Alternatives to traditional algorithms in elementary mathematics instruction

Rolf Hedrén

The purpose of the research project has been to find out what changes ought to take place in elementary mathematics teaching, if we want to consider the changing need of mathematical knowledge and skill in a society with calculators and computers.

The project is and will be carried out in one class followed through years 2 - 5. The children are now (May 1996) in year 3. Traditional algorithms for the four arithmetic operations are not taught, instead the children are encouraged to invent their own methods for written computation and, besides, to use mental arithmetic and estimation whenever appropriate. Each child has also access to a calculator of her/ his own.

The project is mainly evaluated by qualitative methods, e. g. clinical interviews and observations of groups of children. The results that are discussed, are mainly from year 3. They show that the children, when given the chance, invent a lot of different methods to cope with exercises in addition and subtraction, and in so doing they exhibit signs of increasing number sense. However, many children still have difficulties with regrouping in subtraction.

Introduction

Today calculators and computers are getting more and more common in society. A lot of computation is carried out with the help of these devices. In many countries school children also use calculators and computers in their spare time. The problem is, however, that in most of our schools we just go on as usual teaching the same arithmetic we have taught for a long time. I think that it is about time to take the new technological situation into consideration and to ponder about the possibility of other ways to introduce arithmetic to our children.

Besides, according to the theory of constructivism, no child passively receives knowledge, rather every one actively builds, i. e. constructs, her/his own knowledge. Our children' s heads are not empty

***Rolf Hedrén** is Senior Lecturer in mathematics and mathematics education and a teacher educator in Dalarna University College, Sweden. He has a Ph.D. in mathematics education and has carried out research about the use of computers and calculators in mathematics education.*

vessels. The children come to us with a lot of former experiences, and these will influence the new knowledge they are building. Constructivism, especially in the form of social constructivism, also stresses the importance of cooperation among pupils and among pupils and their teachers. Contructivism will be dealt with more in detail later.

We know that if a child has to compute 3 x 132, and she has not learnt the traditional algorithm for multiplication, she will find a way to do that, e. g. by computing 3 x 100 + 3 x 30 + 3 x 2. She uses her old experience and knowledge of place value. We then ought to ask the question, I think, if we do not take better care of our children's experience, knowledge, inventiveness, and creativity by letting them invent their own methods for written computation than if we tell them the methods as some variant of traditional algorithms.

Written computation will throughout this article mean all computation that is carried out by means of pencil and paper.

No doubt, the traditional algorithms for the four arithmetic operations, were once effective and necessary. They made it possible for people to carry out cumbersome calculations. But today calculators and computers have taken over the work. Therefore, I think that written computation should be changed from the algorithms to methods which the children invent themselves, which they can really understand, and which can help them to understand the place value system, magnitude of numbers, and relationships between numbers. Experience also tells us that the methods that the children construct themselves are nearer to mental arithmetic and estimation. I can see no reason that the children should think in one way when struggling with written computation and in quite another way when working with mental arithmetic and estimation.

Finally, I want to emphasise the fact that in many mathematics curricula the authors have tried to consider the new situation. I give as an example the new Swedish curriculum (Kursplaner för grundskolan, 1994) that came into force in autumn 1995. Among goals that should have been reached at the end of the students' ninth school year, that is when they finish compulsory schooling, is stated:

> The student shall
>
> ...
>
> have reached good proficiency in estimation and in computation with whole numbers, decimal numbers, and with per cent and proportions – mentally, with the help of written computational methods and with a calculator. (Ibid. p. 35, the author's translation.)

Written computational methods can and will be interpreted as the traditional algorithms or methods like the one I gave as an example above. Swedish authorities in education have realised that there are other ways to do computations with pencil and paper than the traditional algorithms and that the calculator will be fundamentally important in the society of tomorrow.

In the light of what has been said above, I think it is really important that we do research to investigate the effects of a change in the implemented curriculum in the way I have pointed towards. With a deeper insight, I think we will be able to convince our teachers that with the abandoning of the traditional algorithms for the four arithmetic operations, their children will not lose anything else than the mastery of historically interesting but today obsolete methods of computation. But more important, I think we will also be able to show them that their children will gain much better understanding of numbers and of relationships between numbers, i. e. *number sense*, and that they, besides, will master mental arithmetic and estimation much better.

This article will deal with my research in one class in primary school. It started when the children were in the middle of year 2 (8 years old). The children have had calculators in their desks all the time, and they have not been taught the traditional algorithms. They are now at the end of year 3.

Constructivism

As I mentioned already in the introduction, constructivism plays a vital role in my research, both as one of the reasons for its realisation and as the theory or philosophy of learning in mathematics that helps me plan and design my research and that also helps me interpret my results.

So much has been written about constructivism (e. g. Björkqvist, 1993; Davis, Maher & Noddings, 1990; Ernest 1991a, 1994; von Glaserfeld, 1991) that there is no need to go into details here. Nor will I take up advantages and disadvantages of different constructivist positions. I will only dwell on a few points that I think are especially important for my research, and in doing so, I will especially take advantage of the ideas of social constructivism.

Common to trivial constructivism, radical constructivism, and social constructivism is the so called first principle: "Knowledge is not passively received either through the senses or by way of communication. Knowledge is actively built up by the cognizing subject." (E. g. von Glaserfeld, 1990, p. 22). Radical constructivism goes a step further saying that the learner's previous constructions, her previous experiences, and her view of the outside world play a vital role as she is struggling to build up new knowledge structures. This is expressed in von Glaserfeld's second principle: "The function of cognition is adaptive, in the biological sense of the term, tending towards fit or viability. Cognition serves the subject's organization of the experiential world, not the discovery of an objective ontological reality." (E. g. von Glaserfeld, 1990, p. 23)

Even if von Glaserfeld defines learning as the learner's own organising of her/his experience of the outside world, he also acknowledges that this building up activity occurs as the learner interacts with peers and teachers (Cobb, 1994, p. 14). At school, a child will never be learning in a vacuum. S/he will always be surrounded by class-mates and one or several teachers. In my opinion it is therefore necessary to link the theory of the learner's activity to a theory of the importance of her/his shared practice with others, her negotiating of ideas, problems and solutions with class-mates and teachers. This is done in social constructivism, that "regards individual subjects and the realm of the social as indissolubly interconnected" (Ernest, 1994, p. 8.).

It is interesting to see that Ernest (1991b, p. 106) emphasises the strong parallel between social constructivism and Vygotsky's (1978) social theory of mind. In connection with my research I want to stress that the learner tries to fit new pieces of knowledge into the knowledge structures that s/he has already built up. New knowledge is interpreted by or 'filtered' through existing knowledge. An advancement can most easily be achieved if there is a knowledge structure where the new ability or concept will fit. To me this is a parallel with Vygotsky's (ibid) concept "the zone of proximal development", where the interaction between the learner and more knowledgeable others has a good chance to lead to new insights.

