-saving strategy.

Unfortunately, Cecilia misused the equal sign in her chain of calculations in October in year 4. That happened rather frequently, but I sometimes refrained from pointing out the mistake, because I found it more important to concentrate on the mathematical reasoning than on the formal use of the equal sign. That does not mean, however, that I did not realise the importance of the correct use of this sign, I just thought that a discussion of this should be postponed to a more convenient occasion. I am also aware that the introduction of a special sign for «chain computation», e. g. an arrow, might solve this problem.

Of course, it is a little disappointing that all three girls stuck to the 1010 (split) strategy, as Beishuizen, van Putten, and van Mulken (1997) call it, all through the experiment, instead of using the apparently more advanced N10 (jump) strategy. My belief is that we would have had to give them examples of and hints on the N10 strategy in order to make them use it. It might have been difficult for the girls to get the idea of using it on their own.

## Subtraction

The girls' most common difficulty in subtraction occurred while handling exercises like 413 - 289, where the digits of the second term in some power of ten was bigger than that of the first term, i. e. such exercises, where one generally has to regroup or «borrow». They first met this difficulty in September in year 3, when they were asked to compute 25 - 18. All three girls answered 13. I recommended them to think of a concrete example:

> *«If you have 25 kronor and buy a big bar of chocolate for 18 kronor, how much money will you have left?»*

Then Britta managed to solve the exercise correctly by noting down all the numbers, starting with 19 and ending with 25 and counting these numbers.

In November in year 3, the girls got a word problem that dealt with kilometres driven, and that led to the computation 35 - 16. Both Britta and Cecilia solved it by computing 30 minus 10 is 20; 5 minus 6 is (minus) 1. However, only Cecilia used the expression *«minus one»*, Britta said that *«six is more than five, as we know»* and obviously meant the same thing. Ann, on the other hand, used the method of drawing a mark for every kilometre starting with 17 and ending with 35.

In the clinical interviews in April/May in year 3, both Britta and Cecilia computed 63 - 18 correctly. Britta first got 60 minus 10 is 50, and then she said that she had to take 5 away from 50, but she could not explain why. Cecilia reasoned like this: 60 minus 10 is 50; 53 minus 3 is 50; then she had to take away another 5, as 8 minus 3 is 5; 50 minus 5 is 45. On the other hand, Ann still got the answer 55 by computing 60 minus 10 is 50 and adding the difference of 8 and 3. I had to ask her to draw money to get her understand that her way of doing subtraction was not quite adequate.

I got a similar result in another exercise during the same clinical interview: «In a little school there were 272 pupils. One day 69 pupils were ill. How many pupils came to school?» Ann answered 217, but Britta was able to realise that she had to take away the 7 in the units. Cecilia used an intelligent method of trial and error starting with the answer 217. She checked the answer by adding 217 and 69, but when she saw that the result was not correct, she tried to adjust the would-be answer appropriately.

The difficulty remained for some of the girls in August in year 4. In the exercise 72 - 27, Cecilia correctly pointed out that one cannot take 7 from 2, while Ann and Britta gave the answer 55. However, Britta rather soon got aware of her mistake and realised that 2 minus 7 is minus 5.

In the clinical interviews in April in year 4, all three girls managed to solve the subtraction 134 - 58 correctly and to explain their solutions. Ann used a rather complicated mental strategy: 100 minus 50 is 50; 50 minus 8 is 42; 30 plus 40 is 70; 2 plus 4 is 6; and got the answer 76.

However, the problem came back again, when three-digit numbers appeared. In May in the year 4, I gave the girls the exercise 514 - 237. Cecilia managed to solve it by using negative numbers: 500 minus 200 is 300; 10 minus 30 is minus 20; 4 minus 7 is minus 3. The other two girls both got the answer 323.

It was not until November in year 5 that all the girls started to come to grips with subtraction. In the exercise 535 - 269, Ann and Cecilia used negative numbers, while Britta used a somewhat different method: 500 minus 200 is 300; 330 minus 60 is 270; 275 minus 9 is 266.

Notwithstanding, Ann made mistakes in the exercise 925 - 348 in the clinical interview in March/April in year 5. She got the answer 700 + 20 + 3 = 723. However, she realised that her strategy was wrong, when she was asked to explain her method. The other two girls used different strategies but did not seem to have any difficulties with this exercise.

