Recent research and a critique of theories of early geometry learning: The case of the angle concept

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The van Hieles' theory on levels of development has been supported by many studies, particularly in geometry. An important consequence of the van Hiele work has been the realisation that informal experiences assist students to take the necessary first-step of recognising, for example, shapes. Despite this important influence, some authors have cautioned that visual reasoning can be independent of the levels because there are some difficulties in trying to allot students to levels.

In order to illustrate developments in our understanding of students' geometry learning and the role of visual processing, studies on the difficulties students have with angle concepts are discussed. Based on a large qualitative study, the ability to notice and analyse angles is explained in terms of aspects of problem solving or investigation. Case studies illustrate how the complex conceptualisations about angles were developing concurrently, rather than in fixed levels, as a result of visual (mental) imagery, selective attention, manipulation of materials, and discussion.

Responsiveness, a compound variable resulting from a complex of cognitive processing unique to the individual emerged as important. Responsiveness suggests an empathy or understanding of the problem resulting from cognitive processing, in particular selective attention. Responsiveness affects interactions with materials and with people who, in turn, influence thinking and the continuing cycle of problem solving and understanding of the angle concept.

Introduction

This article briefly reviews literature on the development of geometric thinking in order to establish that conceptual thinking may not be in discrete stages as suggested by some authors. A summary of recent studies on the development of angle concepts is then given before reporting on a study which considered how students learn geometric concepts, particularly the angle concept, through classroom spatial activities. The study is set in primary school classrooms and so consideration is given to the interactions between students and materials as well as their cognitive processing. New constructs emerged from

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the study to assist us in our understanding of how spatial problem solving can assist students to learn new concepts while learning to problem solve.

The Van Hieles' theory and other stage theories

One of the most widely cited theories relating to children's development of geometric concepts is that suggested by the van Hieles (Fuys, Geddes, & Tischler, 1988; Hoffer, 1983; van Hiele, 1986). Most of the studies of van Hieles' theory in relation to explaining development have considered properties of plane shapes. Burger and Culpepper (1995, p. 150) note that it needs further analysis in terms of other "topics such as visualization, measurement, transformations, congruence, similarity, and the relationships between geometry and algebra, all in two and three dimensions."

Burger and Shaughnessy (1986) described the three lowest levels, in order, as:

Visualization. The student reasons about basic geometric concepts, such as simple shapes, primarily by means of visual considerations of the concept as a whole without explicit regard to properties of its components. . . .

Analysis. The student reasons about geometric concepts by means of an informal analysis of component parts and attributes. Necessary properties of the concept are established. . . .

Abstraction. The student logically orders the properties of concepts, forms abstract definitions, and can distinguish between the necessity and sufficiency of a set of properties in determining a concept. (p. 31)

Burger and Shaughnessy (1986) gave indicators for four two-dimensional tasks for each of the levels and were able to classify 14 students (from kindergarten to first year at college) but the number of students was very small. A number of studies (e.g., Denis, 1987; Fuys et al.'s, 1988; Mayberry, 1983; Senk, 1989; Usiskin, 1987) supported the fixed sequence of levels, at least for the first three levels, but generally using students from high school. Crowley (1990) pointed out that the tests used to determine levels may not be adequately reliable or valid. Lawrie (1993), with a larger sample than Mayberry, cast doubt on some of Mayberry's behavioural indicators of levels and identified features which would detract from accurately describing students' thinking.

Another stage taxonomy, the SOLO taxonomy was developed by Biggs and Collis (1982), based on Piagetian theory but considering the learning outcomes of students in the modes of "ikonic" global responses and "concrete symbolic." In each mode, they noted stages: prestructural, unistructural, multistructural, relational, and extended abstract. Initially they placed the ikonic mode lower than that concrete symbolic mode.

Revisions in stage theories

As a result of the analyses made by several researchers using the SOLO Taxonomy, Biggs and Collis (1991) have suggested that both kinds of thinking can occur at the same time and that there can be several cycles of development within each of the ikonic thinking and concrete symbolic modes. For example, Pegg and Davey (1989) found that some students gave both descriptions of ikonic global images as well as concrete symbolic responses which involved analysis of properties.

With the van Hiele theory, several suggestions have been made to overcome difficulties in classifying some students. Fuys et al. (1988), suggested that if a student has reached a level for one concept then earlier stages can be quickly passed through to reach this level with another concept. These researchers also suggested that a plateau may be reached by some students at the lowest level whereas Burger and Shaughnessy (1986) suggested that there is a transition between levels involving small steps.

