The Challenge of Teaching First-Year Undergraduate Mathematics:

Tutors' Reflections on *the Formal Mathematical Enculturation of Their Students*

Elena Nardi

Integrating the findings from a qualitative study of 20 first-year undergraduates' learning difficulties within a tutorial-based pedagogy at Oxford, the tutors' responses to and interpretations of these difficulties were studied in semi-structured interviews. Here the tutors' conceptualisations of the students' difficulties with regard to enculturation into formal mathematical reasoning and their standard teaching practices employed in order to cope with these difficulties are discussed. The conventions of school mathematical writing and journal/textbook/lecture writing and the students' confusion about what knowledge they are allowed to assume (school - university conflict, inter-university course conflict, intra-university course conflict) are identified as major influences on the students' formal mathematical enculturation. The need to transform teaching practices accordingly is highlighted.

Introduction

Axiomatic deduction, hailed by Hilbert (1918), as the most rigorous form of mathematical proof is the norm in the mathematical community. A careful look at the mechanisms of acceptance of a proof however reveals that, similar to the outcomes of all human activities, a mathematical proof is submitted to a context-dependent scrutiny. In other words acceptance of a proof is a sociocultural process (Hanna, 1991).

Above all, proving is convincing and the rhetorics of conviction are subject to a large number of communicational conventions. Moreover on the forefront of mathematical creativity, new mathematical proofs are often presented in elliptic, condensed forms that require a certain amount of suspense of disbelief from the reader. In that sense formal proof is the driving force and the aim of official

Note: The doctorate referred to in this article was supported by the Economic and Social Research Council in the UK. The follow-up study on which this article draws was funded by the Harold Hyam Wingate Foundation in the UK.

Elena Nardi *is lecturer of Mathematics Education at the University of East Anglia, Norwich, UK.*

mathematical communication but it is materialised on the basis of a number of conventions; these conventions are characteristic of the formal mathematical culture and their adoption is synonymous to a learner's advanced mathematical enculturation.

The notion of enculturation employed in this article departs from what is commonly thought in cultural psychology and anthropology as transmission of cultural practices (Bishop, 1991). Contemporary cultural theories move critically beyond a simplistic transmissive perspective. Within the culture of university mathematics, and in order to describe the systemic conventions of mathematical culture — semantic, linguistic and logical — as major determinants of a learner's cognition, the research on which this article draws employs Sierpinska's (1994) use of the cultural theories of E T Hall (1981/1959) and Michel Foucault (1973); in particular Foucault's *épistémê* and Hall's *cultural triad*. For the sake of conciseness and because of the relevance to the particular aspects of the research reported in this article I cite briefly the latter.

Hall recognises 'three types of consciousness, three types of emotional relations to things': the 'formal', the 'informal' and the 'technical'. In the context of mathematical culture the 'technical' level is the level 'of mathematical theories, of knowledge that is verbalised and justified in a way that is widely accepted by the community of mathematicians. At the 'formal' level, our understanding is grounded in beliefs; at the 'informal' level - in schemes of action and thought; at the 'technical' level — in rationally, justified explicit knowledge'.

Central to the purposes of the research reported in this article are processes taking place within the informal level of Hall's triad. This is, in Sierpinska's words, 'the level of tacit knowledge, of unspoken ways of approaching and solving problems. This is also the level of canons of rigour and implicit conventions about how, for example, to justify and present a mathematical result'. A novice's enculturation is seen here as taking place at the informal level: through the accumulation of mathematical experience shared with the expert and in the process of appropriation by an internalising imitation of the expert's cultural practices.

The research presented here is embedded in literature related to the learners' difficulties with formal mathematical proof and, more globally, to the difficulties of advanced mathematical reasoning. I cite briefly some of this literature.

The problematic aspects of the transition from the experimental and intuitive habits of school mathematical reasoning to the formal requirements of advanced mathematical thinking have been studied in the 70s by Bell (1976 and 1979) and consolidated in the work of R C Moore (1994). These findings suggest that students find proof difficult, unnecessary and meaningless. They view empirical evidence as proof and in fact prefer empirical arguments to deductive arguments (Martin & Harel, 1989; Porteous, 1990; Williams, 1980; Yerushalmy et al, 1990). Even when introduced to deductive proof students do not seem to appreciate its 'generic' aspect (Balacheff, 1990; Harel & Tall, 1991). As Fischbein (1982) notes, students are possibly not aware of the distinction between empirical and deductive arguments. Even when they are, says Schoenfeld (1987), who calls learners at this stage pure empiricists, they decline using deduction as a constructive tool for problem solving (Coe & Ruthven, 1994). As Chazan (1993) schematically proposes and elaborates, there seem to exist two types of problematic predispositions towards proof: students either see empirical evidence as proof or deductive proof simply as evidence. Duval (1991) attributes the learners' difficulties with mathematical proof to their confusion of deductive thinking with ordinary argumentation. The difficulty of dealing with the logic behind formal proof lies also in the fact that learners are overwhelmed by the content of the mathematical statements in a proof (Anderson, 1994) and are not able to move beyond content and into the realm of logical manipulation of the statements (Barnard, 1995). These difficulties with mathematical reasoning imply that students may, in view of these difficulties, avoid formalisation but they do not imply that they demonstrate no cognitive need for conviction and explanation (De Villiers, 1991). The issue therefore is to establish the necessary connections between this need and the learners' conscious or subconscious decision making about the explanation means they prefer.