In April in year 6 and after the girls had been taught the standard algorithms, none of the girls wanted to use the subtraction algorithm. Cecilia told me that she computed the exercise 1327 - 859 in the following way: 1000 minus 800 is 200; 300 minus 50 is 250; 27 minus 9 is 18; 200 plus 250 plus 18 is 468. However, she only noted down the numbers 200, 250, and 18 and the answer. Ann, on the other hand, used negative numbers.

## *Comments*

The issue at stake here was the non-commutative property of subtraction. It was amazing to see how difficult it was for some of the girls to realise that 5 minus 8 is not the same as 8 minus 5. Another thing that struck me was that some of the girls resorted to methods, which they had once abandoned as faulty when the numbers involved in the computations became more complicated. We can see this very clearly in May in year 4, when both Ann and Britta gave the answer 323 to the exercise 514 - 237.

It is possible that it would have been easier for the girls to tackle this difficulty if they had been introduced to the N10 (jump) strategy, according to Bieshuizen, van Putten, and van Mulken (1997), at an early stage. These authors state that the strategy commonly used by the girls, the 1010 (split) strategy, and very often causes pupils to apply the commutative property in subtraction as well.

It can be said that the mistakes made in subtraction are due to the fact that the girls had to invent their own methods. However, we know that the same mistakes occur in the standard algorithms, although the terms are placed one above the other. I believe that we have a better chance to discuss what subtraction really means and how the numbers involved should be handled, if we let our pupils work with their own methods. As has been seen, it might take a long time before some pupils come to grips with subtraction, but I think it is worth the struggle.

At the same time, it is very interesting to see how the girls sooner or later managed to use negative numbers, a concept that in my country is not introduced before the seventh year at school. It is difficult to tell why this method became so popular, not only among these three girls but also among all the pupils of the class. A possible explanation might be that the pupils very often tried to split the numbers involved in hundreds, tens and units, which can also be seen in the other arithmetic operations. In such a case the only possible way to indicate certain differences is by using negative numbers.

However, other methods were also used. Ann showed very clearly that she understood what she was doing when computing 134 - 58 in April in year 4, although her method might seem a little odd.

## *Multiplication*

This arithmetic operation did not appear until year 4. The exercises were sometimes solved with the help of the distributive property but as often with repeated addition or a mixture of both.

I gave the exercise 3 x 28 in the clinical interviews in <u>April in year 4</u>. Ann and Cecilia used a mixture of methods, reasoning 20 + 20 + 20 = 60; 8 + 8 + 8 = 24; 60 + 24 = 84. Britta's strategy was a little different: 20 + 20 + 20 = 60; 8 + 8 = 16; 76; 76 + 4 = 80; 80 + 4 = 84.

In the same clinical interviews, I wanted to see if the girls could multiply with 10, 100, and 5 in a fast and effective way. I therefore gave them the exercises 10 x 12, 100 x 12 and 5 x 12. All three girls computed the first one as 10 x 10 + 2 x 10. Ann wrote 100 twelve times in the second one. She had to count these hundreds in pairs, 200, 400, 600 etc., to get the product. Britta and Cecilia wrote 10 x 100 = 1000; 2 x 100 = 200 and gave the correct answer. None of the girls could see that the answer of the third computation should be half that of the first. Instead they started once again to compute the product in different ways.

In May in year 4, to solve the exercise 2 x 212 both Ann and Britta wrote: 2 x 200 = 400; 2 x 10 = 20; 2 x 2 = 4; giving the answer 424. Cecilia wrote: 2 x 2 = 4 = 400; 2 x 1 = 2 = 20; 2 x 2 = 4 and arrived at the same answer.

On the same occasion, however, they resorted to addition when the multiplication facts got a little more complicated. To solve 6 x 27, Britta wrote 20 + 20 + 20 + 20 + 20 + 20 = 120; 7 + 7 = 14; 7 + 7 = 14; 7 + 7 = 14. In the same exercise Cecilia wrote 27 + 27 = 54; 54 + 54 = 108; 108 + 54 = 162. Cecilia told me that she knew the multiplication fact 6 times 7 is 42, but she declared that she was now certain enough to be able to use it. Ann solved the exercise in a manner similar to Britta's.