Gutiérrez, Jaime, and Fortuny (1991) overcame the difficulties associated with a strict hierarchy of modes of thinking by suggesting that students could be scored on each of the van Hiele level. They found that in general, for any level, lower levels were more complete. The specific task itself influences level; for example, deduction of simple congruency of triangles may occur very early in comparison to other deductive proofs (Gutiérrez et al., 1991).

Wilson (1990) found that the hierarchical classification of shapes was more difficult than deductive proofs. In fact, van Hiele and others (see the summaries by Clements & Battista, 1991, 1992; Crowley, 1990) have suggested that a three stage development may be more appropriate because of the difficulties of distinguishing development in levels for higher age groups.

The theories of Piaget, the van Hieles, and the SOLO Taxonomy are based on the premise that each level of thinking is qualitatively different from the others. This may not be the case. As already noted, some studies have shown both modes can occur simultaneously. A non-hierarchical interpretation of development of thinking is feasible; Clements and Battista (1992) have drawn together a number of information-processing theories and thus provided an alternative explanation to that suggested by stage theories. One point to note in this debate is the notion that prior representations of visual features can become sufficiently linked for a shape to be noticed, but the features and links may not be internalised by a student until he or she is actually required to use them. With instruction, property-recognition units (or schema) form and become relatively stronger within the network. Furthermore, dynamic imagery, as encouraged by the use of such computer programs as Cabri Géometrie can become an important way of creating links in recognising shapes. Concrete tasks such as using rope to form a diversity of triangles and other shapes can also assist this process.

Visualising and reasoning in concept development

Battista and Clements (1991) described a number of incidents of children involved in Logo activities in which visualising and reasoning assisted each other. For example, one student used visual imagery in reasoning that a rectangle is a "stretched out square" so the angles of a rectangle are right angles like a square's. Another student suggested that an incorrectly drawn square was a tilted square discounting her own verbalised description of the square with equal sides. A third student distinguished between a rectangle which is tilted and a parallelogram, using visual transformations of the whole rectangle to justify his reasoning. Battista and Clements (1991) encouraged teachers to allow students to use visual reasoning and to foster discussion in which students explained their reasoning and related their concepts and visualisations.

Visual cues can assist students to answer geometry questions if the cues are not bound by some perceptual or schematic connection (Kouba, Brown, Carpenter, Lindquist, Silver, & Swafford, 1988). Concept images need developing through experience of various examples of concepts in order to prevent some students from imposing a visual bias on concept images (Hershkowitz, 1989). Van Hiele (1986) maintained that the level of development in reasoning is more dependent on instruction or informal experiences than on age. Imagery is needed in early developments of concepts but imagery also forms part of the abstraction or mental summary of the concept.

Experience, visual reasoning, and angle concepts

Piaget and Inhelder (1956) concluded from their studies that there is no doubt

> that it is the analysis of the angle which marks the transition from topological relationships to the perception of Euclidean ones. It is not the straight line itself which the child contrasts with round shapes, but rather that conjunction of straight lines which go to form an angle. (p. 30)

Despite this apparent early, holistic impression of angles, students seem to have difficulty with angles. In Outhred's (1987) study of children in Years 3 to 6, children had difficulty in recognising angles in differing orientations, with differing arm lengths, and when embedded in figures. Often right angles that were not aligned with the horizontal axis were not recognised. Similar difficulties have been mentioned by others (Fuys et al., 1988; Mitchelmore, 1989, 1992a; Pegg & Davey, 1989).

Close (1982) encouraged more pre-measurement informal experiences with angles, their size, orientation, and contexts. The importance of mental imagery in estimation of angle size was illustrated by Mansfield and Happs (1992) from their study of sixth and eighth grade students. For their estimates of angles, some imagined a protractor, others used a right angle or half turns as a benchmark, and others used an angle of a polygon.

Mitchelmore (1993, 1994) found that Year 2 and 4 children had a good global understanding of different angle contexts, and that they improved significantly in their recognition of angle-related similarities. Children most easily recognised similarities between situations where both lines are physically present (e.g., crossing) and were nearly as able when one line was present and one had to be imagined (e.g. a slope), but they found it difficult to relate turning (both lines to be imagined) to other angle contexts.