Summarising what has been said, my research builds on three principles:

1. The learner actively builds up her own knowledge.
2. The learner's previous experience plays a vital role during this construction.
3. The learner's interaction and dialogue with others is crucial for her knowledge construction.

The impact of calculators and computers

Calculators and computers exist

Already from a constructivist point of view it is difficult to see why school should go on teaching ready-made methods for written computation, which, to a great extent, is done today. The teaching of the algorithms for the four arithmetic operations has become still more questionable with the advent of calculators and computers. These devices have taken over a big part of the computational work, which was previously carried out with the help of the above mentioned algorithms.

Already in 1979 Plunkett discussed the nature of standard written algorithms. Among others things he stated:

- (The algorithms) are *analytic.*They require the numbers to be broken up, into tens and units digits, and the digits dealt with separately.
- They are not *easily internalised.* They do not correspond to the ways in which people tend to think about numbers.
- They encourage *cognitive passivity* or suspended understanding. One is unlikely to exercise any choice over method and while the calculation is being carried out one does not think much about why one does it in that way. (Ibid p. 3.)

Besides, they are used very little even by children. They are also very often applied unthinkingly to computations like 1000 – 995 or 100 x 26. (Ibid p. 3.) (In this case you had better look upon the numbers *holistically*. You should realise, for instance, that 995 is very near to 1000.)

Many authors emphasise the problem that a lot of children do not understand the traditional algorithms and suggest other methods for written computation, informal methods that the children should invent themselves (e. g. Krauthausen, 1993; Olivier, 1988; Reys, 1994; Sowder, 1992). Krauthausen (ibid) also introduces a suitable term for these informal methods "halbschriftliches Rechnen" (about half-written computation), indicating that in these methods necessary intermediate steps and partial results are written down.

In 1992 the yearbook of the National Council of Teachers of Mathematics (NCTM) in the U S dealt with the impact of calculators on mathematics teaching in elementary school. Wheatley and Shumway (1992) discusses some consequences in our society of calculators and computers. As far as curriculum is concerned, however, we had better start from zero, i. e. without being affected by traditions and curricula from the past:

> Arithmetic would have a place in our "zero based" curriculum, but it would have new goals and emphases. It would be much more important that students *know when to subtract* than that they be able to use a prescribed and complex subtraction algorithm efficiently. Mathematics would be characterized by the search for patterns and relationships rather than fixed procedures to be mastered (Steen, 1990). With the use of calculators, attention would focus on meaning, and mathematics would become a much more exciting activity for students. Mental arithmetic and estimation would become major components of the school mathematics curriculum. Students would be encouraged to create their own algorithms for simple computations, but they would be encouraged to use calculators whenever it made sense to do so. (Ibid p. 2.)

If we really want to introduce such a new and revolutionary curriculum, there is an urgent need for research in this area. Quite a lot has already been accomplished, as I will show in the next section.

Research on alternative computation methods

The most famous research, where children have been allowed and encouraged to invent their own methods for written computation and where they have not been taught the traditional algorithms, is probably the Calculator Aware Number (CAN) project in Britain (Shuard et al, 1991). Beside using their own methods for written computation, the children in CAN always had a calculator available, which they could use whenever they liked. Exploration and investigation of "how numbers work" was always encouraged, and the importance of mental arithmetic stressed. (Ibid p. 7.)

I try to quote and summarise the most important results of CAN :

- The children's enthusiasm for mathematics was often greater than in pre-CAN days. (Ibid p. 56.)
- The children developed a wide variety of methods for non-calculator calculation, where they made an intuitive use of basic mathematical principles. (Ibid. p. 57.)
- The children worked with large numbers, negative numbers and decimal numbers much earlier than in traditional instruction. (Ibid pp. 13 ff.)
- The teachers' style became less interventionist. They began "to see the need to listen to and observe children's behaviour in order to understand the ways in which they learn". (Ibid p. 56.)

Kamii (1985, 1989, 1994; Kamii, Lewis & Livingston, 1993/94) was working together with the children's class teachers in grades 1 - 3 in a similar way in the U. S. She did not teach the traditional algorithms but encouraged the children to invent their own methods for the four arithmetic operations. She also devoted much time to different kinds of mathematical games. She claims the following advantages:

1 they (the children) do not have to give up their own thinking;

2. their understanding of place value is strengthened rather than weakened by algorithms;

3. they develop better number sense.

(Kamii, Lewis & Livingston, 1993/94, p. 201.)

According to Kamii et al. "many of the children who use the algorithm unlearn place value ..." (ibid. p. 202). See also the reference to Narode et al. below.

In one research project she used two experimental classes and one control class. In the last-mentioned (class 1) the children were taught the algorithms in a traditional way. The difference between the two experimental classes was that the parents in one of them (class 3) had been asked not to teach their children the traditional algorithms, which was not the case in the other one (class 2). The evaluation gave a very interesting result.

The pupils of the three classes were given the problem 7 + 52 + 186 to be computed mentally. Not only were the results of class 3 better than those of class 1, but the erroneous answers in class 1 differed much more from the correct one than did the wrong answers in class 3. The results in class 2 lay between those in the other two classes.

Similar research has been carried out by Olivier, Murray & Human (1990; Murray, Olivier & Human, 1991) in South Africa, Harz in Denmark (1993), Sandahl and Unenge in Sweden (Unenge, Sandahl & Wyndhamn, 1994) and Hedrén in Sweden (1995a, 1995b). The results are similar to those described earlier. Above all they stress the children's competence and power to find out their own methods while they consider and take advantage of the numbers involved and use facts about numbers, especially place value, laws of arithmetic and divisibility of numbers.

Narode, Board and Davenport (1993) concentrated on the algorithms' negative role for the children's understanding of numbers. In their research with first, second and third graders they found out that after the children had been taught the traditional algorithms for addition and subtraction they discarded their own invented methods which they had used quite successfully before the instruction. They also tried to use traditional algorithms also in mental arithmetic, they gave many examples of misconceptions concerning place value, and they were quite too willing to accept unreasonable results achieved by wrong application of the traditional algorithms.

According to Beishuizen (1993) there exist two widely used strategies for mental addition and subtraction with ten (or twenty etc.). In the first one the child adds (or subtracts) the tens first and the units afterwards as in: 34 + 10: 30 + 10 = 40; 40 + 4 = 44. Beishuizen calls it the 10 + 10 (1010) or the split strategy. In the second one s/he adds (or subtracts) the tens of the second term to (from) the undivided first term as in: 34 + 10: 34 + 10 = 44. He calls it the N + 10 (N10) or the jump strategy, because the child jumps from the given number in steps of ten or twenty etc.

It should also be pointed out (see e. g. Hedrén, 1995a; Shuard et al., 1991; Thompson, 1994) that the pupils in their own invented methods almost exclusively start their computations from the left hand side, i. e. with the digits that represent the highest value. There we can see a great resemblance to methods used by children and adults when doing mental arithmetic and estimation.