In September in year 5, I could again see a mixture of the two methods. Ann and Britta solved 5 x 44 writing: 5 x 40 = 200; 5 x 4 = 20; 200 + 20 = 220, but they reasoned 80 plus 80 is 160; 160 plus 40 is 200. Cecilia, on the other hand, could do the multiplications involved directly. Britta told me that she knew that it was possible to do the computation in the same way as Cecilia but that she found it too complicated.

Cecilia showed another mixed method in November in year 5. She was asked to compute 7 x 24 and wrote 3 x 24 = 20 + 20 + 20 = 60; 60 + 12 = 72; 72 + 72 = 144 + 24 = 168. She told me that 3 plus 3 is 6 plus 1 is 7. (Again there was a misuse of the equal sign in the first computation.)

When I gave the exercise 10 x 35 on the same occasion, none of the girls found the expected shortcut. Ann and Cecilia wrote 10 x 30 = 300; 10 x 5 = 50; 300 + 50 = 350. Ann explained that 10 x 30 = 300, because 10 x 10 is 100. Britta also computed 10 x 30 and 10 x 5, but she even had to write 30 + ... + 30 (10 times) and to add these numbers in pairs and handle the other multiplication in about the same way. When I asked her about the other girls' way of doing the multiplications, she said that it was a good way. I asked her which way she thought was the fastest one, but she only answered: «Yes, but I *do* like this».

In the clinical interviews in March/April in year 5, Ann and Britta wrote the solution of 7 x 39 as 60 + 60 + 60 + 30 and 18 + 18 + 18 + 9. Cecilia, on the other hand, used the distributive law in the following way: 6 x 30 = 180 + 30 = 210 + 6 x 9 = 54 + 9 = 63 + 210 = 273. She declared that it was easier to multiply by 6 than by 7. This is also a typical example, where I thought it was more important for the pupil to be allowed to work out her own method at ease than it was for me to interrupt her and tell her that she was misusing the equal sign.

When asked to compute 7 x 199 in May in year 5, none of the girls tried the shortcut 7 x (200 - 1). Cecilia noted down 7 x 100 = 700; 7 x 90 = 630; 7 x 9 = 63. She could give the answer of 7 x 9 directly but not that of 7 x 90. The other two girls tried to use a mixture of repeated addition and the distributive law with mixed success.

The same behaviour was repeated in the last observation in April in year 6. I will cite the discussion between Cecilia and me. Obs. is observer and C. Cecilia, ... means a short pause.

C. It was 8 times 298, sure, and then I took ... To make it a little simpler I divided 8 so that it became 4 times 200, and then I double it, as it is half of ... Yes, so it was 1600, and then I took 90 times fo ... or two (inaudible) four so it ... I don't know why, but I took 2, then I took 18 and then 18 plus 18 is 36 and then 36 plus 36, it is 72, and then it became, as there was a zero there behind, it became 72 ... or 720, and then I took 8 (inaudible) 8, it is 64 like that (inaudible) only, then it became 2384.

Obs. Yes, only a little question there.

C. Mmm.

Obs. You took 8 times 90.

C. Mmm.

Obs. What is 8 times 9.

C. 8 times 9? 72.

Obs. Yes. Why didn't you ... Why didn't you take 8 times 90 is 720 at once then?

C. No, 'cause I think it was simpler, 'cause it is the multiplication table itself, sure, and then ... Then it became ... It was simpler so, and then I only put a zero after.

It was probably too complicated for her to use the multiplication table when computing 8 times 90. Ann wrote 4 x 200 = 800 twice; 1600 + 720 + 56 = 2300 + 70 + 6 = 2376. She found her mistake and could give the correct answer while Cecilia was discussing her solution.

## *Comments*

I cannot say that the girls became very skilful at doing multiplication, at least not in finding a fast and accurate way, for instance using the distributive property throughout. There were a lot of good attempts to do it, but as soon as the numbers involved and/or the multiplication facts to be used became more complicated, the girls resorted to repeated addition. We can see this very clearly in the observation of May in year 4, when the girls solved 2 x 212 by using the distributive property but stuck to repeated addition in one way or the other when solving 6 x 27. In the dialogue between Cecilia and me at the end of the section, she clearly declares that she can multiply 8 times 9, but that it was too complicated for her to multiply 8 times 90 in a corresponding way.

