A review of recent research on cognitive, metacognitive and affective aspects of problem solving

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Mathematical problem solving is a complex activity that necessitates thoughtful consideration about the best ways to teach it, since any two individuals can arrive at the same solution to a problem using different, but correct methods.

An identification of common traits that expert problem solvers possess, instruction related problem solving research, the contribution of cognition, meta-cognition and belief systems, are presented in this research review in an effort to weave a coherent picture of the state of the art of problem solving in mathematics today.

Some unrepresented and underrepresented issues are also considered, in order to provide evidence of the diversity and multiplicity of issues concerning mathematical problem solving research and the need for collaboration between researchers in this field of study.

The acquisition of mathematical problem solving skills is very important because it is an ability people need throughout life. Students face many problems with varying degrees of complexity. Problems for students arise when they attempt to understand concepts, relationships and acquire skills. That problem solving is considered important, can be confirmed by the fact that it has been the subject of research by mathematics educators, mathematics teachers, mathematicians, educational psychologists, cognitive scientists and philosophers since the turn of the century.

Recently, "The Curriculum and Evaluation Standards for School Mathematics" (N. C. T. M., 1989), recommended that:

> Problem solving should be the central focus of the mathematics curriculum. As such, it is a primary goal of all mathematics instruction and an integral part of all mathematical activity (p.23).

These sentiments are echoed in other national mathematics statements such as the Cockcroft Report (Cockcroft, 1982), and the Australian National Mathematics Statement (A. E. C., 1991).

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Problem solving has also been a primary stated goal of mathematics instruction since, at least, the 1930's. Schaaf (1979) cited the report of the Committee on the Function on Secondary School Curriculum, published in 1938. That Committee stated that:

> the study of mathematics is of educational value because mathematics can be made to throw the problem solving process into sharp relief, and so offers opportunity to improve students' thinking in all fields (p.1).

Evidence has accumulated over the years and has demonstrated to mathematics educators (see for example Taplin, 1992; Siemon 1988, 1992; Stacey 1987) that there is a genuine need for developing problem solving programs, instructional packages for primary and secondary schools, and research studies in Australia for improving the problem solving abilities of students at all levels of their education through systematic problem solving instruction. According to Taplin (1992) problem solving is an important component of mathematics education since it enables students to achieve mathematical competence through three values: functional, logical and aesthetic. Taplin (1992) further claimed that:

> Approaching mathematics through problem solving can create a context which stimulates real life and therefore justifies the mathematics rather than regarding it as an end in itself (p. 6).

Stacey (1987) shared a similar view by stating that:

> Developing the problem solving performance of students is now recognised as a major goal of mathematics courses throughout Australia (p. 21).

Definition of terms

Before further proceeding, two key terms—*problem*, and *problem solving*—will be defined.

A careful examination of many research papers (Schoenfeld, 1992; Lester, 1980; Bourke and Stacey, 1988; Davis, 1992; Sweller and Low, 1992) reveals the fact that there is no agreement among researchers as to what is a "problem" and even more disagreement is evident when trying to determine the nature of problem solving. Lack of agreement seems mainly due to the fact that mathematical problem solving appears to a certain extent to be so complex and subtle, as to defy definition and description.

Some researchers have defined problem solving as "a chaotic area of investigation" (Lester, 1980; Schaaf, 1979). Lester (1980) proposed that:

> A problem is a situation in which an individual or group is called upon to perform a task for which there is no readily accessible algorithm which determines completely the method of solution (p.287).

Sowder (1985) articulated a distinction between routine and genuine problems, based on schema training. In this model, if a schema is found then the problem becomes a routine exercise. If a schema is not found then the problem is classified as a genuine problem and more search is necessary. What remains unresolved is which schemata should be taught.

In an attempt to develop a "typology" of problem solving Shulman (1985) borrowed a model firstly proposed by Schwab (cited in Shulman, 1985, p. 440). According to this model there are *three* components to a problem: the statement of the problem, ways and means for dealing with it, and a solution. Shulman identified three types of problem solving: exposition, where the problem, ways and means, and the solution are all given, guided discovery, where the problem and ways and means are given but not the solution, and pure inquiry, where neither the problem, ways and means, nor the solution are given. Sweller and Low (1992) proposed a different conceptualisation of what a problem is:

> No task can be classed as either an exercise or a problem simply by referring to its structure and components. Alone, the structure of a task cannot reveal its "problem" status. Its status is revealed fully by the novice-expert distinction (p. 84).

Davis (1992) went further to define the notion of "universal" problems as "those problems that are known not to have been solved at a particular time (p. 183)". Problems of that kind are therefore problems to every person to whom they are posed.

We define a mathematical problem as a task posed to an individual or group, who will attempt to decipher the task and obtain a mathematically acceptable solution by not initially having access to a method which completely determines the solution. The extent to which the task would be a problem or not for a particular individual or group is a function of mathematical knowledge (general and task specific), executive control mechanisms, memory capacity, automation of appropriate skills, mathematical ability, utilisation of potential heuristics, and the mathematical maturity and creativity of the given individual or group. Problem solving can therefore be defined as the set of actions taken to perform a task, assuming that there exists a desire on the part of the individual or group to perform the task.