Mitchelmore and White (1995) explained the development of general angle concepts among young children, in terms of abstraction in mathematics learning. Their interpretation regards a concept as a product of the recognition of deep similarities between superficially different objects, events, or ideas. In particular, the angle concept is thought to develop gradually as children recognise more and deeper similarities between physical angle experiences, going through three successive stages: classification into physical angle situations such as walking up a slope and using scissors; then into separate angle contexts described as *sloping* and *crossing* (e.g., for scissors); and finally into a general angle concept which includes all contexts. Mitchelmore found that primary-aged children related situations involving turning about a fixed point to each other more easily than to turning around a corner and he interpreted this as suggesting that children learn to relate turning to static angle situations by interpreting turning as a static angle rather than by interpreting static angles as turns. This interpretation was not supported by the current study in which students used turning to recognise the static angle because turning is operational and hence a means by which children kinaesthetically appreciated and visualised the concept.

Mitchelmore (1992a) surmised that children normally see right angles only as intersections of horizontal and vertical lines, and therefore do not relate them to angles in general. As the idea of a corner or right angle is understood relatively early, it is not regarded as an angle by some students in the same way as a square is often thought of independently of other quadrilaterals. On the other hand, some students believe that only right angles are angles, and others think that only acute angles are angles. Mitchelmore (1992b) suggested teaching activities need to help children see right angles as special angles. The study described in this paper did in fact do this.

Students need to grapple with the range of meanings and experiences which can be associated with the word *angle*. Davey and Pegg (1991) claimed that there seemed to be a plateau in the development of children's concept of angle after the initial ideas of a *corner* (as in a room where the lines are at right angles) and of *being pointy*.

For two questions on angles, one on recognising a straight angle and the other on the angle of slope to the horizontal, only 74% and 64% respectively of Year 6 students in the state of New South Wales, Australia (NSW) were successful (Owens, in press). These data suggest that students have difficulties recognising angles in everyday situations, and that students may have difficulties understanding the words: *straight angle* (a word sometimes used for *right angle*) and *horizontal*. (The complexity of a picture, in a sales pamphlet, showing the skateboard set on a slope may have caused difficulty.) Language difficulties were noted by Fuys et al. (1988) in their study in which sixth grade students gave descriptions using non-standard vocabulary although more formal language was used during the course of the study. The link between language and contexts for learning needs further discussion.

Language difficulties make it hard to study the development of the concept of angle. Some researchers (e.g., Fuys et al., 1988; Pegg & Davey, 1989; van Hiele, 1986) have claimed that the type of language and the concept images which students use relate to a particular level. Language, however, can act as a mediator to differentiate a particular stimulus and promote attention. The connections between language and image may not be tied by level but, indeed, images and language could be regarded as limited conceptions promoting attention. The study (Owens, 1993) to be described in this paper illustrated the importance of language in concept development and in focusing attention during problem solving and learning.

The study

The theoretical perspective developed in the remainder of this paper emerged from a qualitative study of students working mainly in classrooms in Years 2 and 4 in Australia and Papua New Guinea. The activities given to the students during 11 sessions were open-ended spatial problem-solving activities which had been shown to improve students two-dimensional spatial thinking (1992a) as shown by improvements on the test *Thinking about 2D Shapes* (1992b). The activities can be illustrated by the problems posed for the well-known seven-piece tangram set made of three-sizes of isosceles right-angled triangles, a square, and a parallelogram (see Figure 1).