A Study On University Mathematics Students' Learning Difficulties

The study reported in this article is a small-scale follow-up of the author's doctorate (Nardi, 1996). The doctorate was a psychological study of first-year undergraduates' learning difficulties. For this purpose twenty first-year mathematics undergraduates at Oxford were observed in tutorials (weekly one-to-one sessions in which the student discusses lecture-based mathematical problems with a professional mathematician, the tutor) for two terms. Tutorials were tape-recorded and fieldnotes kept during observation. The students were also interviewed at the end of each term of observation. The recordings of the observed tutorials and the interviews were transcribed and submitted to an analytical process of filtering out episodes that illuminated the novices' cognition. An analytical framework consisting of cognitive and sociocultural theories on learning, as well as literature in the area of Advanced Mathematical Thinking (Tall, 1991) was applied on sets of episodes within the mathematical areas of Foundational Analysis, Calculus, Topology, Linear Algebra and Group Theory. This topical analysis was followed by a cross-topical synthesis of themes that were found to characterise the novices' cognition. The findings were arranged in themes relating to the novices' difficulties regarding their image construction of new concepts as well as their adoption of formal mathematical practices.

As in this article the focus is on the students' difficulties to adopt formal mathematical practices, I summarise some of the doctorate's major findings in this respect (discussed in more detail in (Nardi, 1998) and (Nardi, submitted a)). I note that findings related to specific mathematical topics or concepts are discussed elsewhere (e.g. in (Nardi, submitted b and c)).

The students' difficulties in their encounter with mathematical formalism can be classified as

- avoidance of formalisation and
- uncritical and precipitate adoption of formalisation.

Difficulty in formalising leads to denial of formalisation and regression to more concrete and familiar modes of reasoning. Sometimes this denial is unconscious — the students plainly and uncritically extend their school mathematical practices to university mathematics — or conscious — the students reject formal reasoning as redundant once they are personally convinced. So, for example, they make tacit use of theorems, which they believe are obviously true. This is possibly a perpetuation of A-level attitudes and a regression to familiar from school modes of action.

Even when they have conceptualised the necessity to formalise, they still struggle with the materialisation of this conceptualisation: so, for instance, they assume in their proofs what is to them intuitively obvious or what they are actually being asked to prove. Sometimes, at least in the beginning, their rather over-zealous allegiance to rigour — as they perceive it — yields hesitation towards their school mathematical practices which in turn deters them from, for instance, using some basic arithmetical facts. So they seem to need to clarify the distinction between rigorous and intuitive arguments, legitimate and illegitimate use of knowledge that is thought of as previously established. In other words, they need an explicit articulation of the new didactical contract (Brousseau, 1989) of advanced mathematics.

A Study On University Mathematics Teachers' Perceptions Of Their First-Year Students' Learning Difficulties

The study which this article draws on is a small-scale follow-up to (Nardi, 1996) in which the tutors were invited to reflect and comment upon samples of data and analysis from the doctorate. It is necessary to stress here that, despite its contextual idiosyncrasies, a tutorial is a uniquely intimate learning environment, which offers a naturalistic field for observing novice and expert mathematicians at work. Given that the discussions in a tutorial mostly address the students' difficulties with the various topics, the richness of this source with respect to a psychological investigation on the learning of advanced mathematics, as well as a pedagogical investigation of an expert's response to this learning's difficulties, is self-evident. Moreover it has also been substantiated in the instant recognition of the issues raised in the doctorate by colleagues in international conferences who work in totally different learning environments to the idiosyncratic Oxford one. In sum this is a series of qualitative studies but the methodologies used have secured that these findings are germane to a wider range of settings.

Aims

The primary aims were: to provide feedback to the tutors who participated in the doctorate and enrich its findings by including the participant tutors' point of view; to introduce a pedagogical dimension in the psychological discourse developed in the doctorate; and to inaugurate a collaboration between mathematicians and mathematics educators involved in a subsequent larger-scale project (see concluding remarks) in the development of discourse and methodology.

Methodology of Data Collection.

For the above purposes, three tutors who participated in the doctorate were invited to participate in a series of semi-structured interviews. This choice resides theoretically in the methodological considerations, in particular regarding the interviewing of the students, in (Nardi, 1996) and in the literature regarding the teachers' reflections on their own pedagogical practices (e.g. Brown & McIntyre, 1993). The study espouses Schon's (1990) claim that we 'can learn from a careful examination of artistry, that is, the competence by which practitioners actually handle the indeterminate zones of practice - however this competence may relate to technical rationality':

> *'The central problem inherent in examining artistry in any profession stems from the fact that it is very difficult for an observer of the artist at work to "see" exactly how the artist acts and reasons; neither is the artist usually able to articulate in detail what underpins his thought and action.' (Schon, 1990 p13)*

This is Polanyi's (1967) 'tacit knowledge'. Moreover Schon (1990) asserts that:

> *'[through] countless acts of attention and inattention, naming, sense-making, boundary setting and control, [practitioners] make and maintain the worlds matched to their professional knowledge and know-how. They have .. . a particular professional way of seeing their world and a way of constructing and maintaining the world as they see it.' (Schon, 1990 p13)*

A major consideration here was that 'when teachers plan and prepare their teaching much of what they do is subconscious and draws upon knowledge that has become internalised over the years' (Jennings and Dunne, 1994). In these interviews the tutors were asked to 'bring to consciousness these areas of knowledge by examining their teaching in a structured way' (Jennings and Dunne, 1994). Thus it was intended that tacit 'processes and reasoning that underlie their practice' would become explicit.

Prior to the interviews the tutors were presented with samples of the data, transcribed extracts from the tutorials, and the analysis, presented in the doctorate. The samples were deliberately chosen to trigger tutors' reflection upon the students' learning processes, their own teaching actions as well as their response to the analysis in (Nardi, 1996). The interviewees were informed of this agenda in a note covering the samples of data and analysis that were to be discussed.

Methodology of Data Analysis.

The analysis of the interviews (Nardi, 1998) aimed at juxtaposing the analysis in the doctorate and the tutors' interpretations, as expressed in the interviews; and, moreover, at inaugurating reflection upon the tutors' teaching. The recordings of the interviews were transcribed and the contents of the transcripts were catalogued. Subsequently three analytical perspectives were applied on the data.

