Graphing calculators and students' interpretations of results: A study in four upper secondary classes in Sweden

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Thanks to a drastic change in the availability of graphing tools in the teaching and learning of mathematics, a similar change in written and oral ways to communicate the subject can be expected. In several reports students' participation in mathematics and students' views of the use of electronic tools like graphing calculators are discussed. A study in upper secondary classes in Sweden confirms that test items can be classified according to how students tend to record their written solutions. Logical solutions recorded in a traditional way are not mastered sufficiently, and when a modern approach with graphing calculators is used, at least two difficulties must be taken into account. The student must have technical insight to be able to interpret the information given on a graphics screen, and the student must also have a sufficiently good mathematical understanding to realize the connection between the current problem and the possibilities given by the tool. Some consequences for mathematics education are discussed.

Technology can be a powerful force influencing curriculum, instruction, and assessment in mathematics. Textbooks are rapidly changing to reflect the availability of technological tools, such as graphing calculators. Surveys of schools, teachers, and students document that graphing calculators are becoming more accessible. The capacity of graphing calculators increases as the costs decrease. With the next generation of "super calculators" already on the market, the availability of graphing and symbolic manipulation calculators will obviously continue to increase in the foreseeable future.

The research community has also taken an interest in the advent of the graphing calculator. Reviews of research have started to appear (Dunham, 1993; Dunham & Dick, 1994). Several articles about the effect and influence of graphing calculators on different parts of the

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school mathematics have been published during recent years (Ruthven, 1990; Harvey, Waits, & Demana, 1992; Quesada & Maxwell, 1994). In the present study a special consequence of the use of powerful calculators was the focus.

Students communicational skill, with regard to their use of graphing calculators, is important in different situations. When they talk to other students in the classroom, when they have a mathematical discussion with a teacher, or when solving problems of mathematical nature in other subjects, a certain amount of communicative ability is essential. Especially pronounced is the need of a good ability to argue in written texts, as in presentations of solutions in tests or theoretical discussions in other assignments.

In Sweden, as well as in many other countries, students are expected to write or orally explain and discuss solutions to mathematical problems during regular testing situations. The availability of graphing calculators allows certain types of mathematical problems to be solved in new ways. For example, students often use an experimental approach, which after some trial and error produces a solution. Such a solution process encourages the use of insight and intuition. At the same time this informal approach encourages students not to express their solution in a formal way, which in the long run might influence their mathematical language and the ways they communicate mathematics. On written tests, a solution must be described in its details. The student has to meet this demand even when he or she may not have taken any notes on the steps taken in performing the work with the help of a modern calculator.

A calculator is a powerful tool for a student who knows how to use it. It might set aside such barriers as lack of arithmetic skills or the lack of computational skills at large. Calculators might, however, also be tools for conception. This is even more true for graphing calculators. We have not found any elaborated discussions about to what extent the use of calculators also might act as obstacles of a nontrivial type. The ability to interpret the results from a calculator has always been of importance. Today, with their increasing mathematical power and complexity, the new graphing and symbol manipulating calculators demand higher knowledge. A well developed insight in the mathematics, the technique and the methods may very well be a crucial competence that serves like a springboard or a pitfall. The degree of competence will probably have an influence on the students attitude towards mathematics as such.

In several papers Dunham (1990, 1993) uses the expression *algebraic guilt* to underline the fact that some students see the use of calculators as "too easy." They seem to feel that such an approach is not "mathematical" enough, that it would be more valuable for them to use algebraic techniques. It is like getting away with something to use the graphing calculator—"almost like cheating" (Dunham, 1993). If a student has this general attitude towards the use of a graphing calculator, he or she may also hesitate to describe the role of the calculator in the current solution, and the presentation will consequently be incomplete or misleading.

Graphing calculators in assessment

Providing equitable assessments for all students is an important goal in mathematics education (National Council of Teachers of Mathematics, 1995). Assessments must contain provisions to accommodate and reward the range of solutions coming from students using graphing calculators and from those not using them. The continuous stream of new, more powerful machines with still better graphing capabilities places additional pressure on education to evolve new assessment methods and to reflect the new possibilities in current teaching methods. Traditionally, teachers have to assess students' written solutions as they are produced in different kinds of tests.

Graphing calculators can be used primarily in at least three typical ways (Ruthven, 1995): first, to support certain routine work; second, to give alternatives to symbolic procedures; and finally, to stimulate the student to engage in open-ended activities. Experimenting with a graphing calculator produces a wide range of different screens. Ruthven remarks that students can fail to produce a correct response "through difficulties in translating the calculation into an appropriate form, or in interpreting the results displayed by the machine." Such research suggests that greater conceptual mathematical knowledge facilitates the power of the graphing calculator.