Number Sense

The term number sense has been used often in the foregoing section, and I think that it requires further discussion.

Barbara Reys describes the concept in the following way:

> Number sense refers to an intuitive feeling for numbers and their various uses and interpretations; an appreciation for various levels of accuracy when figuring; the ability to detect arithmetical errors; and a common-sense approach to using numbers. Number sense is not a finite entity that a student either has or does not have, nor is it a unit that can be "taught" then put aside. (Reys, 1991, pp. 3-4.)

In her opinion, a person who is able to value and use number sense:

- will look at a problem holistically before confronting details ...;
- will look for relationships among numbers and operations and will consider the context in which a question is posed ...;
- will choose or invent a method that takes advantage of his or her own understanding of the relationships between numbers or between numbers and operations and will seek the most efficient representation for the given task ...;
- will use benchmarks to judge number magnitude ...;

- will recognize unreasonable results for calculations in the normal process of reflecting on answers .... (Reys, 1994, p. 115.)

Finally I want to quote from NCTM:s *Curriculum and Evaluation Standards* (1989) that gives the following aspects of number sense. Children with good number sense

(1) have well-understood number meanings,

(2) have developed multiple relationships among numbers,

(3) recognize the relative magnitudes of numbers,

(4) know the relative effect of operating on numbers, and

(5) develop referents for measures of common objects and situations in their environment.

(NCTM, 1989, p. 38.)

Quite a lot has been written about number sense (see e. g. Greenes, Shulman & Spungin, 1992/93; McIntosh, Reys & Reys, 1992; Reys, 1994, Reys, Reys & Emanuelsson, 1995). In connection with my own research I want to add some extra aspects. A child with good number sense can also

(6) understand that numbers can be represented in different ways (decimal form or as a fraction, or graphically e. g. on a number line);

(7) know the divisibility of numbers;

(8) recognise the suitability of different numbers in different situations.

Some authors want to include estimation in the concept of number sense. I would, however, prefer to see estimation as one mode of computation, together with mental computation, pencil-and-paper assisted (written) computation, and computation with the help of calculators and computers. Of course, number sense is an important ingredient in the skill to make judicious estimation, but so is also the case with mental computation and non-traditional written computation.

Koyama (1994) emphasises the relationship between computational estimation ability and mental computation ability. He draws the following conclusion: "These results suggest that the reasonable and efficient computational estimation requires flexible rounding of numbers based on a sound number sense as well as mental computation ability." (Ibid. p. 42.)

Purpose and Questions

The traditional drill of the algorithms for the four arithmetic operations should, in my opinion, be questioned for two reasons:

1. The availability of calculators and computers, which can do the computation fast and accurately,
2. Social constructivism, the dominant philosophy for mathematics learning, asserts that a child learns when s/he is actively constructing her/his own knowledge and has a possibility to build on her prior cognizance and experience. Her/his learning is also enhanced by discussions with peers and others, who are more knowledgeable, i. e. teachers and other adults.

Besides it is likely that this drill of the algorithms is one of the causes of the children's declining motivation for mathematics during primary and intermediate years.

In this study I want to investigate whether the algorithms can be replaced by the children's own methods for the four arithmetic operations and what effect such a replacement would have on the children's number sense, their ability to do mental arithmetic and estimation and their motivation for mathematics. Here I mean their motivation for mathematics as a whole, because I think there might well be transfer from a possible dislike of arithmetic to dislike of mathematics. I will also look for possible drawbacks of the abandoning of the traditional algorithms. Will the children lose anything else than the inheritance from former generations, the ability to do sums in an effective but mechanical way?

Earlier research (Ljung & Pettersson, 1990, p. 51-52) showed that girls' skill in computation is higher than boys', while boys had a superior ability to solve problems and a better number sense. We do not know how the computation was carried out, but probably the algorithms were used. The methods for computation used in this experiment will build more on problem solution ability and number sense than the algorithms do. Will there be a risk that the girls are treated unfairly in such a case?

Thus I want to see what changes will occur in a class when the children are given the possibility to invent and develop their own methods for written computation. Especially I will try to get answers to the following questions:

1. How is the children's number sense affected?
2. How is the children's ability to do mental computation and estimation affected?

3. How is the children's motivation for mathematics affected?
4. Is there a difference between girls' and boys' number sense and ability in mental computation and estimation?
5. Is there a difference between girls' and boys' motivation for mathematics?

Methods

Realisation

I have mentioned some earlier research projects (see e. g. Kamii, 1994; Shuard et al., 1991), where the children have not been taught the traditional algorithms for the four arithmetic operations. As far as I can see a comparison between the children taking part in the projects and those being taught algorithms in a traditional way leads to the following conviction: The children in the experimental classes get a better feeling for numbers and relationships between numbers, i. e. number sense, and a better ability to do mental computation and estimation than their counterparts who are taught in a more traditional manner.

However, in this study I want to go into more detail to follow as many children as possible in one experimental class to see what really happens during the children's learning of arithmetic. The questions asked in the foregoing section are followed by a lot of others. What methods do the children use? How do these methods develop during the experimental period? How do the children react when they get to know that their school-mates and their possible older siblings are and have been taught arithmetic in another way? How do their teachers react? How do their parents react? Are they able to help their children in an adequate way, when they cannot use the skill they were taught themselves during their corresponding school years? What are the possible draw-backs with alternative methods for written computation? Of course, I will not be able to answer all these questions in this article.

I have chosen to follow one class from spring term in year 2. (They were then 8 years old.) I hope to be able to go on following the same class until they have finished year 5. The reason for starting in the middle of year 2 is that teaching of algorithms in Sweden generally starts in spring of that school year. The class changed class teachers between years 2 and 3.

The class is an ordinary Swedish class in a middle-sized school for years 1 - 6 in a middle-sized Swedish town. They are now (May, 1996) in year 3, and there are for the moment 9 girls and 13 boys in the class. Unfortunately, there have also been and probably will be changes in class composition.

I have chosen not to have a comparison or control class. As I want to study in depth how the children experience their situation and how their knowledge structure in arithmetic develops during the experimental years, I have found it advantageous to stick to one single class. Besides, I have not seen any reason to use a control class. I could never guarantee that all conditions except those I want to examine would be the same in the experimental and the control class. I am also aware that my experimental class is in some way in a rather fortunate position, given extra attention from a teacher at an education department. Besides, the teachers that want to and dare take part in an experiment like this, are generally especially knowledgeable and interested in developing their teaching style.

In short, the following steps are taken in the experimental class:

1. Written computation

The children are encouraged and trained to use other paper-and-pencil methods than the traditional algorithms to carry out such computations that they cannot do mentally. These methods should as far as possible build upon the children's own thoughts and ideas. Thus, no special methods are taught or forced upon the children. The methods are discussed in groups and in the whole class. The traditional algorithms are not taught and the children's parents are also encouraged to help their children using alternative computational methods and not to teach them the algorithms.