Analytical Perspective 1
This emerged from the need for a transition from looking at episodes from the tutorials from a learning to a teaching point of view, and an immersing in the relevant literature. So the data were used as an empirical basis in which to embed a reading of standard texts on teachers' thinking - for instance (Jaworski, 1994).

Analytical Perspective 2
Adomain oriented perspective. For this, all domain-specific extracts were isolated and cited along with findings in the literature and the thesis - for instance, on the difficulties of the students' formal mathematical enculturation (the issue addressed in this paper) as well as into specific proving techniques. The parts of the interviews referring to specific mathematical topics or concepts will be incorporated into relevant articles based on (Nardi, 1996).

Analytical Perspective 3
A strong incorporation of methodological considerations. Did the interviews trigger the participants' reflection on their students' mathematical thinking and on their own teaching practices? If yes, how? If not, why? The answer to this question is affirmative. A small set of categories emerged from the scrutiny of the transcripts:

R – S:
The tutor describes standard practices, or standard difficulties observed in the students, or standard difficulties in the teaching.

R – D – CEP
The tutor defends their practice in the episode by challenging the critique in the sample on epistemological or pedagogical grounds.

R – D – U
The tutor defends their practice in the episode against the critique in the sample by undermining the representation of the events in the sample.

R – R
The tutor re-evaluates, or even regrets, their practice in the sample either by simply agreeing with the critique in the sample or by engaging in reflection on the students' learning processes (Nardi, 1999).

Data and Analysis

In the following I present an intersection of the analysis within Analytical Perspectives 2 and 3: in particular, I have intersected the analysis relating to the R-S category of Analytical Perspective 3 with the domain-oriented concerns for the students' formal mathematical enculturation of Analytical Perspective 2. The discussion is arranged as follows:

1. The students' difficulty with speaking in a mathematically acceptable way.
2. The students' difficulty with writing in a mathematically acceptable way.
3. Influences on formal mathematical writing
   - of school mathematical writing, and,
   - of journal/textbook/lecture writing.
4. The students' confusion about what knowledge they are allowed to assume
   - school - university conflict,
   - inter-university course conflict, and,
   - intra-university course conflict.

Within each one of items 1-4 the argument unfolds in the following dialectical manner:

A. the tutors' statements regarding their conceptualisation of the students' difficulties and

B. the tutors' statements regarding their standard teaching practices for coping with these difficulties.

1. The students' difficulty with speaking in a mathematically acceptable way

As discussed in more detail in Section 3 the students arrive at a university mathematics course not quite alert to the necessity to speak mathematically accurately. The tutors are aware of this:

Tutor 2:
> Yes! And it's not getting any better. If anything it's slightly worse I think [with the new first year cohort]. And it's something that quite clearly now they are battling with. That they are recognising that this is an area that they find hard and that they are struggling to say things in the... an acceptable way.

Even when their students come up with the 'right' utterance, the tutors are concerned with whether this is merely an appropriation of mathematical form:

Tutor 3:
> Whether it is they learn the jargon — saying the right things about it rather than saying the wrong things about it — or whether they really kind of understand exactly, I am sure it varies.

Interviewer:
> There is a social and cultural element in learning the habit of saying ...

Tutor 3 :
> ... the right things ... Acceptable to the tutors!

This concern is crucial because, as the tutors repeatedly stated, clarity of language reflects clarity of thought.

Tutor 1:
> I try to make sure right from the start that students expound the mathematics clearly. If they are not expounding it clearly to me then probably they haven't modelled it in their own minds. But there maybe something wrong about that, I think that's just a theory.

So beyond awareness of their students' difficulty, are the tutors seeing this shift towards more accurate mathematical expression as part of their enculturating role? The views differ slightly. Tutor 2 is concerned that overtly and regularly interfering with the students' verbal expression may demoralise them and concentrates on doing so on their written work:

Tutor 2:

> *[pause]* What I have done quite a bit with them is to em, concentrate partly on the way they are writing things out. So I haven't worried so much about what they said, because I do not want to keep interrupting them all the time and picking them up 'you don't say that, you say this!', but, if they've got the right sort of idea, I let it pass. But hoping that then, if they manage to write it down more accurately, use the right er, the right language, then this would filter back into the way they talk about things as well.*[We agree that talking is much harder to control anyway.]*

Taking on a different, more active perspective, Tutor 1 outlines an approach, which is highly reliant on exposure to peers and peer explanation.

Tutor 1:

> If I am giving a tutorial with two or three students and ... one or more of the students are having trouble with a problem, but there is one of the students who can do it, has understood it, then, quite frequently, I'd be asking that student to explain the solution on the board, and the reason for that is [...] it is good for the students to learn how to present it at the board, sort of presentation skills em... it is also good for the student who has done it to present it in a way that's not just parroting something that might have been done in auto-pilot from elsewhere. But then thirdly em, generally speaking but not always, the students who have not understood it before, find it easier to understand one of their .... one of the other students than me. I've got all these fixed ideas in my head 'precision', 'use precise language like this', which they are still having difficulty with, so, if they hear it from somebody who is using language in a less em, er, defined way, someone who is using language in a way that they themselves are using it, then, quite often, they can understand it a bit better.

So the benefits of this approach for the students are: practising presentation skills, rehearsing mathematical proofs that may have been reproduced from a textbook and, intriguingly, building up an understanding of a proof on the basis of sharing 'less defined' language. In the learning episodes discussed in (Nardi, 1996), when the students were asked by Tutor 1 to 'defend' themselves on the board and stand a critical examination of their presentation by their peers, this shared 'imperfection' of language has made for paradoxically dynamic discussions. Of course, and in tune with the tutors' major concern that the students are constantly exposed to exemplars of mathematical proof (or, as Tutor 3 observed, '[the tutor] is for giving A correct proof') the expert intervention is inevitable:

Tutor 1:

> Once in a while I do get up there, particularly towards the end of the first year and the end of the third year, in revision classes and things like that, and particularly towards the end of a tutorial, when one has a limited amount of time and a problem the students have not been able to solve that we are dealing with, and then I produce a model answer as an example of what a model answer in [exams] might be. I try not to do that too often though.