It is also important to clarify what is meant by a solution to a problem. Most researchers, with very few exemptions (Polya, 1957; Schoenfeld, 1985a; Holton, Spicer & Thomas, 1995), endeavour to define what constitutes a problem for the purposes of their study but not what constitutes a solution to a problem. In an attempt to decipher what *solution* and *more than one solution* mean in mathematics education, Holton, Spicer and Thomas (1995) formulated a typology of solutions a problem may have in increasing order of sophistication:

(1) Answer incorrect; method incorrect;

(2) no answer or answer incorrect; method unclear;

(3) answer correct; method inadequately described;

(4) answer incorrect; method correct;

(5) answer correct; method correct but elementary or laborious;

(6) answer correct; method correct using standard mathematics;

(7) answer correct using sophisticated ideals; and

(8) generalisation or extension provided to the original problem (p. 347).

An underrepresented issue in research studies is the issue of what is acceptable as successful problem solving. Lester (1985) adopted Mayer's (1982) suggestion that at least four types of knowledge are involved in successful problem solving in mathematics: (i) linguistic and factual, (ii) schematic, (iii) algorithmic and (iv) strategic. Lester (1985) added that the problem solver's "belief system" is important in successful problem solving and that:

> successful problem solving depends also upon knowing when and how to utilise such knowledge and upon having the ability to monitor and evaluate the application of this knowledge both during and after implementation (p. 43).

Dimensions of mathematical thinking and problem solving

During the past thirty years, the mathematics research community has become interested in instructional implications of cognitive theory and research on the learning of mathematics conducted by cognitive psychologists. Cognitive psychologists are seeking to explore and validate theories of human problem solving while mathematics educators are seeking to understand the nature of the cognitive and meta-cognitive interaction between learners, the subject matter of mathematics and the problems they solve. The implications from such research efforts for the practitioner carry significant gravity. These implications should be used as a guide in the design of mathematics curricula and instruction that will result in training students to be better problem solvers and to use their mathematical knowledge more effectively.

One of the goals of the mathematics curriculum should be to teach students to be mathematicians at their own level (Schoenfeld, 1985; 1992), rather than teaching them to just learn to perform routine mathematical operations. That means to guide the students in learning to think mathematically and discover the mathematical truths and relationships rather than teaching them some "techniques" to solve problems. These techniques are most of the time applicable to exercises, and not real problems, namely situations in which an individual or group is called upon to perform a task for which there is no readily accessible algorithm which determines completely the method of solution. Schoenfeld (1987), advocated that the:

> coupling of mathematics and cognitive science is essential to effective curricular and instructional reform (p. 9).

Schooling should try to demystify mathematics by creating appropriate classroom environments where students perceive mathematics as a sense - making activity, participate intentionally (desirably) in the activities and leave the classroom having understood the connections that tie together the procedures that they have studied. Current classroom practices indicate that students gain a fragmented sense of the mathematical subject matter, even if they master some procedures and skills. Recently, Lester (1994) claimed that:

> Although acceptance of the notion that problem solving should play a prominent role in the curriculum has been widespread, there has been anything but widespread acceptance of how to make it an integral part of the curriculum. To date, no mathematics program has been developed that adequately addresses the issue of making problem solving the central focus of the curriculum (p. 661).

Another central issue of research in mathematical problem solving is the emphasis being placed, by contemporary mathematics educators and cognitive researchers, on process rather than product. The main focus should be on how and why a student did what she did, while solving a mathematical problem, (or trying to solve the problem), and not just the correctness of the answer that the student produces.

> Obviously, the correctness of the approach a student finally settles upon is vital and important. What contributed to the ultimate success or failure of one particular search for a solution, or another, what knowledge did the student use, how and why, are vitally important too. Perhaps they are more important for remediation purposes, for gifted and talented instruction, for mathematical competitions training and for a fuller understanding of processes that govern problem solving.

Schoenfeld (1985b) has developed a useful categorisation of dimensions of mathematical behaviours which we will modify slightly in this article to characterise mathematical thinking and problem solving. Four categories of knowledge and behaviour necessary for an adequate characterisation of mathematical problem solving performance are:

- Cognitive resources, including intuitions and informal knowledge regarding the domain, facts, algorithmic procedures, routine non-algorithmic procedures, and understandings about agreed-upon rules for working in the mathematical domain.
- Control and metacognition, including planning, monitoring and assessment, decision making, and conscious metacognitive acts.
- Belief systems—Affect: One's *mathematical world view*, the set of (not necessarily conscious) determinants of an individual's behaviour, including beliefs about oneself, about the environment, about the topic, and about mathematics.
- Heuristics, including drawing figures, introducing suitable notation, exploiting related problems, reformulating problems; working backwards, testing and verification procedures.

Schoenfeld (1992) broadened his framework by enlisting a fifth category—*practices*. The practices of schooling—an anthropological perspective—and classroom environments play a facilitative or debilitating role in the students' mathematical problem solving performance. The first three categories will be reviewed in this paper.