2. Mental arithmetic and estimation

Mental arithmetic and estimation are encouraged and practised. The children are encouraged to invent their own methods, which are discussed in class.

3. Calculators

The children will have calculators in their desks. They will be used for number experiments and for checking computations made in other ways.

The main purpose of the study is investigating what happens when children are not taught the traditional algorithms. Perhaps, I could have stopped at that. However, I think it would be wrong not to introduce the calculator to the children. After all, the reason that the algorithms have become obsolete is the advent of the calculator - and the computer. On the other hand, it is not very natural for a 8 - 11 years old child to do computations with the help of, say, the spreadsheet program of a computer. For this reason, the computer has not been included in the study.

With the exceptions mentioned under 1 - 3 above, the children follow a traditional course. Finally, I have to mention that the ordinary teacher has full responsibility for the mathematics periods. My own task is to design the experiment, to encourage and give advice to the teacher, and to evaluate the project.

Evaluation

As I mentioned before, I found it essential to follow one class very thoroughly in the experiment. Therefore, I try to use as many methods of evaluation as possible, both qualitative and quantitative. The following methods are used:

Qualitative
Clinical interviews
Observations
Copies of children's writing and calculations at the observed occasions
Interviews with children
Questionnaires with open questions
Interviews with teachers

All interviews and observations are tape recorded.

Quantitative
Tests
Questionnaires

Although there are at present only 22 children in the class, I have found it most advantageous to follow six of the pupils more thoroughly. I have chosen one girl and one boy from the middle of the upper third of the class, one girl and one boy as near the median of the class as possible and one girl and one boy from the middle of the lower third of the class. These children were picked out with the help of the results of the very first test battery, in the beginning of the spring term in year 2.

The *clinical interviews* have been, and will be, undertaken with these chosen children at the following occasions:

In the beginning of spring term in year 2,
in the middle of spring term in year 3,
in the middle of spring term in year 4, and
at the end of spring term in year 5.

The problems I have used and will use deal with mental arithmetic, written computation, where the children are free to use any method they like, estimation and number sense.

In year 3 the following problems were given:

1. Compute mentally. Say only the answer. 89 + 12
2. Compute mentally. Say only the answer. 63 – 18
3. Compute. You may use paper and pencil. 267 + 153
4. In a small school there are 272 children. One day 69 children were ill. How many children were at school.
5. Stina buys meat for 37 kronor, a bottle of juice for 28 kronor and cheese for 31 kronor. She has 100 kronor.

   a. Will the money be enough for her to pay for the three things?

   b. Is there a lot missing?/ Will a lot be left over?
6. a. What is the greatest whole number that you can write with the digits 2, 5 and 6?

   b. What is the smallest whole number?

The *observations* are undertaken about once a week during the whole school year. During an earlier research project I have found it most favourable to follow a group of 2 - 4 children. In that way I can discuss in peace and quiet with the children while they are working with exercises in arithmetic, with or without text, and get to know their thoughts and ideas. I can study very proximately how the children cooperate with one another when they are trying to cope with difficult computations or other problems. There is also less risk that I will miss interesting situations than if I try to follow all children when they are working in the classroom. I save copies of the children's writing on each occasion. The questions I ask myself during the observations are:

How do the children use their number sense when dealing with mental or written computation?

Can I observe a factor that facilitates the children's learning, including cooperation with peers and/or me undeliberately acting as a teacher?

Can I observe a factor that makes the children's learning more difficult?

Can I observe a factor that enhances or diminishes the children's motivation for arithmetic and/or mathematics?

Interviews with children are done on the same occasions as the clinical interviews. They are a follow up of the questionnaires. Although the questionnaires give a rough idea of the children's opinions, there is always a risk that children as young as in primary or intermediate school years misunderstand the questions. A discussion in peace and quiet with one child at a time will therefore give additional valuable information about her/his opinions and beliefs.

Even if I try to visit the experimental class once a week, I cannot get a direct picture of what is going on in the classroom, what kind of exercises and problems the children are working with, how they react, and when they exhibit extremely brilliant ways of solution etc. There is, however, one person, who could give a supplementing picture, the class teacher, if s/he is aware of what is going on. I therefore *interview* her/him once a week and more thoroughly at the end of each term.

The following *tests* are given on the same occasions as the clinical interviews are undertaken:

Mental arithmetic,
written computation, where the pupils can use any method (including mental arithmetic),
estimation, and
number sense.

Some of the items given will occur on many occasions, giving me a possibility to follow the children's development. Tests give a rough overview of all the children's knowledge and skill. On the other hand, unless a child has written careful and interpretable notes, they do not say anything about her thoughts and solution methods. Therefore, they are supplemented by clinical interviews, as mentioned above.

Finally *questionnaires* are handed out on the same occasions as the tests. The questions deal with the children's interest in mathematics and in different domains and modes of mathematics such as mental arithmetic, written computation, word problems, and the use of calculators. I also ask the children if they prefer to follow a given rule (algorithm) in written computation or to invent their own methods. In many of the questions there is ample space for children to formulate their own thoughts. As mentioned earlier I find it very essential to supplement these questionnaires with interviews with the selected children.

The reliability of the experiment.

Only one class takes part in this experiment. Of course, it cannot be avoided that the special composition of children in this class affects the outcome. However, judging from my own experience and from the statements of the class teachers who have been involved, it is a pretty typical Swedish class, but perhaps with more social problems than the average class in years 2 and 3.

One problem with the design I have used is my own action. Without doubt, the children are positively motivated by being paid attention to by a lecturer in mathematics in teacher education. As I have used small groups in my observations, they have also had greater opportunities than usual to work in groups under supervision of an adult. Besides it is unavoidable that the teachers that dare take part in an experiment like this, are more interested and more knowledgeable in mathematics and mathematics education than the average primary school teacher. The results I have got must therefore be studied with the reservation that the teaching effort is of more than average quality.

Results

Introduction

In the following sections I discuss answers to the questions above in the same order that they were posed. However, some results give answers to more than one question. In such cases I have tried to put them under the most relevant heading. So far, the children have worked mainly with two arithmetic operations, addition and subtraction. The sums dealt with in multiplication and division have been so easy that they do not require written computation. Thus, I will only discuss the first mentioned two operations here.

Only three girls and three boys took part in the clinical interviews and the interviews. I will name the girls G1, G2, and G3, where G1 was in the middle of the upper third of the class, measured by the results of the tests in year 2, G 2 was in the middle of the class and G3 in the middle of the lower third of the class. Similarly I will call the boys B1, B2, and B3. As a matter of fact, the results of the tests in year 3, confirmed by the teacher's statement and my own observations, showed that the three girls did not hold their positions; in fact, the order of the girls was completely interchanged. The three boys, on the other hand, kept their relative positions among themselves and also fairly well within the class.