Assessment is a particularly troublesome issue when graphing calculators are used in teaching and learning mathematics. Although some parents and teachers may still object to using graphing calculators on assessments, one can argue that when graphing calculators are used in teaching and learning, then these same tools must be available when assessing student knowledge and performance. In Sweden graphing calculators are tools that have been permitted on national tests for several years. They are also permitted in certain international education programs like the International Baccalaureate. The availability of graphing calculators makes some mathematical procedures more important and others less important. This influence has implications not only for the selection of curricular topics, but also for the choice of appropriate test items.

The nature of solutions may also be expected to change if students do not find it important to record all the different steps they have taken in their reasoning. It is likely that when graphing calculators are used the solution process as well as the written record will look very different. Students may consider it less important to provide a written record of what was done. The solutions may become very short, perhaps providing only an answer. Students may find it hard to use text or drawings to describe procedures and give arguments, a type of communication which in the past have provided critical maps and benchmarks for teachers to follow in evaluating performance. The issue is further complicated by the freedom of choice that exists among students. That is, each student must decide whether or not to use a graphing calculator for a particular problem. Once that decision has been made, it is likely that the solution processes of students using graphing calculators will be different from the solutions found, and given, without the use of the tool.

Research questions

In assessing students' written reports, one has to define the limit for argumentation. On what level does the teacher want the student to start an explanation? The teacher and students might very well have a silent agreement based on many discussions in class and usually this unwritten "laws" are developed step by step. This principle of calm development is disturbed if the students integrate the use of modern electronic equipment with established methods for solving equations or in calculus. The large variety of approaches and executions demand high quality written reports. Solutions must be readable, consequent and detailed to clarify the method used by the student. The state of the art in this field of teaching is not discussed in this study. It is, however, important as it implies that a stress on the treatment of the mathematical language in spoken, written, and symbolic form must be parallel to the use of powerful tools.

When graphing calculators are used, students will find shortcuts and new ways to produce solutions. Then they will have the problem of documenting their ideas and solution strategies to get their results assessed by the teacher. This study was an effort to examine this dilemma with a group of Swedish students. We found it especially important to investigate the extent to which students are able to describe a solution. How do they tell the reader how and when the calculator was used, and how do they interpret their results?

Previous research

Visualization is an important part of mathematical learning. In many situations, a graphing calculator provides an immediate "visual" for symbolic representation, which may facilitate understanding. Dunham (1993) reports on one student who said, "Graphs give a lot of information all at once. Functions given in symbol form give information only one point at a time. Graphs give the whole picture" (p. 95). Seeing the "whole picture" may provide a deeper insight into the behavior of functions by means of a greater variety of representations and may also give a greater understanding of the connections between algebraic and graphical representations. It is also possible, however, that once this "whole picture" is seen, many students may become less inclined to elaborate their solutions in their own words.

Students sometimes fail to make connections between graphical and symbolic representations (Dunham & Osborne, 1991). For example, a graph is often seen as an entity and not as a set of ordered pairs. Furthermore, a graph corresponding to a certain function is often traditionally, in a step-by-step approach, treated as consisting of a rather small number of distinct points, such as intercepts and extreme points.

Working pragmatically, students seem to take different approaches to different problems. With some types of problems, a graphic method is obviously found to be very convenient and to provide a quick solution; with others, it is equally clear that some kind of algebraic approach is the best. In the Ohio State Calculator and Computer Precalculus Project (Dunham, 1993), a Calculus Readiness Test was used. The items on the CRT were divided into three subclasses, based on the method of solution: 11 nongraphing items, 4 blatant graphing items, and 7 savvy graphing items. Nongraphing items cannot be solved using graphing utilities. "Blatant items have an obvious solution once the students obtains the graph of the function with the utility. The function to be graphed is also obvious from the statement of the problem ... The savvy graphing items ... required more sophisticated use of graphing technologies than the blatant graphing items" (our remark: Dunham (1993) here refers to personal communication with B. Martin at University of Wisconsin-Madison). (Dunham, 1993, p 91).