Cognitive Resources

Cognitive processes when applied to problem solving, include all thinking done to solve a problem. Process factors include deduction, solving equations, and looking for patterns. A substantial amount of study has been directed throughout the 70s and the 80s toward linking problem solvers to the cognitive processes they employ. As Lester (1980) noted:

> the preponderance of research into problem solving can be classified as the "process tracing" variety. Process tracing approaches....attempt to describe the intellectual processes used by subjects as they render judgements and make decisions or solve problems (p. 301).

Highly competent performance in any complex domain is based on having at one's disposal a large pool of knowledge, experience in problem solving, and a large number of pieces of immediately accessible knowledge. These pieces have been referred to as *chunks* (Simon, 1980).

Chunks are said to be collections of related items of information represented by a single symbol or concept. Simon (1980) estimated that experts in complex domains have "vocabularies" consisting of about 50,000 chunks of knowledge. A large part of their expertise depends on the large number of chunking taking place while solving a problem and almost automatic responses to familiar situations. Research evidence has accumulated (Marshall, 1988, 1989; Sweller, 1988, 1990; Sweller and Cooper, 1985) indicating that competent problem solvers have almost automatic access—using appropriate memory routes—to the appropriate procedures and methods, once they recognise a stereotypical situation in the assigned problem.

There is a significant difference, in this domain, between "expert" problem solvers and "novices" in their characterisation of problems. Novices (students) often perceive only superficial or surface characteristics of the problem, while experts (mathematicians) perceive the deep structure of the problem and they use their working memory more efficiently. Research in this area includes studies on the role subject variables (characteristics of the individual) play in mathematical problem solving and concerns attempts to identify characteristics of good problem solvers (experts), in order to formulate testable hypotheses about what constitutes essential problem-solving skills and how good problem solvers differ from poor problem solvers (novices).

Stacey (1988) studied the influence some subject variables have on various aspects of the problem solving process. Stacey devised a model that portrayed the significance two subject factors were found to have, on the students' problem solving process. The problem solvers' mathematical knowledge dominated all aspects of their problem solving process while the confidence demonstrated by the students in attempting to explain their solutions, was reportedly evident throughout the problem solving process. Stacey (1988) concluded that her model has important instructional implications despite the non-inclusion of process and metacognitive variables in the study.

Suydam (1980) compiled a list of characteristics and relevant clues for teaching. Characteristics of good problem solvers are:

(i) The ability to understand mathematical concepts and terms

(ii) The ability to note likenesses, differences and analogies

(iii) The ability to identify critical elements and to select correct procedures and data

(iv) The ability to note irrelevant detail

(v) The ability to estimate and analyse

(vi) The ability to visualise and interpret qualitative or spatial facts and relationships

(vii) The ability to generalise on the basis of few examples

(viii) The ability to switch methods readily

(ix) Higher scores for self-esteem and confidence, with good relationships with other children

(x) Lower scores for test anxiety (Suydam, 1980, p.36).

One of the most cited researchers of individual mathematical problem solving abilities is Soviet psychologist V. A. Krutetskii. Krutetskii (1976) spent more than twelve years investigating the relationship between problem solving ability and perceptions of problem structure. He found that one major difference between good and poor problem solvers was in their perception of what constituted the most important aspects of the problem. This factor is important for studies on instructional aspects of mathematical problem solving since it presents the researcher with a factor open to improvement through instruction. For Krutetskii, good problem solvers are quicker to see a problem's structure and more able to generalise to problems having a similar structure. Kruteskii also found that good problem solvers are able to skip steps easily, are aware of rational quickest solution paths, and are able to retrace steps more easily than poor problem solvers.

According to Simon (1980) differences in problem solving success may be partly attributable to differences in the problem solvers' knowledge organisation. Simon (1980) postulated that there is no such thing as expertness without domain specific knowledge—extensive and accessible knowledge.

The idea of a memory schema has recently helped explain many aspects of human knowledge organisation and recall. Silver (1987) borrowed a description of schema from Thorndyke and Yelcovich (1980), as representing a prototypical abstraction of a complex and frequently encountered concept or phenomenon. As such, a schema is usually derived from past experience with numerous exemplars of the concept involved. Elaborating on the theory of problem solving schemata, Marshall (1988) hypothesised that a problem solving schema can have four distinct components. The first is a body of facts of knowledge that describes the general situation to which the schema applies. Some researchers have proposed the idea of developing schema-based interpretations of common problem solving phenomena and tried to explain observations of *Einstellung* (mental set) in terms of difficulties encountered in shifting from one schema to another:

> "Locked into a particular approach, the student lacks the flexibility to adapt to new circumstances" (Silver, 1987, p.48).

A second component is a set of conditions that must be satisfied for a schema to be instantiated. A third feature of schematic knowledge has to do with the mechanisms for setting goals related to its instantiation. The fourth is the collection of procedural rules that can be applied as the schema is implemented. As a consequence of the perceived importance of schema representations, the development of a new model of testing by using schema assessment has been proposed by Marshall (1988).