In Section 2 the tutors' reflections on their students' difficulty with writing *in a mathematically acceptable way* are examined. It is worth noting that, as pointed out by Tutor 2, the tutors concentrate their efforts more consistently on their students' development with regard to formal mathematical writing despite potential links between their speaking and their writing. Tutor 3 articulated a rationale for this:

Tutor 3:

> Mmm *[pause]* I am not sure, I don't think... I mean usually the ones who have a problem expressing it verbally are usually the ones who have problems writing it down. And vice versa. So, to that extent, the scale of understanding is pretty much the same verbally and written.

However he continues making a crucial distinction:

Tutor 3:

> ... for one student is there a difference? And there I think it varies because some students are much more concerned with writing down what's acceptable to the tutor and gradually sense and others are more willing to write down what they're thinking. And the second type there won't be a lot of difference between their verbal discussion and what they're writing probably. Whereas in the first case there could be a lot of difference, the ones who are quite concerned with writing the clauses in analysis in the right order.

This statement resonates with this tutor's views quoted earlier where he highlighted a possible discrepancy between the students' verbal attempts at formal mathematical expression and what he called 'exact understanding'. Given this parallel, it is not surprising that the tutors placed particular emphasis on issues related to their students' difficulties to adopt formal mathematical practices in their writing (Section 2).

2. The students' difficulty with writing in a mathematically acceptable way

The evidence in (Nardi, 1996) suggests that the students' difficulty to adopt formal mathematical reasoning is nowhere more overt than in their writing. It is also in their writing that the tutors in these interviews were found to be able to discuss this difficulty in more detail. In resonance with (Nardi, 1996) the tutors suggested that the students quite often equate formal mathematical expression with steering clear of any use of ordinary language. Their writing is then a 'concatenation of formulae' as observed by one of the tutors.

Tutor 2:
> ... some of them seem to write sort of well straight off. Others learn it and then there are others who never really seem to learn it at all. I mean what I am talking about is they would write down an equation and then another equation and there is no connecting, there is no indication of how you connect or whether you are making an assumption. Sort of why you can write something down.

Interviewer:
> Do you think that their mathematical writing is mostly about using the symbols and that they are not very keen on explaining with words?

Tutor 2:
> Certainly initially. Initially they are not. They'll slightly shy away from it ...

Or:

Tutor 3: ...
> the way they are expected to present things is very new to them. Just, as one extreme, you get students who never write an English word if they can get away with it, they write the formula and then in the next line another formula and maybe three dots at the side sometimes. So there again just getting used to the fact that they are expected to write coherent English sentences in mathematics, is quite novel to them.

Year 1 then seems to be a paramount formative experience since, in the exams at the end of it, the students demonstrate a remarkable, if not always sufficient, acquisition of reasoning skills, as noted by Tutor 2:

Interviewer:
> The equation is the end product of a whole thought process. Do you find this in exam papers too?

Tutor 2:

Yes, some are excellent but there are some ... I mean it's not as bad as when they first come up. But they still give you very little idea about why they write things down and generally you have to give them the benefit of the doubt. [...] [in the exams] you accept a lot more than what you would actually do if they were doing it in the tutorial.

The tutors' views on this issue were almost unanimous about their students' accepting the necessity for, understanding as well as constructing a proof. Tutor 3 subtly distinguishes between these three aspects because his experience from his students, unlike Tutors 1 and 2, varies with respect to each one of the three:

Tutor 3:

[...] it's a serious problem in the first year in university that they haven't picked up any idea what a proof is in school.[...] It possibly is a little bit more now than it used to be in specific technical ways, like they might not be familiar with mathematical induction, for example. It's not in the common core anymore. But I don't think you are asking about specific technical points.

Interviewer:

About the notion of deductive proof, axiomatic reasoning.

Tutor 3:

Well, maybe I am just insensitive but I think most of my students kind of know what a proof would look like. It's just they are having difficulty finding it sometimes!

Interviewer:

Yes, which is a different thing. What we are talking about is whether they have a sense of proof and..

Tutor 3:

I think mostly they do.

Interviewer:

... and of the intellectual obligation to provide a proof for any statement you make.

Tutor 3:

Yes, hm, ... up to a point they would accept things as obvious that we re-indoctrinate them to think they should give a proof of. There is that element but most of them have a notion I think, if they know it is something they are supposed to prove, they would recognise a proof if they see it. Not all of them but most.

So his students would not always recognise when proof is necessary and would not always be able to construct a proof but would know what constitutes a proof when given one. The evidence in (Nardi, 1996) differs from this view and is more in accord with the other tutors' views. Tutors 1 and 2 agreed that their students were not very clear about what constitutes proof and that their mastering of the logical tools that are necessary for the production of a proof was limited. Tutor 1 exemplified his views as follows with regard to proving that if df/dx=f then $f(x)=e^x$:

Tutor 1:

> Yes, I think always the aim is to be clear. I mean for example something like this. Em, it maybe that in the question they've been given information about f of a different kind, df/dx=f, and they've been asked to prove this. If they then write what is in the first line, well then I am going to ask what is the status of this assertion. And it emerges rather shamefacedly that this is what they are seeking to prove, it isn't something we know at this stage, it is something that we will know in the future. Alternatively it may be that this is the assumption. Or perhaps they are trying to prove that f of x is not equal to e to the x, which is the conclusion, and in this case they are trying to do it by contradiction, starting from the assumption that f of x is equal to e to x, in which case it must be absolutely clear that this is an assumption.