We can also see that the girls, especially Cecilia, preferred to use what they saw as simpler multiplication facts to more complicated ones. In March in year 5, Cecilia thus multiplied 6 times 30 and added 30 instead of multiplying 7 times 30 directly.

During these observations, they did not try any shortcuts that might come naturally to people more used to computing. For instance, they could not compute 10 times 35 directly, nor could they use this product to find the result of 5 times 35. Another example is the computation of 7 x 199. None of the girls took advantage of the fact that 199 is very near to 200.

Thus, we might ask if the girls learnt anything at all in multiplication. I believe that they did, although they would have needed much more time and practice to be able to understand and master the use of the distributive property. It might also be true that the class teacher and I should have given them more hints and examples to encourage them to use this property and above all to be able to find shortcuts like the ones just discussed. As I have interpreted social constructivism, such behaviour would not be in opposition to the theory, if only we could make sure that the pupils had a chance to understand our suggestions and make them fit in their own previous knowledge schemes.

## *Division*

In division, too, I could see different ways of tackling the problems. The girls got their first division exercise in the clinical interviews in April in year 4, 236 ÷ 4. Ann guessed and managed to see that 50 + 50 + 50 + 50 = 200 and 9 + 9 + 9 + 9 = 36. Cecilia first tried 200 ÷ 4 + 30 ÷ 4 + 6 ÷ 4. When she realised that 30 ÷ 4 was too difficult to compute, she turned to 36 ÷ 4 instead. Also Britta found the partition 200 ÷ 4 and 36 ÷ 4. Neither Cecilia nor Britta could see the last quotient as a division (or multiplication) fact. Instead they had to guess the result.

During the observation in May in year 4, I gave the girls the exercise 128 ÷ 4. Cecilia had no difficulties with it. She simply wrote 100 ÷ 4 = 25; 20 ÷ 4 = 5; 8 ÷ 4 = 2. Britta forgot the second quotient and got 27. However, when listening to Cecilia's explanation she could find the correct answer. Ann had to get a lot of help from Britta to do the computing properly. I call this method «division of parts».

It became a little more complicated, when the girls got the exercise 123 ÷ 3 in September in year 5. Ann tried 40 plus 40 is 80; 80 plus 20 is 100; 100 plus 20 is 120. She wrote 40 + 40 + 40 and then crossed the zeros to change 40 to 41. Cecilia first started with 100 divided by 3. She tried 25 plus 25 plus 25 and then in the same way with 30, 35 and finally 40. She also got some help from Ann's computations. She then divided 3 by 3 and got the final answer 41. Britta solved the exercise in about the same way as Cecilia.

In November in year 5 the girls were supposed to compute 128 ÷ 8. The exercise was given as a word problem of a partitive kind. Again Cecilia tried to divide the different powers of ten by 8, i. e. 100 divided by 8, 20 divided by 8, and 8 divided by 8. She managed to find the second quotient 2.5 and the third one 1. Then she got stuck. Ann did not get anywhere at all. She only wrote 10 eight times. Britta first tried to write 20 eight times, but she realised that it did not work, Then she wrote 15 eight times, added the tens first and probably the fives in pairs. After that she computed 8 divided by 8 is 1 and got the answer 16. Britta explained her solutions to the other girls. I asked her why she tried the number 15, and it seems as if it was by mere chance.

In the clinical interviews in March/April in year 5, two division exercises appeared. The first one, 525 ÷ 5, was to be solved mentally. Ann got the answer 106 by computing 500 divided by 5 is 100; 20 divided by 5 is 5; 5 divided by 5 is 1. Britta immediately saw that 500 divided by 5 is 100 and 25 divided by 5 is 5. Cecilia first suggested the answer 61, but when I asked her to consider her computation, she did a correct one by partitioning 525 in 500 + 20 + 5 and dividing each term by 5, just as Ann did. This is also an example of «division of parts».