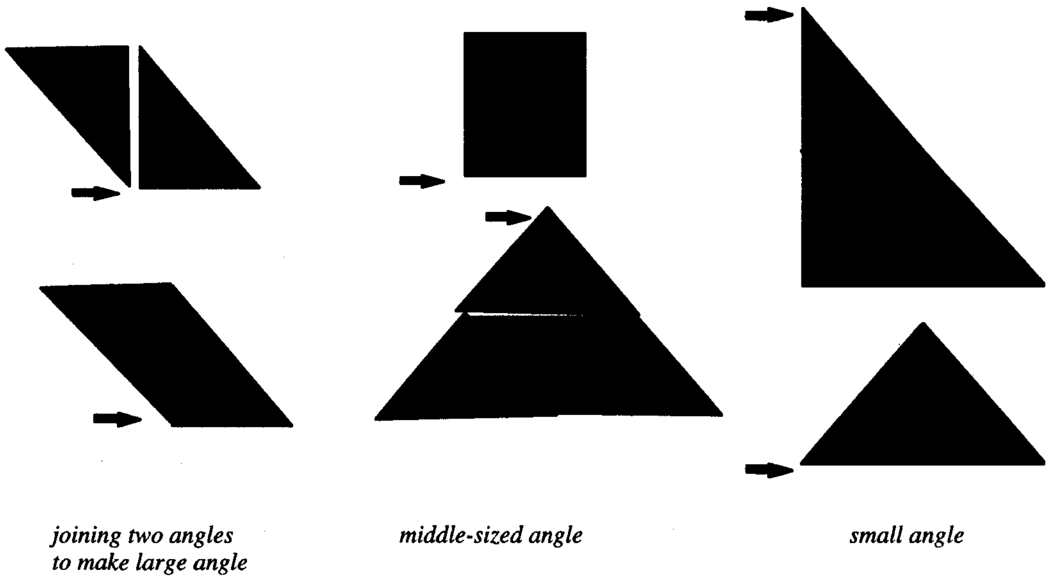

Figure 1. The angles of the seven-piece tangram set.

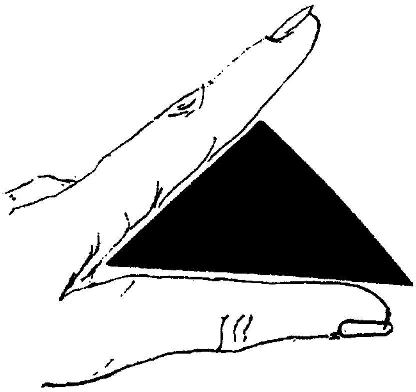

Figure 2. *Using thumb and finger to note an angle on a tangram piece.*

Students were to look for similarities and differences between the pieces, to make the larger pieces from the smaller pieces, to make different-sized squares, triangles, and rectangles. Later they were asked to order the angles in size (small, middle-sized, and large) and to make the shapes with sticks and matchsticks. Similar activities were completed using pattern-block sets containing squares, equilateral triangles, isosceles trapezia and two types of rhombi. Students were assisted in recognising angles by turning their first finger away from their thumb to mark the angles on the pieces as illustrated in Figure 2.

These young students were generally unfamiliar with the word *angle* and seemed to have no school or other experience of the word. For this reason, the word *point* was generally used to refer to angle.

Purposeful sampling

Several different groups of students were purposefully selected as the qualitative aspects of learning unfolded. The study (Owens, 1990; 1993) began with retrospective recall by adults immediately after they had solved the problems. The problems were then given to primary-school students working alone with very little intervention from the researcher; these students made spontaneous comments about their thinking and were asked to recall their thinking. After this initial inquiry, several categories of thinking such as imagining and affect were defined as important.

In the next stage, the effects of interaction were considered. Four groups of three children were given the problems and stimulated to recall how they were thinking immediately after solving each problem or part of the problem. Half these groups were in Year 2 and half in Year 4; one from each year worked as a cooperative small group and the other as individuals but able to talk to their peers. Another sample of students in classrooms took the study further to see the effect of classroom context on problem solving. In three schools, a class from each of Year 2 and Year 4, from one region of Sydney were matched on school, class, and pre-test score and randomly allocated to either the individual or group learning situation and then given the series of eleven problem-solving activities, and retested. Year 2 students, aged seven years in NSW, were included in order to notice early developments in angle concepts.

Two further samples were taken during this grounded-theory research; the story-line was checked and developed further by taking classes in another suburb and in another cultural setting, namely in Papua New Guinea.

Students were video-taped while engaged in the activities and these tapes were analysed. Each incident during which there was a small development in problem solving was categorised. The categories included interaction with materials, student-student and student-teacher interaction, concepts, imagery, heuristics, and affect. The subcategories were developed during the research and from the literature review. Over 120 problem solving sessions were analysed.

Aspects of problem solving during the development of the angle concept

Examples of students learning about angles have been chosen to illustrate how a concept, in this case that of an angle, can be developed in terms that are not associated with levels of development but with cognitive processing, and influences of the context of learning.

For Dora, in Year 2, there was a conflict between her perception of the materials and what she had observed a fellow student doing. Dora set out to compare the angles of the tangram pieces. The following notes described her participation in the activity.