Tutor 1, who also (Section 1) is a fervent advocate of his students' frequently articulating verbally their mathematical thoughts during his tutorials, is quite willing to incorporate telling pictures and diagrams in his teaching as well as to provide sketchy outlines of proofs. When confronted with the evidence that, because of their convincing power, these means are often misconstrued by the students as equivalent to proofs, he admits to the difficulty of conveying a sense for proof especially to his novice students. However he sees this elliptic and condensed approach as inherent to mathematical professionalism and insists that his students are exposed to it on a regular basis:

Tutor 1:

> Yes, yes, that's er, one of the things that I find very difficult to get across in the first year. Later on in the second and in the third year I find it em, somewhat easier, em, there comes a point, doesn't there, where ... you no longer have to give a proof, you have to give a proof that if challenged you could give a proof. Em, and that happens both as a professional technique and, of course, just as an examination technique. An examiner will use the... will use a sketch of a proof or a proof with several steps missing in finals as being a proof knowing that the missing steps are in some sense obvious, that between professionals you can trust each other to fill in those gaps.

Interviewer:

Yes, there is a tacit agreement.

Tutor 1:

And it's that level of professionalism that students have to develop [...]. And graduate students have also, they also have this problem to know to how much detail one must go. Em, with the first year Analysis and the first year Algebra, there is a really quite serious problem because new students don't yet know, don't yet have the experience or the background, the context to know what is acceptable as proof, what is acceptable as proof that if challenged you could provide as a proof, which is why teaching and learning in the first-year is so much harder than later on.

His views on exposing students to how mathematicians actually produce and present mathematics materialise also in his statements about his treatment of mathematical writing regardless of its origin (under or post – graduate students, established colleagues). For him, the rules of mathematical precision are overarching and apply to novices and experts alike. He exemplifies his views by elaborating what the various meanings of $x \in \Re$ can be in a mathematical sentence:

Tutor 1:

... this is something that I do at every level. Em, not just with undergraduates, also with graduates and also I dare to say with colleagues particularly when I am refereeing papers. Em, the language of mathematics is very difficult and em, ... er, ... well, for example [...] making it absolutely clear what quantifier is intended. You see em, if you say em, *[writes on the b/b]* $f(x)=e^x$ $(x \in \Re)$, that would naturally be read as a universal quantifier, wouldn't it? Em, but sometimes, sometimes the context is an existential quantifier, em, and, you see people in lectures and seminars or undergraduate lectures or undergraduates in their writing using exactly the same phraseology, right, to mean for some x, where x is in $\Re$. And then although you can usually ... you can get what was intended from the context, nevertheless the eye has to read it two or three times before it discovers what is meant. So, I try to train undergraduates and graduate students always to make existential and universal quantifiers explicit and besides of which you see although in the displayed formula you can always put a quantifier between the equation and the condition, if it's within text there's only a variable space in between and that can sometimes become very small and then you get this concatenation of formulae. So I would always put in there 'for $x \in \Re$' or 'for all or for some'. As far as that goes I treat undergraduates exactly as I treat graduate students or colleagues.

Analogously to this subtle, microscopic approach, the other two tutors outlined their ways of transforming their students' writing – apart from annotating the students' written work handed in weekly:

Tutor 2:

> ... you say this is good practice, you are telling a story: set the scene, explain what you are going to do and how you are planning to do it.

To illustrate this 'good practice' this tutor, acknowledging also the increased needs of her new students (Section 3) for more instruction on this issue, has recently introduced workshops in which particular mathematical problems are dealt with from the point of view of proof presentation. Her insistence on improving her students' written presentation skills go beyond the purely mathematical:

Tutor 2:

> ... they ought to be learning how to organise their material for and actually write comprehensible reports when they go out to work.

On a more general note, underlain by this tutor's belief in adjusting to his students' individual style and needs, Tutor 3 suggested:

Tutor 3:

> And I think there is a wide range of what you can do: from, at one end, characterise it as brain-washing, just tell them this is the way you got to do it, and the other end, let them reinvent the wheel by the Socratic method which is very much slower. And I think most students are probably somewhere between these two extremes and where you hit the balance is probably the tricky bit. There's always a temptation because of the pace of the thing to tell them 'this is how it is'! [And sometimes I find myself] too far towards the brain-washing end [and] in terms of the scale of the discussion from brainwashing to reinventing the wheel, I think the same probably applies to the discussion of concepts. Probably.

3. Influences of school mathematical writing and of journal/textbook/lecture writing on formal mathematical writing

The students' difficulties to accept the necessity for proof and moreover adopt formal mathematical practices was heavily attributed by the tutors, and in agreement with (Nardi, 1996), to the mathematical reasoning that the students are accustomed to in their A-levels. A number of episodes were discussed in the interviews, mostly in the area of Foundational Analysis and Calculus, where the discrepancy between school and university reasoning mathematical practices was highlighted. In this respect university syllabi are attempting to adjust to the needs of the new cohorts of students (Kahn & Hoyles, 1997):

Tutor 2:

Well, there has been lot of talk about the fact that A' levels are changing and it has become apparent that ... I mean that they are certainly as well prepared for the courses. I mean we have changed the courses because we have made it not progress so rapidly in the first year.

University mathematics teachers cannot assume that a lot of time or energy has been spent in school on mathematical thinking, as they know it. The tutors elaborated this unanimously but Tutor 1 offered a slightly different, more holistic view on the issue:

Tutor 1:

You never could, you know, I think it's a bit romantic to think that we could 30 years ago.

Interviewer:

No, but it's quite certain that they now come here with a lot less experience in formal mathematical thinking from school.

Tutor 1:

That's a serious problem but it's not just formal mathematical thinking. It's not just the mathematics. It's on reasoning of all sorts: students have not learned to use language accurately to reflect reasoning em, nor indeed to use it accurately in descriptive writing. I think that used to be better. Well, you know we used to get an awful lot of students 30 years ago who had technically been trained in that sort of thing but didn't really cope with it. Em, now we've got students who haven't been trained in that sort of thing and don't really cope with it. I don't think it's a really big difference. It's a bit disappointing because I think if one took students from the age of 9 or 10 as a matter of course in all their lessons, their mathematics lessons, their English lessons, their history lessons, their geography lessons, everything, simply expected clear expression, clear reasoning, the understanding of what a hypothetical is 'if this then something', the understanding of the converse isn't necessarily true ...this is an amateur diagnosis but I don't think this is a problem specific to mathematics but it shows up particularly with mathematics. Throughout I would like to see children of all abilities em, challenged a bit more in the way in which they use language in early age and be expected to use it rather precisely.