In the other exercise, 348 ÷ 3, the girls were allowed to use paper and pencil. Ann used a method that I call «addition of parts». She immediately saw that 300 divided by 3 is 100. Then she wrote 10, three times, 4, three times, and 3, three times and gave the answer, 117. She was able to find the correct answer after some discussion with me.

Cecilia tried to divide 40 by 3 but without success. When I hinted to her that it might be more advantageous to divide another number by 3, she found the number 48. She then tried first 17 and then 16 as the quotient and thus arrived at the correct solution. She never discussed the division 300 ÷ 3. Probably, she saw it as self-evident.

Britta also started with the division 300 ÷ 3 = 100. After that she quite simply tried 15, and realised that 3 times 15 is 45. When she saw that only 3 was left in the dividend, she quite simply added 1 to 15 and got the final answer, 116.

The next division exercise was given in May in year 5, 1575 ÷ 15. The exercise was given as a word problem of a partitive kind. Ann could not solve it at all. Britta knew that 100 times 15 is 1500. Then she counted 5, 10, 15, ..., 75 and found that 15 times 5 is 75. She gave the answer 105. Cecilia wrote 100 fifteen times and found that it makes 1500 together. Then she tried to write 10 fifteen times and got 150, which was not an adequate result. She then said:

> *«Yes, and then I thought that if you could take half ... or if you could take 5, and then, it is half of 10, sure, so then it must be half of it, so it was 75, Britta said, so then it was in fact so.»*

On the last occasion, in April in year 6, Ann and Cecilia got a similar exercise, 1509 ÷ 3. Ann wrote 1509 ÷ 3 = 300 + 200 + 3 = 503. She reasoned 1000 divided by 3; 3 times 300 is 900; 100 plus 500 is 600; 6 divided by 3 is 2; 200; 9 ÷ 3 is 3. Cecilia only wrote 500; 503 and thought 1500 divided by 3 is 500 and 9 divided by 3 is 3.

They got another division exercise on the same occasion, 552 ÷ 6. However, it was too difficult for the two girls. Cecilia started with 300 divided by 6 is 50. She then tried 200 divided by 6 and then 250 divided by 6 and got stuck.

## *Comments*

The first thing that struck me was that the girls very often tried to partition the dividends in powers of ten. This is in line with social constructivism, as they could take advantage of such a partition in all the other three arithmetic operations. Unfortunately, this is not always possible in division.

However, the girls could often use an intelligent method of trial and error. For instance, we can see it in May in year 5, when Britta solved 1575 divided by 15. She saw that 1500 divided by 15 is 100, and then tried 5 as the quotient of 75 and 15. Cecilia, on the other hand first thought that this quotient was 10, but when she saw that the product of 15 and 10 was 150, she could realise, with some help from Britta, that the true value should be 5.

I could distinguish two different methods in division: «addition of parts» and «division of parts». In the first case the girls added a certain number as many times as the divisor stated to get a number belonging to the dividend, and in the second case they quite simply divided suitable parts of the dividend by the divisor. It should be added that the exercises that I gave the girls, were either without text or embedded in a context that made a partitive solution natural.

During the course of the experiment, I could see that the girls used more and more refined methods to solve division exercises. However, like in multiplication, I think it would have taken a lot more time to get them really to master even division with a one-digit divisor, to say nothing of a two-digit one. But again, I think we should ask ourselves the question: What kind of computations must our pupils be able to do without the use of calculators?

## The girls' opinions

I am going to discuss two questions from the interviews: *«Which method do you prefer: mental computation or written computation?»* and *«Which one do you prefer to use: a fixed rule that someone has taught you or your own method?»* The questions were given in years 3, 4, and 5. I will also add Ann's and Cecilia's answers to the second question in year 6, as I took the opportunity to talk to them about this issue during the observation.

When the girls were asked to choose between mental and written computation, both Ann and Britta preferred written computation in all three interviews. In year 3, Ann added that she got lost without paper and pencil and that it was easier to see the computations done.

Cecilia, on the other hand, preferred mental computation. In year 3, she made a reservation that it depends on the exercises, simpler ones are better solved mentally, and more complicated ones with paper and pencil. In year 4, she declared that written computation in the way it was done in class was boring, she had to write too much. She admitted that she did computations mentally, even if her teacher had told her to do them with paper and pencil. Still, she noted down very complicated computations.