> 1.01 Dora tells her teacher that her fingers have to be spread further apart for the large angle of the parallelogram. She gathers together most of the right angles for the middle angle. She puts her thumb and finger around them.
> 1.02 She seems a little exasperated for she is unsure about what is expected of them. She discusses with her friend about recording the angles – small, middle, and large – in her book and she draws the two arms of the small angle.
> 1.03 She knows that, beside the small triangle, the big triangle and the parallelogram have a small angle.
> 1.04 She compares the right angle with the small angle but her friend calls the right angle large, so the teacher reassures her that the right angle is in the middle....
> 1.05 When the teacher asks her for the large angle, she picks up the parallelogram and claims she has it before her friend....
> 1.06 She joins up points to make a right angle, and is pleased about doing this.
> 1.07 Then she joins the two triangles to make a parallelogram... and shows this to the teacher as the large angle. (see Figure 1 configurations)

It is clear that this Year 2 child had perceived the differences between different angles on the various shapes. She was able to relate size of the angle to the position of the arms of the angle. The concrete materials helped her to see the same angle in different shapes (paragraphs 1.01, 1.03, and 1.04), and she was able to mark the angles with her fingers and to draw them (paragraph 1.02). She also compared angles with other angles represented by the pieces and drawings (paragraph 1.04). The above account also draws attention to a common affective response to making shapes or angles, that of pleasure and excitement following a measure of success with an activity (paragraphs 1.06 and 1.07). It is as if the concrete materials were often being used by students as a means of confirming and celebrating their abstract idea of an angle.

The materials provided physical representations of angles of different magnitudes, yet they were not sufficient to enable the children to appreciate the teacher's words about size of angles. In other words, the physical representation was not enough, even when combined with the teacher's words, for children to understand the meaning of different sizes of angles. First the children had to focus on or disembed the angle from the rest of the shape and then they had to construct their own meanings. Suggestions made by the teacher encouraged students not only to observe but also to check equivalence and to compare angles by overlaying them. For example, Jodie and James in Year 2, on being given the tangram set, immediately checked the points of the pieces in the same way as the teacher had done during the pretest. (Victor, the third member of their group, was away sick during this first session.)

In the session which will now be described, both Victor and James had to test their ideas of the size of an angle before they could clarify their understandings. The confusion lay in the use of the words *large, middle-sized,* and *small* to refer to the size of the pieces, the size of the side lengths, and the size of angles on the pieces. Victor was also completing the task set the other children in his absence, namely making the larger triangle with the other pieces. (The other children had apparently described the task to him).

2.01 Victor has gathered together some pieces, as if matching all the small angles.
2.02 James has picked up the small triangle for the small angle and points to its right angle.... The teacher asks for the middle-sized angle, he picks up the middle-sized triangle.
2.03 Victor picks up the parallelogram for the middle-sized angle, discards it, and then chooses the middle-sized triangle for the middle-sized angle. He

puts his finger and thumb around the right angle and says "This is the middle angle." James watches.

2.04 Jodie is quite clear about which angle is smallest and which is middle-sized and she has been drawing both the small and middle-sized angles in the book.

2.05 James matches several angles against the drawing of the small angle.

2.06 Meanwhile Victor picks up the parallelogram and runs his finger along the long side. "This is the biggest" but he places it on the large triangle together with the two small triangles to cover the large triangle. He picks up the large triangle as if showing the small angle, "This is the biggest."

2.07 Jodie takes it and says "No, it isn't," and places it on the drawing of the small angle. The teacher confirms, "That's the small angle."

2.08 Meanwhile Victor has picked up the parallelogram, tests it against the drawings of the small and middle-sized angles and draws the large angle into the book.

2.09 He goes back and draws along the whole length of the arms.

2.10 The teacher suggests they make points (angles) by joining smaller points together. Victor puts points on top of each other.

2.11 James puts two small points together. The teacher says he has made the middle-sized one, praises him, and suggests that he can draw it in their book. James nods his head but he doesn't look convinced that he knows what he did. He puts two more together on top of the middle-sized point of the large triangle.

2.12 Meanwhile Victor has been trying to cover the large triangle, and he can see how to make it with the square and two triangles.

2.13 Jodie has now made the big point with the angles of the two small triangles. (See Figure 1.)

The discussion indicated how Victor, who seemed to know what was meant by the size of points (the word generally used by these children to refer to *angle*), temporarily considered that he should be comparing the size of the sides of the shapes (paragraphs 2.06 and 2.09). The interaction between students helped Victor to clarify what was meant by "the point of the same size" (paragraph 2.07). James, who had been able to match points in the first activity, began this later session by choosing the wrong points, largely because he was choosing the small or middle-sized triangles (paragraph 2.02). He soon established the meaning by listening to the teacher and to Jodie (paragraph 2.04) and by checking points with the drawing (paragraph 2.05). Later, although he successfully joined points and the teacher talked about joining angles to make new ones, he was only sure that he had the correct idea when the teacher encouraged him to show his new angle and praised him for his work (paragraph 2.10). Through manipulations, Victor (paragraph 2.12) spent some time relating the operational concept of making angles by joining together two points and the concept of making congruent shapes means making equal angles.