Appreciating logic and precision in the use of language seems to be an issue that transcends mathematical learning, even though it has traditionally been associated with it.

Furthermore an influence on the students' formal mathematical writing that was briefly touched upon was by the writing style of the lecturer or the textbook writer. The tutors, alerted to students' statements from (Nardi, 1996) such as 'I can not-justify myself on paper as long as I can justify myself in the tutorial', agreed that often their students reproduce the elliptic, concise and lacking in inflection lecturer's on-the-board writing style (lectures 'do write up everything in complete sentences but others do give abbreviated notes' noted Tutor 2) or the logical-leap containing style of textbooks. The fusion of the various styles of mathematical writing in different contexts was briefly touched upon in Section 2. This issue was only peripherally discussed in these interviews but it remains an intriguing area for further investigation.

4. The students' confusion about what knowledge they are allowed to assume

A School - University Conflict.

An issue which was quite prevalent in the analysis in (Nardi, 1996) of what constitutes the transition from school to university mathematical thinking problematic was, not only the students' lack of awareness of what necessitates and constitutes proof in mathematics (Section 2), but also, their confusion as to what part of the mathematics they learnt in school they are still allowed to use. While, especially in the first term, still struggling with the idea of building up mathematical ideas on axiomatic reasoning and deduction, the students develop a sensitivity about their previous knowledge which often leads them to take their tutors' cautionary comments to extremist approaches such as 'wipe out all previous knowledge of maths'. The students seem to be totally at sea at this stage:

Tutor 2:

> Even later as well. And it's still a problem with me: certainly when you are presented with a school's question and you think 'well, where am I supposed to start?'.

Interviewer:

> How do you cope with that?

Tutor 2:

> You just have to make your best guess. What seems, what actually producing in an answer that you think is going to be appropriate.

Interviewer:

> Em, how would you cope with a student who said something like 'can I assume the existence of the irrational numbers?' or [...] when they say 'can we use the algebra of limits? Isn't it imprecise?' even though they have seen the proofs in the Continuity course but they don't accept ...

Tutor 2:

> ...that it's going to work in general. [...] They certainly do em, I mean, in the questions I set them, I try to make clear to them what they can assume, or what they can't, or make it clear from the context where it is they are working from. Em, but you can still get misunderstandings where they thought they had to prove something, which was originally there for them to use.

The tutors, even though they acknowledged that they were occasionally troubled by the issue, were not as keen to elaborate. As more generally with regard to clarifying the rules of the formal mathematical game, this is an area where a reconsideration of teaching practices seems impertinent.

An Inter-University Course Conflict

Conflicting perceptions of mathematical validity do not only occur between school and university mathematics; they also occur between different first-year courses. In the case of this study these courses were Continuity-and-Differentiability and Analytical-and-Numerical-Methods: in the former the students are allowed to assume and use only theorems that have been proved; in the latter they use mathematical methods regardless of prior rigorous establishment. The tutors acknowledge this problem unanimously. For example:

Tutor 2:

> ...they wouldn't know what no... yeah. And again I think it's due to the difference between pure maths and applied maths. Em, ... *[pause]* I hope we do make it clear that in the applied areas we are really talking about the methods, it's the method we are worried about, the method we are applying and not justifying it, [that there are] different approaches to the different subjects.

And:

Tutor 3:

> [It is] interesting that there is another game they have to learn: to play some subjects by different rules than others as far as standard of rigour it goes and so on. And yes, certainly I have students who have difficulty with that.

But also they add that making the distinction explicit is part of their standard tutoring role:

Tutor 1:

> ... I am trying to give them methods for evaluation of what is em, better, that is say giving them a critical apparatus, giving them a way of evaluating that these arguments are more satisfying than those because they can be taken back to First Principles, they are much more quicker and so on. Em,...

And:

Tutor 3:

> Oh, I think it should be explained to them. Quite openly. That there are quite different sets of rules. Otherwise how are they supposed to guess that?

Beyond an articulate acknowledgement of the problem and also expressing a willingness to make these 'rules of the game' explicit, the tutors were less inclined to talk about transforming this necessary help to their students in more institutional ways.

An Intra-University Course Conflict.

The tutors touched upon inconsistencies analogous to the School - University and the Inter - University Course ones even within the same course:

Tutor 3:

> And the same phenomenon appears even within a given course that different parts are played with different rules. For instance you might be discussing continuity and differentiability and the Mean Value Theorem in very rigorous terms but then on some examples you maybe using the sine function, say, which you've never defined, and you're still going to assume properties like what the derivative of it is and so on. For the purposes of illustration, you have to learn also ... so that's another business where the rules vary according the different topics or aspects of the same course even.

The tutors agreed that these varying rules ought to be clarified as they seem to contribute to the piling 'fuzziness' (Briginshaw, 1987) about the rules of the formal mathematical game that their students need to adjust to. However they didn't seem to have an explicit agenda of how this clarification takes place in their tutorials and there was little evidence of it in their tutorials (Nardi, 1996). The research mentioned below seeks more evidence on this crucial issue.

Synthesis and Concluding Remarks

The tutors agreed that their students are generally not alert to the necessity to speak mathematically accurately and, even when they do, the tutors were concerned with whether this represents a genuine adjustment of practice. To them this concern is crucial because clarity of language reflects clarity of thought. With regard to their enculturating role in this, one tutor expressed a concern that regular intervention may result in demoralisation; another expressed a preference for exposure to peers and peer explanation for the sake of practising presentation skills, rehearsing mathematical proofs and building up understanding by sharing 'imperfect' language. Exposure to model proofs was also deemed inevitable.