As far as the question about the girls' preferred method, a fixed rule (column arithmetic, a standard algorithm) or the girls' own method, is concerned, it seems as if the girls got to like their own methods more and more during the course of the experiment. In year 3, Britta said that she preferred column arithmetic, while the two other girls thought that their own methods were the best ways for computing. However, Cecilia admitted that her parents taught her column arithmetic.

Ann and Britta changed their minds a little in the interview of year 4, although in different directions. They stated that both column arithmetic and their own methods were convenient. Britta preferred her own methods for complicated computations, while Ann pointed out that column arithmetic is a fast method. Cecilia, however, wanted to stick to her own methods; she said that she had a terrific amount of different methods, and that it depended on the exercise which one she chose. She did not look forward to learning column arithmetic.

In year 5, however, they all unanimously stated that their own methods were more advantageous. Ann said that it was easier to compute by using her own methods than to follow given rules, and Britta found column arithmetic complicated.

In year 6, Ann said that it was fun to learn algorithms but that she only used them in addition. Cecilia, too, found it advantageous to learn them, but she used them very seldom. As mentioned earlier, she did most of the computations mentally.

## *Comments*

It may be stated that these three girls preferred to use their own methods rather than to be taught standard algorithms. Even if we take into account that pupils tend to like the teaching methods that they have experienced, I think the girls' opinions should be taken seriously. At the same time, they could also appreciate the algorithms, although they themselves used them very little. They did not reject their own methods, although they had seen the advantage of the standard algorithms. However, they might have found it more advantageous to use them, if they had had more opportunities to practise them.

As far as written versus mental computation is concerned, the girls held different opinions. As I have stated before, I do not think it matters very much which method they prefer, as the same strategies are used in both cases.

## Discussion

In this paper I have chosen to let the readers follow a part of my project, the performance of three girls, who were generally working together during the observations. This means, of course, that a lot of results of the project have been omitted. On the other hand, it makes it possible for me to shed some light on the cognitive development of these very pupils from their second year at school until and including their fifth year, what they learnt and what special difficulties they came across. It might, perhaps, even be said that it would have been better only to follow these three girls and the corresponding three boys all through the experiment. The most important reason that I did not do so is, however, that of fairness. I thought all pupils should have a chance to talk to me in small groups.

We can see that the girls' solutions could sometimes be very primitive and complicated, e. g. Britta's solution of the subtraction exercise 25 - 18 in September in year 3 and Ann's solution of 35 - 16 in November of the same year. However, it has also been shown that these awkward solutions sooner or later developed into smart methods that could be used for almost all problems of a similar kind.

That did not mean, however, that the girls always used the same method for all exercises in a given arithmetic operation. They often looked at the numbers involved and tried to adjust their methods to them. In

October in year 4, Cecilia, for instance, recognised that 148 in the exercise 148 + 74, was very near to 150. She took advantage of this by adding 2 to 148 and subtracting the same amount from 74. In the interview of year 4, she also very clearly declared that she had a supply of different methods and that it depended on the exercise which one she chose.

Sometimes the girls were hesitant about changing methods, although they were shown more effective ones. Britta made this very clear when she was asked to multiply 35 by 10 in November in year 5. She stuck to her method of adding 30 ten times, although the other girls directly computed 10 times 30 is 300. I see this as a demonstration of the ideas of social constructivism. Britta felt a need to construct *her* method. Although she could understand and presumably even appreciate her peers' method, she was not ready to make it her own.

Although, as stated above, the girls' methods were sometimes rather primitive, we could see many examples of how they practised and developed their number sense. Britta gave a typical example when she was supposed to divide 128 by 8 in November in year 5. She saw that 20 times 8 was too much and therefore tried 15 times 8, which brought her very near to the dividend 128. It is tempting to compare the methods shown here with the standard algorithms, where you, in general, handle only units and sums and products of units. The girls, whom we have followed here, were forced to look at the numbers much more holistically than pupils following the given rules of the algorithms.