Nongraphing type: Simplify $\dfrac{y-\dfrac{1}{x}}{x-\dfrac{1}{y}}$

Blatant graphing type: The graph of $y=\dfrac{x}{x^2-2x-3}$ is best represented by:

[5 graphs are given as choices]

Savvy graphing type: Solve for *x:* $|x-1|>3$

In the study now presented a test was used which will be discussed in detail below. General ideas and concepts as equations, functions, graphic respectively algebraic problem solving were used when the test was constructed. A control of what specific types of examples the students actually had worked with before they participated in the test was however not done. Some of the problems were especially suitable for the use of a graphing calculator. Either an exact method is not discussed in school mathematics, or an alternative method demands an extensive effort in the solving process. This can be seen as responding to what CRT labels Blatant graphing type and Savvy graphing type. One problem of Nongraphing type was chosen to find if the students were able to use the larger display of a graphing calculator in an exploratory way.

A study to map students' interpretations of graphing calculator displays

This study reports an effort to examine the types of responses Swedish students provided to six mathematical questions. The analysis focused on how the graphing calculator influenced the solutions they expressed. Four mathematics teachers, at four different schools, involved as supervisors in the teacher education program at the University of Gothenburg were individually contacted. They all agreed to participate with one of their classes. Two classes were in Grade 11 and two in Grade 12 in the Swedish upper secondary school science program, the secondary program with the largest mathematics component. The curriculum is divided into five sub courses (named A to E), and the students who participated in the study were following the second or third (B or C) of these courses. They had done initial work in calculus and had studied most of the algebra they would meet.

The purpose of the analysis as described above has many aspects. The impact of the calculator might concern the student's choice of method or way of working. It might also have some weight on the disposition to formulate oneself in text or to illustrate one's solutions with graphs or tables. In this study we concentrate on the extent of the students' use of calculators, without trying to find the single student's argument for this use.

A six-item test in mathematics was constructed. The items were chosen from different core parts of the mathematics curriculum for Grades 11 and 12. The items were to be solvable in a traditional way as well as with the help of the graphing calculator. The use of graphing calculators to solve a problem should not be too far fetched but should also not clearly be the obvious way to solve it.

The six items are presented below in the order in which they were given on the test. The general background for each is discussed, together with some expected approaches and possible mistakes.

1. Solve the equation $x^3 - 3x = \ln x$

Traditionally this type of problem is discussed only to a restricted extent in the secondary school. It can be done by plotting the two sides of the equation as functions in the same coordinate system. The point of intersection can be found with any degree of accuracy if the crucial part of the diagram is sufficiently enlarged. This process takes considerable time and skill when done with pen and paper. Expected 'modern' solutions with the help of the graphing calculator would be to examine either the intersections of $y = x^3 - 3x$ and $y = \ln x$ or the zeroes of $y = x^3 - 3x - \ln x$.

2. Given the function $y = \dfrac{x^3}{x^2 - 1}$.

Draw the graph and describe the function with as complete a reasoning as possible.

Quite clearly this item can be solved in a traditional but rather laborious way. Equations, derivatives, arguments concerning asymptotes, the behavior of the function for small and large values of x, local maxima and minima, and eventually a table of additional values are all details that must be discussed in such a solution. A typical solution using a graphing calculator *starts* with the graph and gives all other characteristics based on that. A certain lack of mathematical arguments can be anticipated in that case.

3. Solve the system of simultaneous equations: $\begin{cases} 2x+3y=1 \\ -3x+2y=2 \end{cases}$

Traditionally, two kinds of solutions are discussed in the secondary curriculum, an algebraic and a graphic approach. In a common algebraic solution, the two equations are added to eliminate one of the variables. In another algebraic solution, one equation is solved for one of the variables, which is substituted into the other equation. In a graphic solution, the graphs corresponding to the two equations are drawn in the same coordinate system. A solution is then defined as the coordinates of the point of intersection.

With the help of a graphing calculator a graphic solution is close at hand. The two coordinates are found with the TRACE function, if necessary combined with ZOOM for greater accuracy. With a simple linear system like the one given here, the algebraic methods are, however, very simple, and one cannot predict how students might prefer to solve the problem.

4. Solve the equation $\sin x = \frac{1}{2}$

Students are usually taught a pattern to obtain the two basic solutions. The teaching is often based on the 'unit circle,' which makes it easy to explain that the sine has a period of 360° or 2π. The problem is simple to solve with a calculator that can handle the inverse sine function, often symbolized $\sin^{-1}$.

With a graphing calculator the graphs for $y = \sin x$ and $y = \frac{1}{2}$ can be drawn and the solution found as an intersection, but students will probably prefer to use the inverse sine function. In this case, there is a certain risk that only one solution is found and the period omitted.