Later the group made shape outlines and Victor explained that James had not made a right-angled triangle as James had thought but that he had just made an equilateral triangle in another orientation. Victor himself had made the right-angled isosceles triangle with the long side horizontal and he checked it with the tangram piece which he put on top ("a lid," he called it). The teacher asked the children what was meant by bigger. Jodie replied more spread out and picked up the tangram right-angled triangle and the pattern-block equilateral triangle, put one on top of the other and said, "See it is bigger."

The expectation of the students was to decide on the different sizes of the angles and they focused their attention on this responding by manipulating and discussing. With the later experiences there were still clarifications to be made especially when Victor was intent on completing the previous activity for which he was absent. When asked about his answers to the questions on angles in the posttest (the shapes now represented on paper), he had not initially noted the right-angle on a triangle in the same turned position which he had made with matchsticks and sticks. Later, when the page was turned for him, he said he could now see it was the same shape so it had to be the same angle. For another question, he noted that two right-angles were the same, "It is blunt like this one." His concepts of equality of angles involved the congruency of shapes, an informal description of the angles, visual imagery of the angle in different orientations, representations of angles by opening fingers and using sticks, an angle as the joining of smaller ones, and use of the word, for example, *large* to refer to different aspects of a triangle (an angle, overall size, length of side).

Students frequently noticed the equality of angles more that of sides. The tendency to notice angles but not sides was due to holistic imagery and to an inability to disembed sides from the rest of the shape (Owens, 1993; 1994; Owens & Clements, in press). The following is an extract from Year 2 children working next to each other with their own tangram sets.

> 3.01 Jonah makes a large parallelogram from the two large triangles. Sam says it is like the parallelogram piece which he picks up.
> 3.02 Lois makes her own parallelogram at a distance and watches as Sam matches the various angles of the parallelograms and she does likewise.

Students generally recognised angles on pieces which were likely to fit an angle to complete a jigsaw shape.

Although angles were perceptually strong, they were often difficult to describe. Students often said they were corners (right angles) or sharp. "The sharper it is the bigger it is" thought some, but Jodie, in fact, deliberately took on the opposite word to give the correct sense of size and stated that "the flatter it is the bigger it is." Students commented on the spread of arms of an angle, represented by thumb and fore-finger, to explain why two angles were not equal. The teacher's interaction with the Year 2 students sparked a high degree of understanding of the size of angles and overall shapes. In the groups of three, who were asked to explain their thinking for every session, all the children (including the Year 2 children and the lower ability children) grasped the comparison and size of angles, and made other angles. In the classroom situation, fewer children were really sure about what they were doing, suggesting that not only did they have difficulties in disembedding the angles from the shape but also that student-teacher interactions assisted learning of the angle concept. This interaction was one of the influences on students' cognitive processing, especially on their selective attention. Words also gave students a means of marking aspects of their manipulations and what they were noticing.

Selective attention was important when a group of Year 4 students had made a square from the two large triangles and were trying to cover it with the rest of the tangram pieces. The students had been trying numerous ways of putting the pieces together.

4.01 Tess tries the medium triangle in the corner and places the parallelogram against it. Shen then slides them across to the corner. She flips the triangle as she moves it away. She continues to reposition the parallelogram on the large square.
4.02 Natalie says " perhaps if you turn them over." Tess places the parallelogram in the corner as she had before, looking at the various spaces that are left.
4.03 Natalie gets bored and wants to use the book.
4.04 Tess picks up the square and fills in the top section carefully. She then collects the small triangles.
4.05 Damien moves closer and tries to put the medium triangle correctly. Tess restrains him, taking the triangle from him. She is not sure of how to put it but eventually flips it and places it so it completes the angle and matches the side of the square.
4.06 All three students get excited, Damien moves closer and Natalie says, "Yeah, yeah." They all see where the final two small triangles can be placed. Tess sits back, content.

Tess had shifted and narrowed the focus of her attention once she noticed that a right angle could be made using two angles (45°). At this point she not only solved the problem but became aware of how to make an angle from two smaller angles.