In elaborating on mathematical problem solving research mainly concerned with the cognitive domain, Sweller and his colleagues, (Owen and Sweller 1989; Sweller 1988, 1990; Sweller and Cooper 1985) conjectured that skilled problem solving in mathematics is determined by the students' acquisition of problem solving schemata, and the automation of rules, the latter being important in transfer performance. Cooper (1988) further claimed that novice problem solvers:

> need to solve problems using means-ends analysis and this imposes heavy constraints on cognitive processing capacity and misdirects attention away from aspects of problem structure that may be necessary to facilitate schema acquisition and rule automation (p. 56).

Sweller's work has received considerable attention from the mathematics education community, as an ongoing work that according to Putt and Isaacs (1992):

> continues to challenge one conventional mode of mathematics teaching which relies on presentation of new material followed by worked examples and a large number of practice problems or exercises" (p. 215).

Not all mathematics educators have embraced Sweller's work, however. A recent example is Lawson (1990) who juxtaposed his view that although the research evidence at that time (1990), was apparently not strong enough to support the thesis that instruction in general problems solving heuristics could enhance the problem solving ability of students, research evidence cited by Lawson (1990) justified the inclusion of general problem solving heuristics in the mathematics curriculum.

Control and Metacognition (Meta-processes)

Metacognition and metacognitive functions such as managerial functions, control processes, executive functions, executive schemes and reflective intelligence have received increased emphasis from researchers during the 80s and the 90s (Lester 1985; Silver 1987; Siemon 1988, 1992; Schoenfeld 1985a, 1985b, 1987, 1992). One of the problems about metacognition is that it means different things to different people, resulting in a confusion on what is and what is not metacognitive. Flavell's (1976) definition seems to be generally accepted, having incorporated two important aspects of metacognition, monitoring and regulation of ones own cognitive processes:

> Metacognition refers to one's knowledge concerning one's own cognitive processes and products or anything related to them, e.g., the learning-relevant properties of information or data. Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects on which they bear, usually in the service of some concrete goal or objective (p. 232).

Another reason for confusion is that it is not always easy to distinguish what is cognitive from what is metacognitive. Lester (1985) offered a simple explanation on how to distinguish between cognition and metacognition: Cognition is involved in doing, whereas metacognition is involved in choosing and planning what to do and monitoring what is being done.

It has already been claimed in this paper that extensive domain-specific knowledge is vital to success in problem solving. Teachers of mathematics need extensive training in using metacognition to regulate, monitor and evaluate their own problem solving representations and activities, if they are going to model mathematical problem solving behaviours in their classrooms. Another difficulty in using the domain specific knowledge potential at their disposal effectively is that individuals, while engaged in a problem solving task of some intellectual complexity, may be subconsiously or unconsciously monitoring and evaluating progress and feel that things are going fine. They may decide to do something different if they have evaluated the current state of affairs and found that the current approach must be abandoned for another. Keeping track of current state, or using a different approach if it is feasible, are both aspects of self regulation.

A critical issue not adequately researched is how we control or regulate subconscious processes embedded in our decisions about mathematical problem solving. Krutetskii (1976) in his seminal work on the investigation of mathematical abilities of talented high school children, attempted to clarify the problem of a sudden solution in mathematics. Krutetskii reported that:

> Often underlying incidents of sudden guessing or 'inspiration' was generalisation: the unconscious application of general methods of operation (or of an individual device) or general principles of an approach to the solution, based on the common (but at times very remote) properties of various mathematical objects, schemes, or problems (p. 306).

P. Sevarev (cited in Krutetskii, 1976), noted that:

> when solving complex problems the examinee usually tries to subsume a problem under a type he already knows, but he is not aware that this device is being implemented, and is not conscious of the general principle by which he is operating (p. 306).

It seems that there is in some way, an unconscious (or subconscious) activation during complex problem solving, still not well understood by mathematics educators and researchers. Research on metacognition indicates that successful problem solvers can reflect on their problem solving activities, have available powerful strategies for dealing with complex and unknown problems, and regulate (even subconsciously) powerful strategies efficiently. Novices, in contrast, have acquired fewer problem solving strategies, are less aware of the utility of them and do not use them effectively in the acquisition of new learning. Of particular importance to researchers in expert-novice differences is the psychological research on meta-memory, and the individual's awareness of the storage and retrieval of information. Psychological research has considered problems associated with the activation of inaccessible stored information and the incubation effects in human problem solving.

> By definition, the term incubation refers to an increased likelihood of successfully solving a difficult problem, a result of placing a delay between an initial period of intensive work on the problem and another subsequent period of conscious effort toward completing the problem's solution (Yaniv and Meyer, 1987, p.188).

Incubation is obviously related to research on memory, since as Yaniv and Meyer (1987) advocated in the same paper:

> An initial failure could occur because the stimulus configuration at the start of processing provides inadequate retrieval cues and/or because components of the memory traces are too weak for present purposes (p.189).