How is the children's number sense affected?

I turn first to the tests. Instead of giving tables with results of the different tests, I will concentrate on the notes and drawings that the children have made on the test papers. In the test in written computation the children were given ample space to make notes or drawings to help them. In year 2 quite a lot of children took advantage of this and drew money to help them. No other kind of pictures or notes were seen, with one exception, and that will be mentioned later. I will give just one example of the use of money drawings: In item 7, 37 – 13, G1 drew this picture to help her:

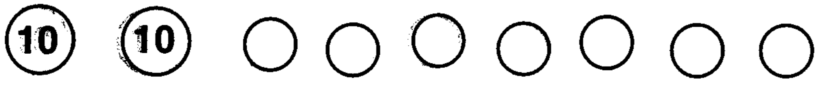

Figure 1. A picture to help a girl compute 37 – 13.

She probably started with the computation 37 – 10 and arrived at the answer 27 as seen in the picture. After that she only needed to subtract 3. The final answer was correct.

Also in multiplication items some children made pictures to help them arrive at the correct answer. When computing 3 x 4 and 2 x 12 respectively one girl drew 3 rows with 4 circles and 2 rows with 12 circles.

In year 3 the children wrote notes instead of drawing figures, although in most cases the space between the item and the place for answer was left empty, indicating that the child had done the whole computation mentally. To help her compute 31 – 12 one girl, for instance, wrote:

30 – 10 = 20
1 – 2 =

Although she did not write an answer to the last subtraction, she got the correct answer 19. In accordance with many observations, she most probably thought – though, expressed in her own words - that the result was negative one, and that she thus had to subtract one from 20.

The same girl unsuccessfully tried the same strategy in the following, more difficult item, 371 – 269:

$$300 - 200 = 100$$
$$70 - 60 = 10$$
$$90$$
$$1 - 9 = 8$$

She wrote 90 below the result 10, probably mixing up positive and negative numbers. The final answer was given as 81.

It is interesting to see, though, that there are also a few notes in the test in mental arithmetic in year 3. One girl wrote 40 – 4 to help her compute 53 – 17, and one boy wrote 43 – 7 in the same item. In both cases the answers were correct, and I saw no reason to turn down the solution in spite of the heading "Mental arithmetic".

It can also be stated that some simple problems in multiplication and division (3 x 132; 64 ÷ 2) were solved mentally in year 3 by most children, although the operations had just been mentioned but practised very little.

There was a special test in number sense in both years. One item was given both years: "Write the number that consists of one hundred and three tens" in Swedish "Skriv det tal som består av ett hundratal och tre tiotal". In year 2 14 out of 21 children solved this item correctly, in year 3 only 12 out of 22 children.

The last problem in the clinical interviews (*a. What is the greatest whole number that you can write with the digits 2, 5 and 6? b. What is the smallest whole number?)* was solved directly by B1, B2, G2, and G3, and these children could also explain why 652 and 256 respectively are greatest and smallest. The other children seemed to have some difficulties in finding the answers, but they arrived at the correct answers after I asked them to compare some numbers they had already written down.

How is the children's ability to do mental computation and estimation affected?

There are a few common items in the tests of year 2 and year 3. I discuss the test in written computation here, as most children did the computation mentally.

In the item 125 + 14 the result changed from 10 correct out of 21 to 21 out of 22;
in the item 31 – 12 from 8 out of 21 to 14 out of 22; and
in the item 2 x 12 from 10 out of 21 to 21 out of 22.

Although we can see an obvious improvement of the results, subtraction with regrouping remains a tangible obstacle.

In the test of estimation there were two common items, where they pupils had to tell how many ten kronor coins they need (as a minimum) to pay the amount of 58 kronor + 39 kronor and 24 kronor + 13 kronor respectively. I could see a great improvement, from 6.5 correct answers out of 21 in both items in year 2 to 16 respectively 13 out of 22 in year 3.

In the first item of the clinical interviews in year 3 *(Compute mentally. Say only the answer. 89 + 12)* the six pupils picked out for clinical interviews had no difficulties at all.

In the second one *(Compute mentally. Say only the answer. 63 – 18)* a problem arose in the subtraction of the units, 3 – 8. G1 and B2 quite simply subtracted 8 – 3 and got the final answer 55. G2, G3 and B1 also said "8 from 3 equals 5" (G2) or "3 from 8 equals 5"(G3, B1), but they understood that they had to take this 5 away from 50 (after 60 – 10 = 50). B3, on the other hand, solved the problem in quite another way: 60 – 10 = 50; 50 – 8 = 42; 42 + 3 = 45.

In the third exercise *(Compute. You may use paper and pencil. 267 + 153)* the three boys refrained from writing and managed to solve the problem mentally. All the girls did the following written computation:

200 + 100 = 300; 60 + 50 = 110; 7 + 3 = 10; 267 + 153 = 420.

G2 and G3 first tried to find the answer without the notes but were unsuccessful. On the other hand they had very little difficulties when they had written down the sum of the hundreds, the tens and the ones. (Compare their attitudes to mental viz. written computation below.)

The boys continued to solve the fourth problem *(In a small school there are 272 children. One day 69 children were ill. How many children were at school?)* mentally and successfully. B2 wrote 210, though, to help him remember. B3 used the same method as in problem 2 to compute 212 – 9 (after 270 – 60 = 210). He said 10 – 9 = 1, 1 + 2 = 3. G2 wrote 200; 70 – 60 = 10; 2 – 9 = 7, but she understood that she had to take 7 away from 210. G1, on the other hand, gave the answer 217. She started like G2 but went on with 210 + 7 = 217. Even when I changed the number of children in the school to 72, she persisted in giving the answer 17 and could not see the unreasonableness of the answer. G3, finally, used another method. She started with a guess and added 208 + 69 in a written computation. She got the answer 277 children, and with this result as a starting point she eventually realised that the correct answer was 203 children.

None of the children seemed to like the idea of estimation. In the fifth problem (*Stina buys meet for 37 kronor, a bottle of juice for 28 kronor, and cheese for 31 kronor. She has 100 kronor. Will the money be enough for her to pay the three things?)* all the children except G2 added the prices of the three items exactly. The boys and G3 did the computation mentally. G1 did the addition with paper and pencil. G2 did a written, exact subtraction. However, all the children got the correct answer.

I made observations in the class almost once a week. This gave me a lot of possibilities to examine the ways the children made mental computation and, indirectly, also to get to know their number sense. I will concentrate here on three types of problems, which I think are of special interest, and I will only give examples of especially interesting strategies:

- Addition and subtraction of 10 and 20.
- Addition of two two-digit numbers with regrouping.
- Subtraction from a two-digit number with regrouping.