Aspects of the problem-solving model

From the analysis of observations of students and their comments about how they were thinking, it seemed that imagery was supported by conceptualisation which, in turn, was aided by comments from the teacher or from other students, and by observations of other students' work. As a result, students manipulated the materials in certain ways. After observing the results and making decisions about the suitability of certain actions, further manipulations were made. Other interactions with materials such as matching pieces or parts of pieces to compare the size of angles assisted the development of the concept of angle and its size. Students were able to test their concepts and images against the materials, and in this way they were able to assess their own progress. They frequently turned the forefinger away from the thumb in order to consider angle size.

The above descriptions of students learning about angles also serve to introduce two key aspects in the situation which emerged as important aspects of problem solving—responsiveness and selective attention.

Responsiveness

Students' responsiveness during active engagement in problem-solving activities is precipitated by their own thinking and feeling. Their responsiveness affects the immediate social and physical environment which, in turn, influences the person's thinking. It may, for example, be a change in position of concrete materials or a verbal reply by another student. This is illustrated in Figure 3.

Responsiveness is a compound variable; its components are dependent on a balance of cognitive and affective processing. It is a mental action for a specific context; a part of an interaction. It is manifest in physical responses such as expressions, movements, and words which, in turn, influence the context. Part of the context for which responsiveness is occurring consists of taken-as-shared expectations and stable classroom interaction patterns.

Responsiveness is the movement forward, the risk-taking of problem-solving. Often multiple thoughts have to be held for consideration and action over several seconds or minutes until the context reacts to the development. The context change might be a comment from a friend or the new position of materials when acted upon.

Responsiveness implies a degree of understanding of the situation as well as involvement and interest in the activity. There is an ongo-

Responsiveness
Person ...
Imposes concepts and imagery on materials
Manipulates materials
Applies heuristics
Records, displays, describes
Notices aspects of materials / people
Expresses feelings
Communicates with the teacher / student

Context	**Cognitive Processing**

Teacher
Materials
  set problem
  availability
  placement
Other Students
  Comments
  Cooperation
Classroom
  groupings
  seating
  expectations
  time constraints
	Selectively attending
	Perceiving, listening, looking
	Intuitive thinking
	Heuristic processes
	  Establishing meaning of problem
	  Developing tactics
	  Self-monitoring
	  Checking
	Imagining
	Conceptualising
	Affective processes
	  response to organisation, success,
	  confidence, interest,
	  tolerance of open-ended situation

Influence
Context ...
Influences perceptions especially seeing and hearing
Affects feelings
Affects the opportunity to manipulate
Disrupts / prompts thinking
Encourages / discourages communication

Figure 3. Aspects of problem solving (Owens, 1994).

ing dynamic relationship between students and their environment (i.e., other students, the teacher, the classroom, classroom expectations, and the task).

Changes in cognitive processing and the learning environment occur throughout the period of a student's engagement in a learning experience. The student is continually perceiving, thinking and feeling, and then responding; this responsiveness dynamically affects the learning context. There is often a "snowballing" effect, not only on participation, but also on the extent and quality of imagery, concepts, understandings, and problem-solving tactics. The cyclical interaction pattern shown in Figure 3 incorporates growth and continuity like a spiral and yet cycles may overlap like a double helix.

Responsiveness results from a combination of cognitive processes which include attending, perceiving, listening, looking, visual imagining, conceptualising, intuitive thinking, and heuristic processing (such as establishing the meaning of the problem, developing tactics, self-monitoring and checking). Cognitive processing also incorporates affective processes such as reactions to the organisation of the classroom and to success, confidence, interest, and tolerance of open-ended situations. Problem-solving episodes or points involving critical change in thinking are likely to involve both changes in affect and changes in understanding.

Selective attention

One particular aspect of cognitive processing, namely selective attention, gradually emerged from the data of Owens' (1993) study as extremely important in problem solving. It is listed among the cognitive processes in Figure 3. Analysis of the data indicated that many students had selectively concentrated on particular aspects of a problem situation and that the choice of aspect was often highly idiosyncratic. A student's attention may have been focused, for example, on one of the pieces, on a part of a piece, on a configuration of pieces, on a comment of a friend or the teacher, or on the result of an action (Owens, 1993; Owens & Clements, in press).

Selective attention relates to that part of Osborne and Wittrock's (1983) model that initially links long-term memory with sensed perceptions, and then relates the perception to what is stored in memory. It may be linked to focus of awareness as discussed in phenomenographic studies (S. Booth and D. Neuman, personal communications, see Marton & Booth, in press). For example, Ahlberg (1992) noted the effect of discussion, taken-for-granted product-intentions or process-intention on students' number problem solving.