5. Solve the equation $\sin x + 2\cos x = \frac{3}{2}$

Several solutions are possible.

a) Transform the angle to $x/2$ and use $\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} = 1$ on the right side of the equation. The result is a second degree equation for $\tan x/2$.

b) Introduce the polar coordinates R and φ defined as $R \cos \varphi = 2$, $R \sin \varphi = 1$, and write the equation as $R \cos(x - \varphi) = \frac{3}{2}$.

c) Use a graph-drawing tool and draw the graphs corresponding to each side of the equation. Then find approximations for the intersections and the period.

The traditional types are (a) and b), and the student must have established these techniques to solve the problem. The third type may be expected if the student uses a graphing calculator and wants to find a solution but cannot remember any of the traditional ways to do it.

6. A person borrows 20 000 kronor at the beginning of each year for a period of six years. At the beginning of the seventh year, the person starts paying. A yearly payment of 20 000 kronor is made at the beginning of each year. In how many years is the loan fully repaid, and how large is the last payment? The rate of interest is 10 percent throughout.

The problem displays a typical use of a geometrical sequence applied on compound interest. The solution becomes more or less complicated depending on which year the student takes as a reference. If the interest is not given, the solution is open ended and needs a much fuller discussion. A full solution also ought to discuss how different interest rates influence the answer. In the test two versions of this item were used, only one class was informed about the rate of interest. The three other classes consequently faced a more difficult problem. With a graphing calculator, or any other calculator, some students may try an experimental approach, which may make it difficult to document the solution. Although a systematic solution working year by year can be developed, it is rather time consuming.

The study

The teachers gave the test to the students at a time convenient to each class during a few weeks in February-March 1995. The students were given the test during a period of 60 minutes. The researcher gave instructions to the contact teachers. The test was given in Swedish. After the test was taken, the papers were returned to the researchers.

The six items were arranged two to each sheet of paper, with space for the students to give their solutions directly on the test paper. Prior to the test, the students were given instructions by the teacher. They were informed that the test was part of a research study. Their performance on the different questions would not count for their grade, but it was important for the study that they perform as well as possible.

Some of the instructions were also written on the front page of the test. A translated version follows:

- There are six items in this booklet, and you should try them all.
- You may use your usual mathematical tables, drawing tools, and a calculator of your own choice.
- You are not expected to solve the problems by any special method. A solution of any kind is better than no solution at all.
- Write your solutions directly on the test papers. Mark clearly your answer, and please explain what you have done.
- If you need more space, please write on the back of the sheet.
- If you used a calculator during the test, please record the type of calculator you used.

Before the tests were received, the researchers proposed a set of categories of answers and solutions that were expected to be found. When all student papers were received, the first 10 tests from the first class were corrected separately, and the solutions were categorized. These categories were compared to the hypothesized categories, and a few minor changes were made. This experience also facilitated some common agreement on how to treat the definitions of the categories. The categories provided the framework for assessing the remainder of the tests. The rest of the tests were then assessed by the authors separately. The individual test papers were coded to preserve the identity of the student and the class. About 600 student solutions were assessed. In 31 solutions a disagreement was found. They were examined in detail. In most of these cases one or the other of the authors would change his category after a discussion. For fever than 10 solutions this agreement could not be achieved, mainly because the solution was very short or in some cases hard to understand. These 10 solutions are not discussed further. They are estimated not to have influenced the results discussed below as they represent less than 2 percent of the total number of solutions.

Classification of students' solutions

The hypothesized general structure of solutions was anticipated from traditional methods as well as from when graphing calculators might be used. We also identified key processes in the student solutions that we thought were associated with using graphing calculators. Four categories of solutions were anticipated, two of them divided into subcategorizes:

1a A logical solution, presented stepwise and followed by a correct answer.

1b A logical solution, presented stepwise but followed by a wrong answer.

2a A very short solution or none at all followed by a correct answer.

2b A very short solution or none at all followed by a wrong answer.

3 A complete misunderstanding or otherwise unsatisfactory solution.

4 A blank space or no response.

The "no response" alternative in Category 4 includes only a few incoherent words or a repetition of some of the data given in the problem but no attempt to solve it.

Comments on the items and the results of the test

Two items, Problems 3 and 6, were not suited to distinguishing between students who preferred the use of graphing calculators and those who used other methods. Problem 3, the two simultaneous linear equations, was normally treated in a traditional way. The students used substitution or addition methods. Some students, all from the same class, used a matrix method with the help of their calculator. None used a graphical method, perhaps because the traditional methods are simple and efficient for these equations, and were well known to the students. Problem 6, which was given in two different versions, with and without information about the interest, was solved without any hint of a graphical representation. Most students worked with a year-by-year decrease and normally found at least the number of years. In quite a few cases, however, either the number of years or the final payment was incorrect. This type of problem is not very well known by students in the science program, and none of the students successfully used a traditional method with geometric progressions.