I am quite aware that most of the strategies shown below are known from earlier studies. However, I want to give an account of how the children in this experimental class worked and reasoned.

Where the children, who were especially selected for clinical interviews and interviews are involved I will use the same symbols for them as above.

Addition and subtraction of 10 and 20.

34 + 10 = 44. Counts on their fingers 35, 36, ..., 44.
(Two girls, March year 2.)

34 + 10 = 44. Gives the answer directly. (Jump. Girl, Febr. year 2.)

37 + 20: 30 + 20 = 50; 50 + 7 = 57. (Split. G2, Sept. year 3.)

37 + 20 = 57. Gives the answer directly. (Jump. Boy, Oct. year 3.)

44 – 10: 40 – 10 = 30: 30 + 4 = 34. (Split. G3, Sept. year 3.)

44 – 10 = 34. Gives the answer directly. (Jump. Girl, Oct. year 3.)

The words "split" and "jump" are used in the way Beishuizen (1993) used them, described under the heading *"Research"*.

Addition of two two-digit numbers with regrouping

27 + 15: 20 + 10 = 30; 7 + 5 = 12; 30 + 12 = 42. Explanation: "Then, you cannot have the whole of twelve on 30 something" ... "Then you have to add ten.". (G2, March year 2.)

In a group of three girls, two of them calculated 49 + 22 = 61. The explanation was simple: 40 + 20 = 60; 9 + 2 = 11; just skip the first digit one, and you get 61. G2, who was a member of the group, got 71. Her explanation was a little more sophisticated: "Well, you cannot continue like this, sixty ten, sixty twelve, or sixty eleven, like that, you cannot do that, then you must start on seventy, seventy one." (March year 2.)

37 + 89: 80 + 20 = 100; "He (seven) gives one to it (nine), so it makes ten, too"; 10 + 10 + 6 = 26; Answer: 126. (Boy, May year 2.)

Subtraction from a two-digit number with regrouping.

15 – 8 = 7, because 8 + 7 = 15. (G3, Sept. year 3.)

25 – 18 = 13. 20 – 10 = 10; 5 – 8 = 3; 10 + 3 = 13. (G1, G2, and G3, Sept. year 3.) All three girls were absolutely convinced that the result was correct. I had to challenge their result by giving them an everyday problem: "You have 25 kronor in your purse and buy a bar of chocolate that costs 18 kronor, how much is left?". Then G2 finally wrote 19, 20, ..., 25 and convinced herself and - with great difficulty - also her group mates of the correct result. These girls found this discussion and the final discovery of the correct answer so thrilling that I had great difficulty getting them to finish working and have a break.

25 – 18: 20 – 10 = 10; 10 + 5 = 15; 15 – 5 = 10; (8 – 5 = 3); 10 – 3 = 7. (B1, Oct year 3.)

33 – 17: 30 – 10 = 20; 3 – 7 = (–) 4; "And then you take away these four from 20. Then it will be 16." (B1, Oct year 3.)

25 – 18: 25 – 5 = 20; 20 – 3 = 17 "And then you have three left from the eight."; 17 – 10 = 7. (Girl, Oct. year 3.)

33 – 17: 33 – 10 = 23; 20 – 7 = 13; 13 + 3 = 16. (Girl, Oct. year 3.)

25 – 18: 20 – 18 = 2; 2 + 5 = 7. (Boy, Oct. year 3.)

37 – 19: 19 + 19 = 38; thus 19 + 18 = 37, and 37 – 19 = 18. (Boy, Oct year 3.)

35 – 16: 30 – 10 = 20; 5 – 6 = (–) 1 ”Then I took six. Six is more than five, you know. Well, then you put minus. Then it will be 19.” (B2, Dec. year 3.)

35 – 16: 16 + 16 = 32; 16 + 17 = 33; 16 + 18 = 34; 16 + 19 = 35, thus 35 – 16 = 19. (B3, Dec. year 3.)

How is the children´s motivation for mathematics affected?

So far, I can only answer this question with the help of the questionnaire and the interviews with selected children in year 3. The most interesting question in the questionnaire is *What do you think is best when you are supposed to carry out an addition or subtraction with paper and pencil: 1) to follow a certain rule that someone has taught you, 2) to think out yourself how to do to make it correct?* All children except one boy thought that alternative 1) was the best one. A boy belonging to the majority said: ”Then it is easier to do sums, I think”.

This attitude was confirmed in the interviews. With the exception of G2 (thus not the child mentioned above) all the children preferred choosing their own methods to using traditional algorithms. B3 wanted, however, to learn the algorithms later.

The interest in calculators went down considerably. In year 2 all children except one were looking forward to using this device, whereas in year 3 expecially the boys were pretty indifferent to the calculator. The children (in year 3) also made comments about the possible advantages and disadvantages of calculators in primary school. I give some examples:

What do you think you can learn by using a calculator in primary school?

You learn to handle a calculator. (One girl, one boy.)
You learn the calculator and the digits + arithmetic operations. (Boy.)
Then it is boring, if you look at the calculator, you don’t learn anything. (Girl.)

Do you think that there are any disadvantages with using calculators in primary school? Write them if that’s the case.

You don’t learn anything if you always use it. (Girl.)
I think you should compute yourself. (Girl.)

> Well, you cannot always use the calculator. You have to use your head, too. Otherwise it is cheating, you know. (Girl.)
> You don't learn anything and use it too much. (Girl.)
> Perhaps, you don't learn to compute mentally as well. (Boy.)
> You cannot learn to compute mentally. (2 boys.)
> No. (Boy.)
> Only shit. (Boy.)

The interviews, too, gave the same result. G1 and B1 just mentioned that the calculator is a good thing. G2, G3, and B3 stressed the importance of learning to compute "in the old way" (B3), which as far as I could understand did involve mental arithmetic and written computation without using algorithms. B2, who is a very quiet boy, did not make a decision.

It is also interesting to see if the children stick to their own methods for written (or mental) computation or if they are taught and use traditional algorithms in spite of the experiment's intention. According to the interview with the class teacher and my own observations, 2 girls and 2 boys are using the algorithms from time to time. One of the girls has changed classes and was taught the algorithms for addition and subtraction in her former class, and she persists in using them. Her teacher and I cannot see a reason to force her to abandon them. The other children have, so far, had some difficulties with mathematics, and their parents think they can help them by teaching them the algorithms.

With the just mentioned exceptions all the parents accept the avoiding of the algorithms. Interestingly enough there has just (May, year 3) been a parents' evening, requested by the parents, where they, the class teacher and I discussed different alternative computation methods.

So far, it is not possible to judge how the children's motivation for mathematics might be affected. I would like to mention, though, that B3 told me in the interview that mathematics was "heavy" in years one and two, but that it now was fun. His attitude has been confirmed by his class teacher and by earlier observations I made.

Is there a difference between girls' and boys' number sense and ability in mental computation and estimation?