Siemon (1992) developed a theoretical metamodel in an attempt to explore the role of metacognition in primary school children's mathematical problem solving, in a series of two studies. The first study was a ten-week teaching experiment involving grade three and grade six classes. Siemon (1992) reported that:

> results of the initial study supported the view that metacognition as it was defined, is a "force" governing mathematical problem solving behaviour, and that for some students enhanced metacognition can be achieved by training, the complexity of the problem solving behaviour observed suggested that metacognition is not a single or even bi-value "driving force", but a multiplicity of complex "forces" and relationships not all of which may be operating in the same direction at the same time (p. 1).

The second study was a year long teaching experiment, involving fourth grade children and conducted from a constructivist perspective. The second study set to explore the role of metacognition on students' mathematical problem solving, and the viability and sufficiency of the meta-model. According to Siemon (1992), the meta-model, in its four generic approaches, can describe the student's behaviour, in mathematical problem solving through a cognitive/metacognitive continuum. The meta-model's four generic approaches vary according to two dimensions—high-low conceptual and high-low procedural. Siemon concluded that contextual setting, specific content knowledge, beliefs, and motivations and values play an important role in students' mathematical problem solving. Another conclusion of the study according to Siemon (1992) was that:

> many problem solving efforts failed not necessarily because of a lack of monitoring ability per se or even a lack of knowledge as to what strategies to use and when, but because of a lack of access to specific content, and the inability to apply monitoring strategies (p. 8).

Absence of self-regulation or the inability to apply monitoring strategies while solving a mathematical problem could lead to catastrophic results. Students can exert themselves in wild goose chases exploring a very limited sector of the multidimensional problem space, and fail to obtain a solution to the problem they are dealing with. A common byproduct of wild goose chases, observed almost everyday in mathematics classes, is students' disappointment. Schoenfeld (1987) analysed an extensive series of video tape sessions of students solving a non-standard geometric problem. Students spent most of the time available (20 minutes) *doing* rather than *thinking*. Schoenfeld (1987) contrasted the attempt of a mathematician solving a difficult geometric problem, to that of these novice students. The professional mathematician was not a geometry expert, but he considered various approaches in attempting to solve the problem by continuously generating and rejecting ideas. He spent most of his time *thinking* rather than *doing*. Schoenfeld (1987) concluded that:

> the difference between the mathematician's success and the students' failure cannot be attributed to a difference in knowledge of subject matter. Indeed, the students started off with a clear advantage over the mathematician. They knew all of the procedures required to solve the problems they were given, whereas he did not remember them and had to figure them out for himself. What made the difference was how the problem solvers made use of what they did know. The students decided to try something and went off on a wild goose chase, never to return. The mathematician tried many approaches, but only briefly if they didn't seem to work. With the efficient use of self-monitoring and self-regulation, he solved a problem that many students—who knew a lot more geometry than he did—failed to solve!! (p. 195).

In an attempt to provide mathematics teachers with techniques that focus on metacognition, for use in virtually any mathematics instructional setting, Schoenfeld (1985b, 1987) advocated a 'kitchen sink' approach, consisting of four techniques which would arguably enable the development of metacognitive skills in students. The four techniques are presented in order from least interventionist to techniques calling for increasingly deeper interactions between teacher and students. In summary these techniques are:

- Watching video tapes of problem sessions.
- Teacher as a role model for metacognitive behaviour.
- Whole-class discussion of problems with teacher serving as "control".
- Problem solving in small groups.

Schoenfeld considered his students to be culturally immersed in a microcosm, being trained in heuristic strategies, managerial strategies and self-regulation, while solving problems. Work on beliefs was also considered to be important. Beliefs will be discussed in the next section of this research essay.

Problem solving in small groups was a natural environment with the students experiencing mathematics in a way that made sense, similar to the way mathematicians experienced it. Schoenfeld (1987) further proposed that we need a program of "cultural design" for schooling, since understanding enough about the social contexts that promote the need to develop and understand mathematical ideas may allow us to create classroom environments where students do mathematics naturally. In another paper, Schoenfeld (1985) identified two major difficulties (limitations) of research on metacognition. The first is that descriptions of competent executive behaviour characterises ideal behaviour. The second is that discussions of meta-behaviour tend to isolate this kind of behaviour from other levels of cognition.

To recapitulate, it is our contention that research on metacognitive aspects of mathematical problem solving, both theoretical and empirical, should be on top of the priority list of mathematics educators, since according to Silver (1987):

> no process model of problem solving in any domain can be complete without an adequate account of the role of metacognition and belief systems (p. 50).