Selective attention is closely related to responsiveness and might be thought of as both the engine room driving the responsiveness, yet something arising from responsiveness. Actions are performed one at a time although several features can be perceived at the same time. Imagery, in particular, provides multiple simultaneous inputs (Kaufmann, 1979). Selective attention guides a student's responsiveness to a problem because of the "too large capacity" (Van der Heijden, 1992) of the mind at the perceptual input stage of mental processing; not all perceptions can be responded to immediately.

The role of language in this processing cannot be overlooked. The intention or purpose of the problem, as this was understood by the student, was important in directing attention. For example, when Tess and her friends were making the square, anything that they thought would not help them achieve that end was disregarded. Tess was busy trying to use the two angles to make the pieces fit into the corner. By contrast, another member of Tess's group was, at this same point in time, considering writing in their book, and this other group member's intention and attention was focused on completing the recording part of the task. Tess and this other group member were, temporarily, attending to different things even though both were on-task.

Conclusion

The data gave support to the contentions that: (a) images are not mere pictorial representations of concrete materials but part of analytical thinking, (b) words do not necessarily convey the meanings intended by the speaker, and (c) students do not learn from concrete materials "embodying" certain concepts (as some believe Multibase Arithmetic Blocks do), but the use of materials can generate problem-solving strategies which assist the interplay of cognitive processes.

Students were learning through attending to certain aspects of the problem. Students who had adequately shown recognition of different angles may not have noticed them later until interacting with the teacher or materials. The students were not necessarily restricted by apparent levels of development or order of constructs as suggested by other studies reviewed earlier. Unlike Mitchelmore's study, the study looked at angles of shapes rather than those represented by two lines but the idea of turning or parting of two lines was used by the students. In particular, the right-angle was contextualised as one of many angles. Like Mitchelmore's study, students in Year 2 (aged 7 years) were able to grasp the angle concepts and this is much earlier than some curriculum documents would encourage. The main difficulty seems to be in the use of the language, especially that of angle. Nevertheless, the angle concept is complex and will take time to develop in all its many manifestations.

The problem solving situations encouraged students to focus on angle size. Students checked their developing concepts by interacting with peers or the teacher or by manipulating materials. It was through problem solving that students were truly being responsive and engaging in learning. Students were noticing and attending; some intention was gained from the words of others. Students were analysing and noting size as part of their early recognition of angles although it was not initially a stable concept. Further focusing, checking, and experiencing of different orientations and contexts assisted the concepts to develop. The meaning of angle was embedded in numerous conceptual frameworks which were developing concurrently and integrally with visual imagery. Mason (1992) coined the phrase "doing and construing" but this study has shown how responsiveness, acting within a specific context as a result of a complex interaction of cognitive processes including affect, has encouraged not only problem solving but the development of angle concepts.

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Tidig geomtriinlärning. Vinkelbegreppet

Van Hieles teori om utvecklingsnivåer har fått stöd från många studier, speciellt i geometri. En viktig konsekvens av van Hieles arbete är att man insett att informella erfarenheter hjälper eleverna att ta det första steget att känna igen t ex geometriska former. Trots denna viktiga inverkan har en del forskare hävdat att visuella resonemang kan vara oberoende av nivåerna eftersom det finns problem att knyta elever till nivåer.

För att illustrera utvecklingen av vår förståelse av elevers inlärning av geometri och rollen av visuella processer diskuteras här elevers svårigheter med vinkelbegrepp. Med utgångspunkt i en omfattande kvalitativ studie förklaras förmågan att identifiera och analysera vinklar i termer av problemlösning eller undersökande verksamhet. Fallstudier visar hur den komplexa begreppsbildningen går till, snarare i växelspel mellan än inom fixa nivåer, som ett resultat av visuella (mentala) bilder, riktad uppmärksamhet, laborerande och diskussioner.

Öppenhet och förmåga att reagera på ett konstruktivt sätt (responsiveness) är en sammansatt och viktig variabel som framträder som ett resultat av en komplex kognitiv process unik för individen. Den ger associationer till inlevelse eller förståelse av problem som ett resultat av sådana processer, speciellt i form av selektiv uppmärksamhet. Variabeln påverkar samspelet med material och med individer som i sin tur påverkar tänkandet och det ständiga kretsloppet av problemlösning och förståelse av vinkelbegreppet.

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