In addition, we can see very clearly how the pupils tried to help each other and how they learnt from each other. As early as in March in year 2 (addition) for instance, Britta tried to convince her peers that it did not do just to add 40 and 20 and add a one when doing the sum 49 + 22, it was necessary to change to the next ten. In November in year 3 (subtraction), Ann had taken over Britta's method to mark every number from 17 up to and including 35, although she did not note down the specific numbers.

Finally I will try to sum up the most important points that I think I saw when following this group of three girls.

- Although the girls sometimes had difficulties with some of the arithmetic operations, after a longer or shorter period of time, they could overcome these difficulties and find methods, which they understood and which could help them solve the exercises.

- The methods that the pupils used, were mostly less effective than the standard algorithms. On the other hand, they were more like those used for effective mental arithmetic and computational estimation. Thus, the pupils could avoid thinking in one way when doing written computing and in quite another way in mental arithmetic and estimation.
- From the methods the girls used, it could be seen that they acquired and developed good number sense.
- Even after they had been taught the standard algorithms, the girls preferred their own methods.

Admittedly, the girls' preference of their own methods in year 6 might be due to their greater experience of these methods during the foregoing years.

Anyhow, I think we have to consider whether, in the age of calculators and computers, it would not be wiser to take the chance to let our pupils develop their number sense and their skill in mental computation, even if it might cause a deficit in the use of the most effective written computational methods. However, we will have to pose the same question as I did in the comments earlier: What kind of computations must our pupils be able to do without the use of calculators? Personally, I believe that mental computation is not enough, I think our pupils need to be able to carry out at least simple written computations with some kind of support notes. In this project I have tried to find out what our pupils are able to do without resorting to mechanically following the directions of some standard algorithms.

At the same time, I think it is very important that our pupils feel that the methods which they use are really their own. To teach our pupils alternative methods instead of the standard algorithms, would, in my opinion, be totally wrong. In such a case, the algorithms would only be replaced by less effective methods, which would most probably be as unfamiliar and difficult to understand to our pupils. We would land up in the situation that Bauer (1998) fears, i. e. that these new methods would turn into new algorithms, albeit far less effective than the traditional ones. See the end of «Previous Discussion and Research».

## References

Bauer, L. (1998). Schriftliches Rechnen nach Normalverfahren – wertloses Auslaufmodell oder überdauernde Relevanz? *Journal für Mathematik-Didaktik, 19(2/3)*, 179 - 200.

Beishuizen, M., van Putten, C. M. & van Mulken, F. (1997). Mental Arithmetic and Strategy Use with Indirect Number Problems up to One Hundred. *Learning and Instruction, 7(1)*, 87 - 106.

Duffin, J. (1996). *Calculators in the Classroom.* Liverpool: Manutius Press.

Emanuelsson, G. & Emanuelsson, L. (1997). Taluppfattning i tidiga skolår. *Nämnaren, 24 (2)*, 30 - 33.

Hedrén, R. (2000). *Social konstruktivism i elementär aritmetik. Kan elever i år 2 - 5 göra skriftliga beräkningar utan de traditionella uppställningarna?* Högskolan Dalarna: Kultur och lärande. Rapport 2000:1.

Kamii, C. (1985). *Young Children Reinvent Arithmetic. Implications of Piaget's Theory.* New York (USA): Teachers College Press.

Kamii, C. (1989). *Young Children Continue to Reinvent Arithmetic. 2nd Grade. Implications of Piaget's Theory.* New York (USA): Teachers College Press.

Kamii, C. (1994). *Young Children Continue to Reinvent Arithmetic. 3rd Grade. Implications of Piaget's Theory.* New York (USA): Teachers College Press.

Kamii, C.; Lewis, B. A. & Livingston, S. J. (1993/94). Primary Arithmetic: Children Inventing Their Own Procedures. *Arithmetic Teacher, 41(4)*, 200 - 203.

Kilpatrick, J. (1988). Editorial. *Journal for Research in Mathematics Education, 4(4)*, 274.

Klein, A. S.; Beishuizen, M. & Treffers, A. (1998). The Empty Number Line in Dutch Second Grades: Realistic Versus Gradual Program Design. *Journal for Research in Mathematics Education, 29(4)*, 443 - 464.