Item 4 involving the sine equation was usually solved directly with the arcsine function. Eight students, six of them in the same class (at Grade 12), referred to a mathematical table. Very few (19) gave both groups of solutions, 30° and 150°, and ever fewer (15) also included the correct period 360°. In one class (at Grade 11) no correct solutions were found

Problem 1 does not lend itself to traditional algebraic methods. The two sides of the equation can be plotted separately with the help of a number table and the intersections found, or if all terms are moved to one side the zeroes are found. On the test, many students tried vainly to find an algebraic solution. A minority (19) used the graphing calculator and found intersections or the zeroes mentioned above. Many of these students got an acceptable result, although most solutions were very short and contained little explanation. A few (5) in one of the classes (at Grade 11) referred to a stored program. Newton-Raphson's method was used (6), but it is not clear whether the individual solutions were based on the method or whether a stored program was used. In three cases, the student had made a table and found one of the solutions.

Item 2, the function $y = \dfrac{x^3}{x^2 - 1}$ can be discussed traditionally in a step-by-step solution. A few students tried the traditional approach, but most of them were unsuccessful in producing an acceptable result. The item was, however, most often solved with the graphing calculator. It is easy to recognize influences from the picture you see on the screen of a graphing calculator.

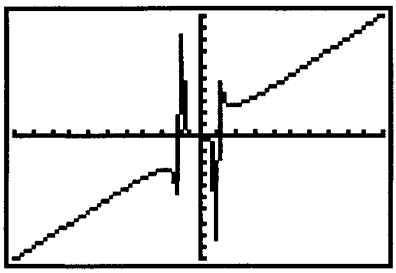

Figure 1. The graph of $y = \dfrac{x^3}{x^2 - 1}$
TI-81

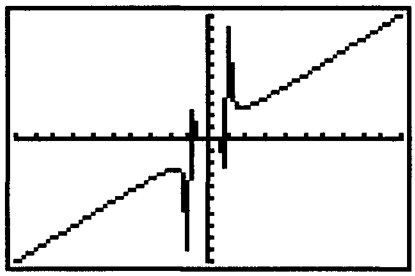

Figure 2. The graph of $y = \dfrac{x^3}{x^2 - 1}$
TI-82

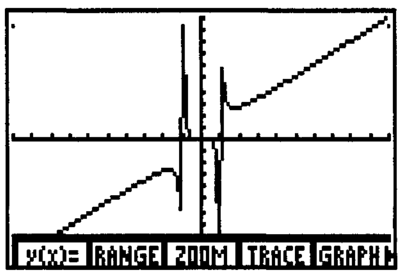

Figure 3. The graph of $y = \dfrac{x^3}{x^2 - 1}$
TI-85

In each of the graphs from the Texas Instruments calculators TI-81, TI-82, and TI-85, the standard window was used. As a comparison, we let the same function be drawn by a more powerful graphing calculator and also by a computer, figure 4 and 5.

Students using the graphing calculator on the test showed a certain insight into the behavior of the function, but their comments showed that the function was peripheral to the normal knowledge of the students in these courses. "This function gives an advanced graph." "One has to lift the pen to draw the curve, whatever that is called." Misunderstandings were frequent, like "The function gives three graphs." The origin as a point of symmetry was expressed as follows: "The curve is the same on both sides of the y-axis although it is turned upside down on the negative side." Several comments revealed that the function was understood as a polynomial function of the third degree: "A typical 3-degree with the x-axis as a tangent at the origin."

In many cases the asymptotes were seen as part of the curve. A typical approach seemed to be used by this student: "First I drew the function with the graph drawer to get to know what it looks like." It was obviously possible to copy the graph shown on the display

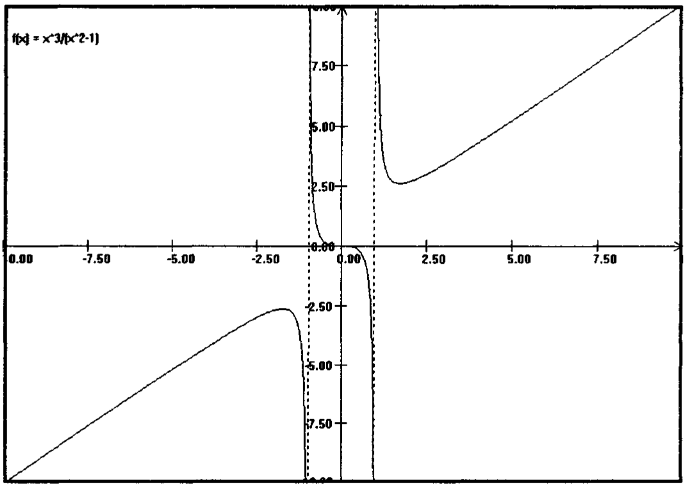

Figure 4. The graph is here constructed with the aid of The Mathematical Workshop A. With the good resolution provided by a computer monitor and an accurate graphing routine, we are able to separate the graph from its asymptotes.