No difference was identified between weakness in speaking and weakness in writing mathematically. However 'thinking aloud' in writing was valued as more revealing and attracting efficient intervention by the tutor. The tutors also agreed that students often steer clear of using ordinary language and their writing resembles an unjustified 'concatenation of formulae' – often as imitation of the writing witnessed in lectures. This improves towards the end of Year 1 but not always sufficiently.

The tutors distinguished and identified their students' difficulties with accepting the necessity for, understanding and constructing a proof as well as their own difficulty to convey a sense for proof to the students. However exposure to and emulation of the commonly elliptic and condensed format of mathematical proofs was deemed a valuable exercise in mathematical professionalism. In this respect the need for extra help was recognised (workshops on proof were suggested) also because improving students' written presentation skills relates to their writing of reports in their professional lives. An adjustment to the students' individual needs was also highlighted as important.

Difficulties to adopt formal mathematical practices was largely attributed by the tutors to their students' previous mathematical experience at A-level but a more holistic explanation – mathematics is only the most obvious area where a more general lack of rigour and precision in the acquisition of skills at school is reflected – was offered. The tutors agreed with (Nardi, 1996) that the students are not offered a clear framework with regard to what parts of their school mathematics can be assumed and that this extends to inter- and intra- university course conflicts concerning these assumptions. The need for clarification was acknowledged but very little substantive teaching practice was discussed.

This study exemplified the need for re-examining and partly reforming standard practices in the teaching of undergraduate mathematics with regard to formal mathematical enculturation; it also highlighted that it is crucial that this reform is grounded in a scrutiny of the learner's cognitive needs. The aims of these interviews, as outlined earlier here, were certainly fulfilled. In sum – and I offer these also as an evaluation of the methods used in this project:

- The tutors engaged in an articulation, justification and often reassessment of their teaching actions in the discussed episodes, or even more generally. Occasional inconsistencies in their practices were also highlighted.
- The tutors engaged in a scrutiny of the evidence on their students' thinking, a task for which any time is rarely allowed. This often amounted to their gaining an awareness of existing research literature, for instance, in specific areas of learning difficulties in undergraduate mathematics.
- Certain analytical themes from (Nardi, 1996) were enriched by the tutor/practitioner's point of view, e.g., in the extracts illustrated here, regarding the difficulties of the students' formal mathematical enculturation.

Offering the voices of professional mathematicians reflecting on their students' difficulties serves a bilateral purpose: gaining crucial insight into the learner's and the teacher's minds as well as evaluating reflection as a method for conceptualising and potentially reforming one's teaching practices. Both these components were acknowledged by the interviewees in this study in their emphasising that the impact of this exchange, in particular when taking place on a regular basis, on the tutors' perception and enactment of their role can be significant. These considerations have been built into the formation of the aims and the methodology of a project, funded by the Economic and Social Research Council in the UK, currently in progress at Oxford. In this, 6 tutors are observed in their weekly tutorials. They are subsequently interviewed about their teaching practices in specific learning incidents from the observed tutorials. The aim is to characterise these practices and to elaborate the developing realisation that a reform of the university mathematics curriculum should mostly be focusing on teaching.

References

Anderson, J. A. (1994) The answer is not the solution: inequalities and proof in undergraduate mathematics. *International Journal of Mathematics Education in Science and Technology*, ***25(5)***, 655-663.

Balacheff, N. (1990) Beyond a psychological approach: the psychology of mathematics education. *For the Learning of Mathematics*, ***10(3)***, 2-8.

Barnard, T. (1995) The impact of 'meaning' on students' ability to negate statements. In L. Meira & D. Carraher (Eds) *Proceedings of the 19th Annual Meeting of the International Group for Psychology in Mathematics Education*, Vol.2, 3-10. Recife, Brazil.

Bell, A. W. (1976) A study of pupils' proof explanations in mathematical situations. *Educational Studies in Mathematics*, *7(1)*, 23-40.

Bell, A. W. (1979) The learning of process aspects of mathematics, *Educational Studies in* Mathematics. ***10(3)***, 361-385.

Bishop, A. J. (1991) *Mathematical Enculturation.* Dordrecht / Boston / London, Kluwer: Academic Publishers.

Briginshaw, A. (1987) Myth and reality in teaching undergraduate mathematics. *International Journal of Mathematics Education in Science and Technology*, ***18 (3)***, 327-334.

Brown, S. & McIntyre, D. (1993) *Making Sense of Teaching.* Buckigham, Philadelphia: Open University Press.

Brousseau, G. (1989) Le contrat didactique: le milieu. *Recherches en Didactique des Mathématiques*, ***9(3)***, 309-336.

Chazan, D. (1993) High School Geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics*, ***24(4)***, 359-387.

Coe, R. & Ruthven, K. (1994) Proof practices and constructs of advanced mathematics students. *British Educational Research Journal*, ***20(1)***, 41-53.

Duval, R. (1991) Structure du raisonnement deductif et apprentissage de la demonstration, *Educational Studies in Mathematics*, ***22(3)***, 233-261.

Fischbein, E. (1982) Intuition and proof. *For the Learning of Mathematics*, ***3(2)***, 9-18.

Foucault, M. (1973) *The Order of Things: An Archaeology of Human Sciences.* New York: Vintage Books.

Hall, E. T. (1981/1959) *The Silent Language.* New York: Anchor Press, Doubleday.

Hanna, G. (1991) Mathematical Proof. In D. Tall (Ed.) *Advanced Mathematical Thinking.* Dordrecht / Boston / London: Kluwer Academic Publishers.

Harel, G. & Tall, D. (1991) The general, the abstract, and the generic in advanced mathematics, *For the Learning of Mathematics*, ***11(1)***, 38-42.

Hilbert, D. (1918) Axiomatisches denken. *Mathematische Annalen*, 78.