Murray, H., Olivier, A. & Human, P. (1994). Fifth Graders' Multi-Digit Multiplication and Division Strategies after Five Years' Problem-Centered Learning. In J. P. Ponte & J. F. Matos (Eds), *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education*, 399 - 406. University of Lisbon.

Narode, R.; Board, J. & Davenport, L. (1993). Algorithms Supplant Understanding: Case Studies of Primary Students' Strategies for Double-Digit Addition and Subtraction. In J. R. Becker & B. J. Pence (Eds.), *Proceedings of the Fifteenth Annual Meeting, North American Chapter of the International Group for the Psychology of Mathematics Education.* Pacific Groves, CA (USA).

Plunkett, S. (1979). Decomposition and All that Rot. *Mathematics in School, 8(2)*, 2 - 5.

Ralston, A. (1997). A Zero-Based Curriculum: What It Is and How It Might Be Used. In H. Neill, & A. Ralston, (Eds.), *A Zero-Based Mathematics Curriculum. Proceedings of a Working Group at the Eighth International Congress on Mathematical Education, Seville, July 1997*, 1 - 8. The University of Nottingham: The Shell Centre for Mathematical Education.

Reys, B. J. & Reys R. E. (1995). Perspektiv på number sense och taluppfattning. *Nämnaren, 22(1)*, 28 - 33.

Reys, B. J.; Reys R. E. & Emanuelsson, G. (1995). Meningsfulla tal. *Nämnaren, 24(4)*, 8-12.

Reys, B. J. et al (1995a). Vad är god taluppfattning? *Nämnaren, 22(2)*, 23 - 25.

Reys, B. J. et al (1995b). Svenska elevers taluppfattning. *Nämnaren, 22(3)*, 34 - 40.

Rockström, B. (1996). *Matteboken.* 4A och 4B. Stockholm: Bonnier Utbildning.

Rockström, B. (1997). *Matteboken.* 5A och 5B. Stockholm: Bonnier Utbildning.

Shuard, H. et al (1991). *Prime. Calculators, Children and Mathematics.* London (UK): Simon & Schuster.

Vermeulen, N.; Olivier, A. & Human P. (1996). Students' Awareness of the Distributive Property. In L. Puig & A. Gutiérrez (Eds), *Proceedings of the Twentieth International Conference for the Psychology of Mathematics Education*, 379-386. University of Valencia.

**Abstract** (in Swedish)
I denna artikel diskuterar jag ett forskningsprojekt kring användning av alternativ till standardalgoritmerna för de fyra räknesätten och särskilt hur tre flickor, som deltog i projektet, arbetade. Flickorna blev inte undervisade om standardalgoritmerna under sina första fem skolår. I stället blev de uppmuntrade att använda sina egna skriftliga räknemetoder, inklusive användning av bilder, för alla slags beräkningar, som de inte kunde göra muntligt. De tre flickorna arbetade oftast tillsammans i en grupp. Inte förrän under sitt sjätte skolår fick de undervisning om standardalgoritmerna.

De resultat, som jag fick, visade huvudsakligen att flickorna klarade av att finna sina egna metoder, ofta på egen hand men ibland med kamraters eller lärares hjälp. Jag fann också att de metoder, som de använde, mestadels var mindre effektiva än standardalgoritmerna, men att de mer liknade effektiva metoder för huvud- och överslagsräkning. Flickorna skaffade sig god taluppfattning och god förmåga i huvudräkning, och de föredrog sina egna metoder även efter det att de undervisats om standardalgoritmerna.

**Acknowledgement**
The research project, which is partly accounted for in this article, was realised by courtesy of Högskolan Dalarna that let me work with the project as a part of my duties as a teacher educator.

I also want to thank the reviewers whose valuable points of views and advice have, in my opinion, helped to improve the article to a great extent.

***Author***
Rolf Hedrén is professor emeritus and a teacher educator in mathematics education in Högskolan Dalarna, Sweden. He has carried out research about the use of computers and calculators in mathematics education.

***Address***
Högskolan Dalarna, SE-791 88 FALUN, Sweden.
Tel: (+46) 23 77 82 97
Fax: (+46) 23 77 80 80
E-mail: roh@du.se