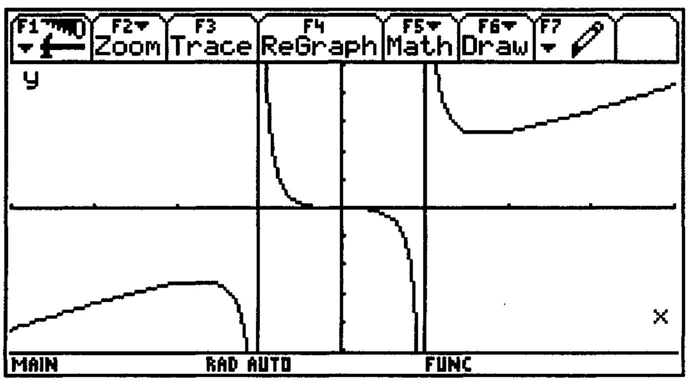

Figure 5. By experimenting with the choice of window limitations, one can get the same picture as in figure 4, with a powerful calculator, here illustrated with the TI-92. The asymptotes are not drawn, but the image now gives the impression of a more correct treatment.

The graph is drawn with the window setting $-4 < x < 4$, step 1 and $-5 < y < 5$, step 1.

without understanding the mathematical implications: "I used the TRACE function to get the top values, and then I copied the graph from the calculator." From the copy the student gave, one can see that the student saw the "asymptotes" on the calculator screen as integral parts of the graph. This type of solution was common, with the asymptotes being reported as part of the curve. Thirty-one students, however, presented an illustration in which the vertical lines drawn by the calculator were not included. Most of these graphs were very poor and without any written comment at all. Although 11 students used the word *asymptotes,* only a few (5) gave a proper explanation of how they were defined and drew them correctly.

Discussion of some typical results

Three items on the test, Problems 1, 2, and 5, were well suited to graphical solutions. Problem 3 can be solved with a graphing calculator, but the students preferred a traditional method. Some typical results from these problems are discussed below. The other two items, Problems 4 and 6, will not be further commented on. Problem 4 proved to be too trivial; the students used the inverse function arcsine. That many of them did not produce a full solution has nothing to do with calculators. All efforts to solve Problem 6 were made without the help of the graphing facilities and, as far as we know, without using the ANS key.

Sample solution of Problem 1

Graphical solutions were often referred to but frequently without a corresponding illustration on paper. The student *informed* the reader that one or two graphs had been drawn on the calculator. With the use of the TRACE function, zeroes or intersections were found. Often only one root was found, probably because the student missed one of the two locations because the standard window covered too narrow a part of the coordinate system.

Another approach used a test based on numerical values presented in a table. The student checked when the two sides of the equation were equal. A calculator might have been used in this work but not necessarily a graphing one. Often one root was lost because only one change of sign was found in the table, which spanned too small an interval.

In a few cases the student referred to the Newton-Raphson method, to a calculator program of his or her own making, to the interval method, or to the use of the solving tool ROOT, which comes with the TI-85.

Any attempt to produce a traditional algebraic solution, of course, failed. There were also various incorrect approaches and conclusions connected to a weak knowledge of mathematics rather than to the misuse of a calculator.

Sample solution of Problem 2

Several students plotted the curve with the help of a table of values. The most common approach was, however, to draw the curve with the help of a graphing calculator. Usually only a few words of explanation were added, such as: "The graphing calculator has been used." Obviously, the student copied the figure from the screen to the paper. In this procedure a variety of mistakes were found, ranging from poor drawing ability, lack of experience with graphs, inattention to accuracy, and other habits that seemed to have nothing to do with the use of graphing calculators as such.

The most common mistake in the students' "explanations" was in interpreting the curve as a function of the third degree (Figure 6 a). The graph was often wrong despite the fact that it had obviously been drawn using the graph on the calculator as a blueprint. The passage through the origin was often wrong, the curve was not drawn tangent to the x-axis, the two vertical asymptotes were often drawn as integrated parts of the curve, and the curve was consequently shown as a continuous function within the interval (Figure 6 b). Sometimes the points at which the asymptotes were connected to the curve were even reported as local maxima and minima. Thus, despite the use of the graphing tool, many incorrect interpretations resulted.