The girls were better than the boys in most tests in year 3. The only exception was the test in number sense. See table 1. However, none of the differences was statistically significant.

Table 1. Tests in spring term in year 3. Results for girls and boys.

Test	Girls			Boys			Max
	n	x	s_x	n	x	s_x	score
Mental computation	9	10	2.60	13	8.46	3.36	12
Estimation	9	4.94	1.98	13	4.19	2.02	8
Written computation	9	7.33	2.24	13	6.92	2.33	10
Number sense	9	7.11	2.07	13	7.96	1.48	10

Is there a difference between girls' and boys' motivation for mathematics?

I will first answer this question with the help of the questionnaire and the interviews with the six selected children in year 3. The only differences I can see in the questionnaires, occur in the motivation for mental arithmetic and for the use of calculators. Boys are more interested in mental computation, 11 out of the 13 boys find it good fun or quite fun, only 4 of the 9 girl make the same choice. This difference is almost significant, judged by Fisher's exact probability test. ($p = 0.06$.)

The interviews showed that all the three boys preferred mental computation to written computation. The three girls answered more circumstantially. They realised that more complicated computations are better done with paper and pencil and that it might be difficult to keep all numbers involved in their heads, when they do not note anything. Although the few children interviewed make this result rather unsure, it is in accordance with the foregoing one.

It was quite the other way round with the interest in calculators. In the questionnaire 8 out of 9 girls found it great fun or fairly great fun to use a calculator in primary school, the remaining girl answered neutrally. Only 7 of the 13 boys were very or fairly positive to the use of calculators in school, and 3 of the boys found it even very boring. If I use the same method in the statistical test as in the foregoing question, the difference is, however, not significant.

All the six selected children except B2 have a calculator of their own. None of the children use it very often, though. G3 told me that she might sometimes use it to do a part of a written computation. She might for instance add the thousands, the hundreds, and the tens mentally, but the ones with the help of the calculator.

Summary of results

Already in the simple exercises of adding or subtracting 10 or 20, there are differences in the methods different children use. Many children keep to the less effective split method even in year 3, while some see the possibility of making a jump (without removing and afterwards adding the unit) much earlier. In the more difficult exercises there is a big amount of different but successful methods.

The problem of regrouping in addition caused some trouble in year 2 due to the children's lack of number sense. What is interesting, however, is the way some children try to handle the difficulty. "Then you cannot have the whole twelve on 30 something", "Well, you cannot continue like this, sixty ten, sixty twelve, or sixty eleven ...".

Many children give examples of a good knowledge of the connection between addition and subtraction: e. g. "15 – 8 = 7, because 8 + 7 = 15". Others use ideas from equations, often starting with doubles: e. g. 16 + 16 = 32, thus 16 + 17 = 33 etc., "He (seven) gives one to it (nine), so it makes ten, too".

With very few exceptions the children do all their computations mentally, and they have shown that most of them can solve addition problems with two two-digit numbers in their heads. Some children can even solve exercises with three two-digit numbers or with two three-digit numbers mentally.

As far as motivation is concerned, I have to restrict myself to stating that the children, with very few exceptions, prefer to invent their own methods for computation to being given rules to follow.

Discussion

I first discuss the answers to the specific questions. However, as there are very few comments to the last two and they depend quite a lot on each other, I have put them together.

How is the children's number sense affected?

In my opinion, the best way to measure the children's number sense is from the way they do the computations. We can see a difference among them already in the use of the jump or the split method when they add or subtract 10 or 20. I do not see any reason to force the idea of jump onto the children. In accordance with constructivism they will detect this method themselves, when they have seen more examples and got more experience.

The problem of regrouping in addition caused some trouble in year 2. When the children have to invent their own methods, they are forced to tackle this problem, and as far as I have seen, they master it sooner or later. They cannot just put one "on the shelf" as in the algorithm.

I would also state that the children's lack of number sense causes trouble in subtraction with regrouping. Sadly enough, many children have yet (May, year 3) not overcome this problem. They still have difficulties to distinguish between 8 take away 3 and 3 take away 8 in a computation. Others are on their way to master it, although very few talk about negative 5 in the case of 3 take away 8. (See for instance B1, Oct year 3). They are building a knowledge structure where the concept of negative number will fit, or with Vygotsky (1978), this concept belongs to their zone of proximal development.

Other children go round the difficulty and subtract the bigger unit bit by bit or else subtract it from a ten instead of from the unit in the first number. It might be a matter of discussion which way is the most effective one. According to constructivism, however, it seems to me to be more important that every child has the possibility to choose a way that s/he is confident with. The continued experiment will show when those, who do not yet master the regrouping difficulty in subtraction, will learn from their class-mates, who do.

The examples, where the children use the connection between addition and subtraction or ideas from equations, and many more that have been left out, show that the children are given ample possibility to practise and enhance their number sense. The diversity of strategies used also points to the fact that the each child tries to take advantage of this possibility.

I made another important observation in the clinical interviews. All the selected girls showed very clearly that it was a great advantage for them to write down some notes when doing more complicated computations. (See also the last section of the discussion "Final remarks".)

Item 6 in the interview *(a. What is the greatest whole number that you can write with the digits 2, 5 and 6? b. What is the smallest whole number?)* was also solved confidently by four of the selected children. The other two could not use the relationship between hundreds, tens and units directly, but in my judgment, some further practice will soon give them full insight into place value and its relevance.

The decline in the children's ability to write the number 130 *(Write the number that consists of one hundred and three tens)* was, of course, a disappointment. However, the explanation is probably very simple: Presumably, the class had just been talking about hundreds and tens (in Swedish "hundratal" and "tiotal") in year 2, while the children had forgotten these words in year 3. The answer of one bright girl points distinctly in this direction: "100 hundredths are in any case one second". As constructivism tells us, the children try to build on the experience they have.

How is the children's ability to do mental computation and estimation affected?

Of course there is a close relationship between the answer to this question and that to the foregoing. As we have seen, most exercises have so far been done mentally. When we go on, the children will have to write notes to a much greater extent than before, but the data I have gathered up to now, points to their willingness to do as much of the computation as possible in their heads.

We can see that the children's enhanced number sense helps them solve addition problems with two two-digit numbers mentally. As mentioned briefly in the results, even simple multiplication and division problems seem not to cause any difficulties for a majority of the children. However, subtraction with regrouping remains a difficulty for some. Although the proportion of right answers increased from year 2 to year 3, only two thirds of the children solved a problem as simple as 31 – 12 on the later occasion. As stated above this problem has to be tackled thoroughly.

Another problem we have to approach is the issue of estimation. Even if the results of the items common to year 2 and year 3 gave a much better result in the latter year, the fact remains that none of the six selected children tried to make an estimation for the problem (*Stina buys meat for 37 kronor ... She has 100 kronor. Will the money be enough ...?*) It could be said that the numbers were too simple to encourage estimation, but we can still see that the idea of estimation is not very natural for these children. This is another issue that we have to tackle in the continued project.