Belief Systems—Affect

Affective issues in mathematical problem solving have attracted considerable attention by mathematics educators and cognitive psychologists in the past decade (Putt and Isaacs, 1992; Leder, 1993; Mason, Burton and Stacey, 1985; McLeod, 1988, 1989, 1992; Lester & Garofalo 1987; Schoenfeld, 1985, 1987, 1992; Dreyfus & Eisenberg, 1986; Mandler, 1989; Silver, 1994). Some investigators (McLeod, 1989; Schoenfeld, 1985, 1992; Lester & Garofalo, 1987) have postulated that inclusion of affective aspects of mathematical problem solving is necessary for any useful theory of problem solving in school contexts. A certain obstacle to progress in this domain is the non-alignment of terminology used by mathematics education researchers. Schoenfeld (1992), in an attempt to assemble a theory of thinking mathematically and problem solving, listed *beliefs* and *affects* as one of five aspects of cognition. McLeod (1992) adopted a slightly different stance, stating that

> beliefs are largely cognitive in nature, and are developed over a relatively long period of time. Emotions, on the other hand, may involve little cognitive appraisal and may appear and disappear rather quickly. Therefore we can think of beliefs, attitudes and emotions as representing increasing levels of affective involvement, decreasing levels of cognitive involvement (p. 579).

Recently Leder (1993) used Corsini's definition of affect as a term

> used to denote a wide range of concepts and phenomena including feelings, emotions, moods, motivation and certain drives and instincts (p. 1-46).

Despite lack of consensus among mathematics education researchers on the use of terminology, the affective domain is generally regarded as referring to constructs that, according to McLeod (1992), go beyond the cognitive domain, and that beliefs, attitudes and emotions can be considered as *subsets* of affect.

In a research project, initiated in 1981, Lester and Garofalo (1987) attempted to provide some theoretical and research considerations on the influence affects and beliefs (along with metacognition) have on the cognitive activities of problem solvers. The project was initially designed to explore seventh graders' metacognitive awareness in mathematical problem solving. The scope of the project was later extended in an attempt to account for affective factors and students' beliefs about mathematics and problem solving. Lester and Garofalo (1987) postulated that a student's failure to solve a problem successfully cannot be attributed only to an inadequate knowledge base—either formal or informal—but also to non-cognitive and metacognitive factors, namely beliefs, affects, control, and socio-cultural conditions. The position held by these researchers was that attitudes are 'transient traits' of the individual as opposed to emotions, which are situation-specific. Two attitudes appeared to attract the interest of the investigators: confidence and perseverance. Lester and Garofalo (1987) postulated that perseverance is the resultant of three components: "desire to obtain correct answers, resistance to premature closure, and persistence" (p. 7). Taplin (1992) adopted a similar stance on perseverance.

A critical research issue in the affective domain is the role that beliefs about mathematics, or about mathematical problem solving, or about oneself, play in skilful problem solving.

> The idea here is that student's understandings regarding the nature of mathematics establish the psychological context within which they do mathematics—and in consequence, these understandings shape the students' mathematical behavior. The results can often have strong negative effects on performance (Schoenfeld, 1985, p. 375).

Counter-productive beliefs must be identified and be dealt with on an individualised basis. It is important to train our students to realise that their beliefs that most problems can be solved in five minutes, and copying the teacher's solution from the board, do not constitute mathematical thinking in any way. Researchers working on mathematical problem solving have tended to ignore the role of beliefs in "expert" problem solving, with the exception of Schoenfeld (1985, 1987, 1989), Silver (1987, 1994), Lester (1985, 1988), and Lester and Garofalo (1987). Lester and Garofalo (1987) stated that "beliefs shape attitudes and emotions and direct the decisions during problem solving" (p.7).

The above mentioned studies of students' learning indicate that affect plays an important role in their mathematical performance.

Stacey (1990), investigated students' capacity to utilise their own mathematical knowledge in unfamiliar situations. Stacey concluded that there is a dependence between the students' attitudes and beliefs about mathematics and its learning, and their mathematical knowledge and understanding.

Another area of research in mathematics education that can be considered to be related to affect is the area of problem posing. Problem posing refers to students posing their own problem or reformulating problems given to them by teachers or other sources. Silver (1994), posited three major conclusions that can be drawn from research studies in problem posing:

> First, it is clear that problem-posing tasks can provide researchers with both a window through which to view students' mathematical thinking and a mirror in which to see a reflection of students' mathematical experiences. Second, problem-posing experiences provide a potentially rich arena in which to explore the interplay between the cognitive and affective dimensions of students' mathematical learning. Finally, much more systematic research is needed on the impact of problem-posing experiences on students' problem posing, problem solving, mathematical understanding and disposition toward mathematics (p. 25).

It is interesting to note that all three conclusions involve aspects of affect.

Finally, an area of the affective domain that has received very little attention by researchers and curriculum frameworks and reports (A. E. C., 1991; N. C. T. M., 1989), is the role aesthetic influences play on mathematical thinking and problem solving. A number of well known mathematicians and psychologists (Hadamard, 1945; Krutetskii, 1976; Poincare, 1946) have endeavoured to investigate the role aesthetics plays in mathematical thinking and problem solving. Hadamard (1945) and Poincaré (1946) suggested, according to Silver and Metzger (1989), that mathematical discovery is guided by aesthetic emotions, which in turn guide mathematicians (at a conscious and an unconscious level) to decide the route to a particular solution. Krutetskii (1976), who conducted a twelve year-long study on the problem solving abilities of mathematically talented high school students, reported that they consistently monitored and evaluated their solutions, in search for an elegant path to the solution.