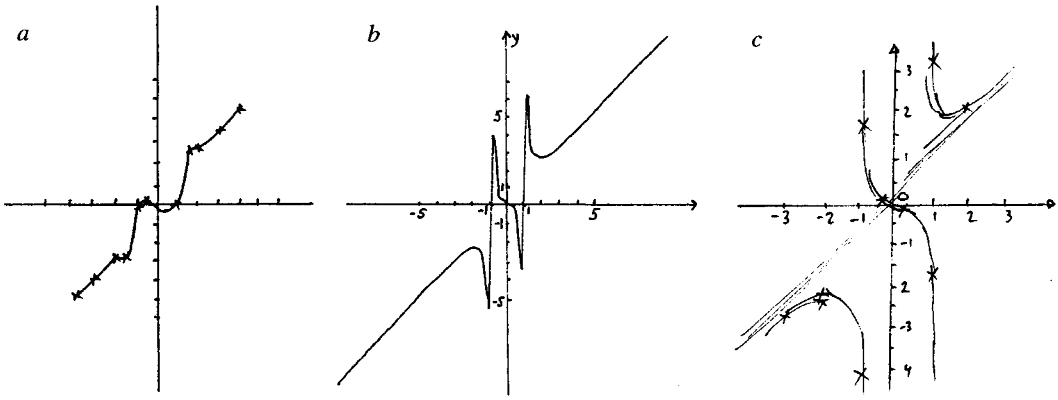

Figure 6. *Three typical results, copied from tests, illustrate the discussion.*

Some solutions (12) were found with correct comments on the vertical asymptotes and the passage through the origin. Four of these were based on a number table and a curve plot. In 6 cases the solution was a combination of work with the graphing calculator and comments based on traditional reasoning. No student found the third asymptote $y = x$ although a few (3) sketched a line in the proper area without comments (Figure 6 c). In a couple of cases comments were given on the behavior of the curve for large values of x, and the student offered the opinion that it looked like a straight line. No student reported the coordinates for the two local extreme points after correct reasoning.

A few of the 12 students mentioned above tried to analyze the function in a traditional way, but only one gave a fairly complete solution of this kind. To produce a logically complete discussion the student must have an experience that includes the differentiation of functions and solving equations. This is necessary in order to find zeroes and extreme values and finally to plot graphs with all the proper characteristics. This research suggests that very few students have an overall picture of how this is done.

Sample solution of Problem 3

In one of the classes the students showed general knowledge of how to use the matrix tool in the graphing calculator to solve systems of simultaneous equations. Presumably the technique had been discussed in that class either in lessons or between the students. Several students successfully produced a solution of this kind. However, the only argument given was "I used the matrix in the calculator." One student declared that he knew there was a method for using the calculator but he had forgotten it. He did not give any other solution. In the other three classes, Problem 3 was correctly solved using non calculator techniques.

Sample solution of Problem 5

Several traditional solutions introduced an extra angle defined by its tangent. In general, a correct answer was found in this way except that the period was often omitted.

Students using graphing calculators drew one or two graphs and found the zeroes or intersections with the help of the TRACE function. In the latter case (6 students), one of the solutions was often (in 5 cases) missing, and as a rule the period was omitted. If an illustration was given it was often poor and did not underpin the answer.

Among the 27 students who gave an acceptable graphical solution for Problem 1, 13 also used a graphical method in Problem 5, but of these only 5 reported both angles and correct periods.

Occasionally, the students successfully used the Newton-Raphson method (3 cases) or wrote a program to solve the problem (4 cases). The optimistic comment "I quickly wrote a program that calculated the answer for me" was, however, given along with an incomplete solution. In one case, the student used vectors with a reference "as we do in physics" and got a correct answer.

Information from the teachers in the four classes

In telephone interviews the teachers answered three questions about the use of graphing calculator in the classes.

1. In this class, have you ever used the overhead-projector equipment that is available for graphing calculators to present or discuss mathematical problems or the use of the calculator?

The answers were "Never" (1), "A few times" (2), or "Not this year" (1). One teacher reporting "A few times" had used both Casio and Texas Instruments calculators in this way with the students.

2. Do you discuss the calculator in an organized manner concerning techniques or methods for using it?

The four answers were "No," "We have had some discussions about the menus and the possibilities for handling parameters," "Sometimes the students get more demanding problems which they are not expected to solve without the help of a graphing calculator," and "Not regularly, only in the first year in upper secondary."