Jaworski, B. (1994) *Investigating Mathematics Teaching: a Constructivist Enquiry.* London / Washington D.C.: The Falmer Press.

Jennings, S. & Dunne, R. (1994) Training teachers of mathematics: the role of the mathematics subject tutor. *Teaching Mathematics and Its Applications*, ***13(2)***, 57-67.

Kahn, P. E. & Hoyles, C. (1997) The changing undergraduate experience: a case study of single honours mathematics in England and Wales. *Studies in Higher Education*, ***22(3)***, 349-362.

Martin, W. G. & Harel, G. (1989) The role of the figure in students' concepts of geometric proof. *Proceedings of the 13th Annual Meeting of the International Group for Psychology in Mathematics Education.* Vol 2, 266-273.

Moore, R. C. (1994) Making the transition to formal proof. *Educational Studies in Mathematics. 27(3)*, 249-266.

Nardi, E. (1996) *The Novice Mathematician's Encounter With Mathematical Abstraction: Tensions in Concept-Image Construction and Formalisation.* University of Oxford: Unpublished doctoral thesis.

Nardi, E. (1998) Tutors' reflections upon the difficulties of learning and teaching mathematics at university level: a report of work-in-progress. *Proceedings of the Conference of The British Society for the Research into Learning Mathematics,* Birmingham, 79-84.

Nardi, E. (1998) The Novice Mathematician's Encounter With Mathematical Formalism: A Sample of Findings From a Cross-Topical Study. *The Teaching and Learning of Undergraduate Mathematics Newsletter.* Electronic publication Issue No 9 available at <http://www.bham.ac.uk/ctimath/talum/newsletter>.

Nardi, E. (1999) Using Semi-structured Interviewing to Trigger University Mathematics Tutors' Reflections On Their Teaching Practices. In O. Zaslavsky (Ed), *Proceedings of the 23rd Annual Conference of the International Group for Psychology in Mathematics Education.* In Volume 3, 321-328. Haifa, Israel.

Nardi, E. (submitted a) Rigour Versus Intuition: Extremities of an Unjust War. Under review by the *Journal for Research in Mathematics Education.*

Nardi, E. (submitted b) In Search of a Meaning for Order of an Element, Cosets and F.I.T. for Groups : Mathematics Undergraduates' Use of Linguistic and Geometric Metaphors and Multi-Level Abstractions in Group-Theory. Under review by *Educational Studies in Mathematics.*

Nardi, E. (submitted c) Zeros and Ones in Advanced Mathematics: Transcending the Intimacy of Number. Under review by the *International Journal For the Learning of Mathematics.*

Polanyi, (1967) *The Tacit Dimension.* New York: Doubleday and Co.

Porteous, K. (1990) What Do Children Really Believe? *Educational Studies in Mathematics, 21(6)*, 589-598.

Schoenfeld, A. H. (1987) Confessions of an accidental theorist, *For the Learning of Mathematics. 7(1)*, 30-38.

Schon, D. (1990) *Educating the reflective practitioner: towards a new design for teaching and learning in the professions.* Oxford: Jossey-Bass.

Sierpinska, A. (1994) *Understanding in Mathematics.* London / Washington D.C.: The Falmer Press.

Tall, D. (1991) *Advanced Mathematical Thinking* (Dordrecht / Boston / London: Kluwer Academic Publishers.

Villiers, M. d. (1991) Pupils' need for conviction and explanation within the context of Geometry, In F. Furingetti (Ed) *Proceedings of the 15th Annual Meeting of the International Group for Psychology in Mathematics Education.* Vol 1, 255-263. Assisi, Italy.

Williams, A. (1980) Brief reports: High School students' understanding of proof, *Journal for Research in Mathematics Education, 11(4)*, 239-241.

Yerushalmy, M., Gordon, M. & Chazan, D. (1990) Mathematical Problem Posing: Implications for facilitating student inquiry in the classroom, *Instructional Science, 19(3)*, 219-245.

Forfatteren diskuterer problemer studenter opplever ved overgangen mellom skole og universitet. I artikkelen presenterer og diskuterer forfatteren funn fra en kvalitativ studie av vansker knyttet til formelle matematiske resonnement som 20 studenter i sitt første studieår møter innenfor et spesielt veiledningssystem ved University of Oxford og veiledernes tolkninger og reaksjoner på disse vanskene. Artikkelen rapporterer og tolker informasjonen fra semi-strukturerte intervju med tre veiledere. Forfatterens utgangspunkt er å betrakte møtet med matematikkutdanningen på universitetsnivå som en tilpassing til en ny kultur for studentene.

I artikkelen diskuteres veiledernes forestillinger om studentenes vansker med denne tilpassingsprosessen til formelle resonnement og deres bruk av tradisjonell standard undervisningspraksis for at studentene skal klare å overvinne disse vanskene. Et av problemene som studentene synes å ha er en uklar oppfatning hva et bevis er og på hvilket grunnlag det er legitimt å basere et bevis på i forskjellige sammenhenger. Dette blir sett på som et problem ved overgangen fra en didaktisk kontekst i skolen til en mer avansert matematikk ved universitetene.

Tre forskjellige analytiske perspektiver blir brukt til på dataene og to av disse, matematikkperspektivet og de metodiske overveielsene til veilederne, blir diskutert og analysert i artikkelen. Fra denne analysen blir det utviklet 8 kategorier til å beskrive observasjonene i studien.

Author
Elena Nardi is lecturer of Mathematics Education at the University of East Anglia, Norwich, UK.

Research Interests
Teaching and learning of undergraduate mathematics, psychology of mathematical thinking.

Address
Dr Elena Nardi
School of Education, University of East Anglia, Norwich NR4 7TJ, UK
Phone: 44 1603 592631
Fax: 44 1603 593446
E-mail: e.nardi@uea.ac.uk
Website: http://www.uea.ac.uk/~m011