The question of what constitutes mathematical aesthetics and how it is related to mathematical thought and problem solving is another area where research has not as yet provided the mathematics education community with a unanimously acceptable answer. Dreyfus and Eisenberg (1986) have attempted to answer these questions. They quoted Birkhoff (1956), who attempted to quantify aesthetics. Birkhoff proposed that aesthetics could be determined by

> the formula M = O/C, where O is a measure of order, C a measure of complexity, and M a measure for the aesthetic value of the object or argument under consideration (p. 3).

Hofstader, (cited in Dreyfus and Eisenberg, 1986) however, held a diametrically opposite opinion:

> there exists no set of rules which delineates what it is that makes a piece beautiful, nor could there ever exist such a set of rules (p. 3).

Despite the disagreement among researchers however, Dreyfus and Eisenberg (1986) stated that factors contributing to an aesthetic appeal of a solution or proof are interconnected. Those factors were clarity, simplicity, brevity, conciseness, structure, power, cleverness, and surprise. Elegance was not listed although it can be argued that some of the factors in the list such as simplicity, brevity, and conciseness, could be considered to partly represent the essential ingredients of an elegant solution.

Although training students on mathematical aesthetics seems to be last on the list of curriculum developers of mathematics programs, Halmos (1980) insisted that students should be trained to look for aesthetically appealing solutions in mathematical problems. The issue of how mathematics teachers can facilitate their students' appreciation of qualitative differences between pedestrian and elegant solutions and an appreciation of the beauty of mathematics, remains a largely unexplored and unresolved issue, since it represents a very complex area of human learning that requires a lot more research. Training students to appreciate mathematical aesthetics could become a prominent feature of mathematics curricula, if and when mathematics curricula designers adopt the rationale that students and teachers of mathematics should be embedded in a *mathematical culture* in the way mathematicians experience it, which according to Davis (1990):

> has two aspects to it: the largely unconscious part, so internalised that is difficult to see if any thing but obvious truth, and another part that is distinguished by an individual's conscious choice to become part of a broader culture (p. 4).

This culture values highly "elegance, parsimony, symmetry, coherence, simplicity, beauty and similar attributes", according to Silver and Metzger (1989, p. 71).

One of the main obstacles of research efforts into the affective domain and the role it plays in mathematical problem solving performance, is the absence of an adequate theoretical framework (Lester and Garofalo, 1987). Recently McLeod (1988, 1989) has attempted to formulate a theory, based on Mandler's (1989) theory of affect and emotion. The dimensions of the theory take into consideration characteristics of affective states such as magnitude and direction, duration of the emotion, the level of awareness, and the level of control. Consideration is given to the relation of affect to instructional issues such as types of cognitive processes, types of instructional environment, and belief systems. McLeod (1988, 1989) offers a detailed exposition of the theory.

There is a need to probe further into the affective domain and to achieve an integration of research on the affective and the cognitive domains, on affect and learning, and on affect and teaching. There is also ongoing debate over which research methodology should be used to measure and evaluate affective aspects of mathematics learning and problem solving. McLeod (1992) has claimed that the debate is almost over and that the use of clinical interviews is the way ahead in research on the role affective issues play in mathematical problem solving.

Discussion

Successful problem solving in mathematics requires the convergence of a number of critical elements. This review has focused on cognitive, metacognitive and affective aspects which are considered to underpin mathematical thinking and problem solving. Emerging areas of research such as the practice of mathematics teaching and problem solving, enculturation, mathematical problem solving assessment and the needs of prospective teachers of problem solving, have not been reviewed. Mathematics teachers however, may refer to the following excellent resources: Leder (1992); Schoenfeld (1992); Charles (1988); Bishop (1988).

Schoenfeld, whose model of mathematical thinking and problem solving (1985a, 1985b, 1992) has provided the scaffolding for this review, argued (Schoenfeld, 1992) that metacognition, beliefs, and mathematical practices are the critical components of his model, and that "domain specific knowledge plays an altered and diminished role, even when it is expanded to include problem-solving strategies" (p. 363). We postulate—along with Schoenfeld—that the only way mathematical knowledge, metacognitive skills, beliefs, aesthetic appreciations of mathematical work, and mathematical practices of individuals can fit together to form a unified network, is by developing individuals who can visualise, represent and analyse the world through a mathematical spectrum—the way mathematicians experience it.

The first issue that results from our review and needs clarification in further research efforts, concerns the definitions of the terms 'problem', 'problem solving' and 'problem solution' when the terms are used in research studies. An attempt was made to address this fundamental and currently unresolved issue. We contend that if we are to avoid confusion—especially within the mathematics teaching community—the mathematical problem solving research community should adopt *one* operational definition for each one of the terms, or reach an agreement that, at the least, every study will provide definitions of the terms followed by specific and detailed examples.