3. Do you have any other comment on the graphing calculator connected to this study that you would like to express?

The teachers gave a short overview of earlier work. The first teacher had taken the class for the first year and could not describe what had happened earlier but hoped that the graphing calculator had been discussed. The second teacher had commented in class on problem solving with the help of the overhead projector and had also used a CBL (Computer-Based Laboratory) equipment unit from Texas Instruments. The students were currently building a workshop in physics and chemistry in this way. In the third teacher's school, the teachers were trying to develop new courses in computer science and also hoped to find ways to use peripheral equipment for the graphing calculator in science and chemistry. The fourth teacher expressed a slight ambivalence towards calculators, contending that it is still important to be good at defining problems and that sometimes students' solutions now tend to be short and a little "sloppy." The students were encouraged to work together, to exchange experiences, and to develop their general knowledge. "We do not force the students to buy a graphing calculator, but all have access to one. A few students borrow them from the school, and very often they soon buy one of their own."

Conclusions

Computers are sometimes viewed as the cause of some effects in our modern society. We, however, would prefer to look at the computer as a catalyst of already present characteristics of the society. The same perspective may very well be used on the role the calculators may have in the mathematical classroom. We do not see the calculators as creators of problems in the process of learning and teaching mathematics, but neither are they simple problem solvers. Nevertheless, they illuminate educational problems already present in the classroom. By way of example, individualization will be stressed when students in the same class quite probably use different types of calculators.

In this study we were able to confirm the opinion (Dunham, 1993) that mathematics problems are seen by the students as coming in different types. Some are treated as more suited than others to solution with the use of graphing calculators. Our results, however, show that students and consequently teachers need to develop a double competence to get the best benefit from a graphing calculator. First, it is important to develop an understanding of what the graphing calculator can do and what technical limits it has. Many students copy the screen without trying to uncover any hidden problems. A critical approach has to be developed.

Second, a basic mathematical knowledge is necessary for a correct interpretation of results obtained from a graphing calculator. In our opinion, the importance of written language cannot be stressed enough. If the use of a graphing calculator now slowly changes the way in which students present their solutions, then the assessment of mathematics must change too. Teacher education must take these new forms into account, and ideas of how mathematical knowledge might be assessed must be discussed.

Future teachers may be confronted with the situation that any assessment in mathematics includes the difference between a graphing approach as such and a skillful graphing technology approach. It is of great importance that teachers explain the difference between what we may see on a screen or even a monitor and the "real" behavior of a function. The fact that some students sees a graph as an entity may explain some of the mistakes we found. Students copy a graph from the graphics screen onto the paper and do not reflect on the validity of what they see. Any machine-constructed representation of a graph may include "nonexistent" part such as asymptotes.

In a way, it is a language problem: to translate from the "screen language" based on the capabilities of the calculator to the "paper-and-pencil language" based on traditional ideas of how to draw graphs and how to take into account the mathematical behavior of a function. Students who experiment with their graphing calculator need to have good experiences in mathematics to grasp all that the screen reveals.

Traditional methods will most certainly change, but it should be obvious that a core of mathematical knowledge must remain unchanged. All the same, very definite advantages are clear when graphing calculators are used, and instruction must take care of these. The better one masters the technique and the better knowledge of mathematics one has, the more powerful a tool the graphing calculator is.

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Om grafiska miniräknare och elevers tolkning av resultat.

En studie i fyra gymnasieklasser i Sverige

Under senare år har en dramatisk förändring av tillgången till grafritande hjälpmedel i undervisning och inlärning av matematik ägt rum. En liknande förändring kan förväntas avseende det sätt på vilket muntlig och skriftlig kommunikation sker inom ämnet. I ett flertal rapporter har olika forskare diskuterat elevers matematikstudier och deras uppfattning av elektroniska redskap såsom grafiska miniräknare. Den nu presenterade studien, som genomförts i svenska gymnasieklasser bekräftar att frågor på prov kan klassificeras i enlighet med hur eleverna tenderar att formulera sina skrivna lösningar. Logiska lösningar dokumenterade på traditionellt sätt behärskas inte i tillräcklig grad och när en modern ansats med grafisk miniräknare används kan åtminstone två skilda svårigheter tas i beaktande. Eleven måste ha en teknisk insikt för att kunna tolka information som lämnas på en grafisk skärm och måste också ha tillräckligt bra matematisk förståelse för att inse sammanhanget mellan det aktuella problemet och verktygets möjligheter. Några konsekvenser för matematikundervisningen diskuteras.

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