The results point clearly to the variety of successful methods that the children use both in addition and subtraction. In this experiment every child has been allowed to use her/his former experience and knowledge structure to tackle the problems and exercises. Of course, to each child her computational strategy can also be a form of algorithm. In contrast to the use of various kinds of traditional algorithms, however, she has invented the method herself, it his her "private property".

Again, I want to stress that the children do not always choose the most effective way for their computations. As I interpret the philosophy of constructivism this is not a problem. As times goes on and the children get more experience and see their class-mates and their teacher use strategies more effective than theirs, they will change their own ways of tackling a computation. After all the most important thing is that the children can find at least one way to do reasonably simple computation with mental arithmetic, written computation, or estimation, respectively.

How is the children's motivation for mathematics affected?

As stated already in the results it has been difficult so far to get an overall picture of the children's motivation for mathematics. Later I will consider the children's answers to the question *Do you think mathematics is fun or boring?* in several consecutive years. Then it will not be difficult to compare the development in this experimental class with similar results from other classes followed in primary and middle school.

I also consider it an advantage that the children are aware of the possibility of misusing the calculator, and I cannot see the fall of the calculator's popularity as a failure.

The parents' attitudes will, of course, affect the children's motivation. Therefore, it is a big advantage that so far the parents have been positive to the experiment. As mentioned earlier, there are a few who still want to "help" their children, who have some difficulties with mathematics, by teaching them the algorithms. Of course, if the child learns the rule, s/he will no doubt arrive at a temporary apparent success by being able to do some more sums. However, we try to convince the parents in question that in the long run their children gain more from being encouraged to overcome their difficulties with place value and other aspects of number sense. Even if every child cannot reach equally far in mental and written computation, all should be given the chance to reach as far as their ability allows without using either the algorithm or the calculator as a crutch.

Is there a difference between girls' and boys' number sense and ability in mental computation and estimation, and between their motivation for mathematics?

I think it is too early to draw any conclusions from the girls' superiority in computation (including estimation) and the boys' better results in the test of number sense. There might be a connection between the test results in number sense and the boys' higher motivation for mental arithmetic, but so far I dare not draw any conclusions. However, up to now I have not seen any hints that the girls (or the boys) would be treated worse in the experiment than when traditional algorithms are taught.

Beside the one mentioned above there is another difference in girls' and boys' attitudes, their interest in the calculator. However, I cannot see that this discrepancy has any importance.

Continuation

The results I have arrived at so far, have, of course, given rise to a lot of new questions and ideas for the continued investigation. For example, I will try to follow some of the children that have been especially selected, to see if and in that case how they change their computation strategies during the years. I will also follow the interaction between a group of children and between the children and me as participant observer to see how they affect each other and possibly are affected by an adult when choosing computation strategies.

Another matter to be investigated will be whether the children choose different strategies for exercises that are similar but differ in some respect, that might be important to the children, e. g. 317 – 74 and 317 – 274.

Final remarks

Some researchers (Harz, 1993; Sullivan 1990) have maintained that we need only two modes of exact computation: mental arithmetic and the use of the calculator. As can be seen from the results and the discussion above I propose a third one: written computation, but without the restriction of strict rules determined from outside. There are at least three reasons for the practice of this written computation:

1. We can carry out more difficult computations when we make some notes than when we use pure mental arithmetic. After all, there might be occasions where it is too cumbersome to fetch a calculator and we need an exact answer.

2. Written computation makes it easier for the learners to follow what they are doing and, thus, how place value and other aspects of number sense are used.

3. Written computation (compared to mental computation) makes it easier for the learners to communicate their ideas to their classmates and their teachers.

Thus, I do not suggest that we should teach our students to use a calculator for all computations that are too complicated to be done mentally. Using a calculator for mere calculation is a very effective method, but it is just as mechanical a way as the use of a traditional algorithm.

It has also been said that we can make room for some new topics in mathematics learning, if we abandon the teaching of algorithms. I think we have to be careful here. In the class I am studying, we have devoted as much time to mental computation and written computation using the children's own methods as others do to traditional algorithms. I believe this is necessary, if the children are to acquire good number sense and good skill in mental computation and estimation. My hope is, though, that time will be saved in future mathematics education, when the teaching of new topics can be built on more solid foundations.

For reasons of fairness, it should also be stated that the use of algorithms for the four arithmetic operations should not always be avoided. There are occasions where they are still very practical to use, e. g. when you make notes about your bank or cheque account. To me it is not a very important matter, but if teachers and children (or children's parents) insist, the algorithms could be taught later, e. g. in year 6 and 7. By then, the children have, hopefully, already acquired good number sense. Experience from earlier research projects shows that if they can build on such knowledge, they will learn to handle the algorithms in a very short time.

If you maintain the position that the algorithms should be abolished or postponed, you very often meet the questions: "Can you really do that, do we not deprive our children of a lot of valuable skill and knowledge? What will happen in lower and upper secondary school?" Of course, we have to be very careful and see to it that there will be a lot of research to find out possible disadvantages with the ideas put forward in this article. But I think it is up to the advocates of the algorithms, too, to show proof of their advantages not only as a substitute for the calculator but also for the childrens' mathematical development.

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Alternativ till traditionella algoritmer i den elementära matematikundervisningen

Syftet med detta forskningsprojekt har varit att se vilka förändringar som borde äga rum i den elementära matematikundervisningen, om vi försöker ta hänsyn till de ändrade behoven i ett miniräknar- och datorsamhälle.

Projektet äger rum och kommer att äga rum in en klass, som följs genom andra t o m femte skolåren. För närvarande (maj 1996) går eleverna sitt tredje år i skolan. Traditionella algoritmer för de fyra räknesätten lärs inte ut. I stället uppmuntras eleverna att finna på sina egna metoder för skriftlig uträkning och att använda huvudräkning och överslagsräkning, när så är lämpligt. Varje elev har tillgång till en egen miniräknare.

Projektet utvärderas huvudsakligen med kvalitativa metoder, t ex kliniska intervjuer och observationer av grupper av elever. De resultat som diskuteras, kommer huvudsakligen från tredje skolåret. De visar på att eleverna, när de ges möjlighet, uppfinner en mängd olika metoder för att klara av uppgifter i addition och subtraktion. Eleverna visar tecken på allt bättre talförståelse och taluppfattning. Många elever har emellertid fortfarande svårigheter med subtraktion med tiotalsövergång.

Författare

Rolf Hedrén är universitetslektor i matematik och matematikdidaktik och lärarutbildare vid Högskolan Dalarna. Han är fil dr i matematikdidaktik och bedriver forskning om användning av datorer och miniräknare i matematikundervisning.

Address

Dalarna University College, S-791 88 Falun, Sweden