In the area of cognitive research two issues require further clarification, according to Schoenfeld (1992). Firstly, an adequate framework for the description of cognitive mechanisms must be developed, and secondly, the extensively researched but currently unresolved issue of the relationship between cognitive resources, strategies, affect and beliefs. Regarding metacognition, Lester (1985) argued that one of the issues that requires attention from researchers is the role metacognition plays in successful mathematical problem solving. He argued that metacognition guides cognitive processing during problem solving. Two further critical issues on metacognition require further exploration and clarification. The first issue is the role of aesthetics in metacognitive activity during problem solving. Silver and Metzger (1989) postulated that:

aesthetics can serve as a link between one's monitoring and evaluating behaviour and one's emotional response. In this capacity aesthetics provides a link between cognitive and metacognitive activity and emotion" (p. 71).

The second issue is lack of an adequate theory of the mechanism of metacognition (Schoenfeld, 1992). That is, for example, how does expert mathematical knowledge affect metacognitive and aesthetic decisions during problem solving and vice-versa.

There has been renewed interest in the past two decades, by mathematics educators and cognitive psychologists, to conduct research on affective issues and belief systems. Schoenfeld (1992) however, was not satisfied that these results clarify the vexing issue of an adequate theoretical model, despite the efforts of McLeod (1988, 1989, 1992) mentioned earlier.

With regard to mathematical practices and enculturation, Schoenfeld (1992) has warned us that we know very little in what "may ultimately turn out to be one of the most important arenas of understanding the development of mathematical thinking" (p. 365). Schoenfeld (1992) also identified instruction and assessment of problem solving as inadequately researched areas, with a host of questions still remaining unanswered. Some instructional implications of the research will be reviewed in the next section.

A recent book however, edited by Leder (1992), may be seen as an attempt to bridge the gap between instruction, assessment and learning of mathematics. Leder's title of the last chapter of the book: "curriculum planning + assessment = learning?" encapsulates the dilemma with which the contemporary mathematics education community is faced with respect to assessment and curricular/instructional issues. It is evident from the previous exposition of underexploited issues, or issues that require further clarification and theoretical support, that there exist some discernibly difficult theoretical and practical questions to be tackled by researchers in the years to come.

Directions for further research

It can be concluded from the review of cognitive studies on expert and novice problem solvers that the novice-expert transformation is not a continuous process. We have argued earlier in this paper that expertise in mathematical problem solving requires extensive theoretical and practical training, mathematical maturity and a multitude of traits and skills. Research studies could attempt to resolve the issue of why this discontinuity occurs and how instructional and curricular programs might be delivered to remove the discontinuity and achieve a smooth transition in the novice-expert transformation process.

We have argued that metacognitive decisions could be considered the 'driving forces' in mathematical problem solving. Research studies could investigate the teachers' own monitoring, regulation and evaluation mechanisms during problem solving instruction and how it might be used as an instructional method to enhance the students' metacognitive activation during problem solving.

In this review an attempt has been made to present some views on the constituents of successful problem solving. Is it possible however, to develop a taxonomy of *indicators of success* in mathematical problem solving?

It seems that we are a long way from the development of a unified model of mathematical thinking and problem solving despite the efforts of Schoenfeld (1985, 1992); and Goldin (1992). Considering that a partial unification has been achieved via the Lester (1985) model, and Schoenfeld's (1992) and Golding's (1992) attempts for the development of unified models, is it possible to develop a Grand Unified Theory of Mathematical Problem Solving, which will incorporate as its constituent components: a refined Polya model; the approach advocated by Sweller and his colleagues; Schoenfeld's (1985, 1992) and Lester's (1985) models; and Goldin's (1992) attempted unification model?

Conclusion

Students' mathematical problem solving is a very complex phenomenon. It can be considered as a nexus of cognitive, metacognitive, affective, instructional, environmental and cultural attributes, at the least. Studies reviewed have generated a plethora of theoretical and practical data on each one of the abovementioned attributes. Some researchers have endeavoured to formulate a chart of the problem solving terrain, with limited success. An articulation of the tacit component of what the problem solving experts do remains an elusive dream, much like the 'ghost particle' of high energy physics—the neutrino. Although research progress has been made, we do not know enough to formulate the necessary and sufficient conditions for an adequate description of mathematical problem solving expertise acquisition. It appears that problem solving researchers are embarked on an exciting journey, into a galaxy whose manifold topology necessitates systematic further research.

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En översikt över aktuell forskning kring kognitiva, metakognitiva och affektiva aspekter av problemlösning

Problemlösning i matematik är en komplex aktivitet som ger oss anledning att noggrant tänka över de bästa sätten att undervisa, eftersom olika individer kan komma fram till samma lösning av problemet med olika men korrekta metoder.

I syfte att ge en sammanhängande bild av forskningsläget idag presenteras i denna forskningsöversikt kvaliteter gemensamma för goda problemlösare, undervisningsrelaterad problemlösningsforskning samt bidrag från olika system av kognitiva och metakognitiva s k "belief systems".

Några hittills obearbetade och underrepresenterade frågeställningar tas också upp för att visa på olikheter i och mångfalden av angreppspunkter i forskning i problemlösning i matematik och behoven av samarbete mellan forskare inom detta fält.

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