A re-examination of the rôle of instructional materials in mathematics education

Robert P. Hunting & Susan J. Lamon

Two current phenomena, the prevalence of computer-based learning environments and renewed interest in "hands on" learning as a popular response to recommendations of national curriculum statements such as the NCTM Standards, occasion a revisitation of issues surrounding the use of physical materials. We discuss the efficacy of several manipulative materials in instructional settings, noting both their strengths and their limitations.

Adopting a modeling perspective on the use of instructional materials in mathematics learning, we draw critical distinctions between internal models and external embodiments, and between teacher-intended meanings and the subjective nature of student interpretation. We explore the consequences of a narrow interpretation of active learning and caution against relying on a single manipulative to capture the richness and connectedness of mathematical ideas. As an alternative to asking students to discover mathematical ideas as perceived by adults, we endorse the perspective that materials should be used within a larger pedagogical framework in which students individually and collectively negotiate mathematical meaning. Remaining unanswered questions are posed as an agenda for future research.

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What is a square and how do we know?

The notion of a square is a well entrenched item of mathematics curriculum from before the days of Euclid. As teachers we probably have two related ideas of a square: some kind of formal definition like *a quadrilateral having equal sides and equal angles*, and a mental image of a particular geometric shape. Before children in the elementary school are aware of a formal definition, our guess is they

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Robert P. Hunting *is senior lecturer at La Trobe University, Australia. Research interests include rational number learning, clinical assessment methods, early number learning, constructivist research methodologies.*
Susan J. Lamon *is assistant professor at Kansas State University, USA. Research interests include ratio and proportion learning, authentic assessment in mathematics, and mathematical modeling.*

have a fairly good mental image of the geometric shape – or do they? What do they really know? How do they come to know it? And what does it mean to know the concept "square"? These are open questions for which we do not have definite answers. However, in a classroom a common teaching strategy would be to demonstrate by drawing a picture on the chalkboard or overhead projector. Perhaps the teacher might invite the children to portray what they understand a square to be. Of course the picture is not a square even though it might in some sense represent a square. It is some kind of approximation indicating "squareness". Indeed, such an action can only point crudely to an organizing principle that draws together related phenomena. As Davis (1991) has suggested, in relation to circles:

> ... when a teacher says "let C be a circle", what does the teacher mean? In practical terms the teacher means that C will be a name for any one of the phenomena related by the relation of circularity: in other words the name of anything that fits the definition of 'circle'. But what then of the 'concept' of a circle? It is easy to claim, and we do claim, that the concept does not exist as such. To put it more positively, what exists is a constructed relation; the idea of a concept is then a generic name for an object in the class of related phenomena (p. 233).

What about a fraction like, say, 1/2?

The fraction one half is an interesting example to choose, for of the fractions of school mathematics it singles itself out as being quite special (see for example, Behr et al., 1984; Hunting & Davis, 1991; Polkinghorne, 1935; Pothier & Sawada, 1983). The verbal utterance "one half", "half", and similar terms, as well as the written symbols 1/2 and 0.5, are, according to Davis (1991), simply names for any one of the phenomena related by the relation of "half-ness". (We accept that some might argue that terms like one-half are also symbols.)

But what kinds of phenomena are connected by the relation one half? Are they purely numerical – such as the equivalence class of fractions of which one half is the standard representative? Or does acceptable phenomena extend to physical quantities with which we engage? Observant teachers of children know that the concept of one half develops from qualitative conceptions, such as 'you have the little half and I have the big half', to quantitative conceptions in which precision and equality are key factors, and later to meta-quantitative knowledge whereby the relation common to a possibly infinite number of numerical relationships is constructed (Hunting

& Davis, 1991). It seems a sensible aim to use didactic settings and materials which will assist students to advance to more sophisticated (i.e. powerful) conceptions. How this might be done in practice is something with which we continue to wrestle. As Fischbein (1987) says:

> The main problem is to learn to live with the intuitive loading of concepts – necessary to the productive fluency of reasoning – and, at the same time, to control the impact on the very course of reasoning of these intuitive influences. For this, the student has to learn to become aware of the exact, formal meaning and the implications of the mathematical concepts, on the one hand, and the underlying intuitions on the other (p. 207).

But what does Fischbein mean by "the intuitive loading of concepts", and another thing, how do we develop materials that stimulate students' intuitions in such a way that the signal is able to be separated from the noise?

Case of the CopyCat: A computer–based fraction learning setting

The CopyCat is an operator-like computer based learning tool developed in Hypertalk 2.0 for use on Apple Macintosh computers. Experiments performed using the CopyCat program may be directed at determining what fraction is responsible for observed numerical inputs and corresponding outputs, or at determining the numerical value of inputs or outputs for given fractions governing the CopyCat's behavior. Part of the graphic display can be seen in Figure 1.

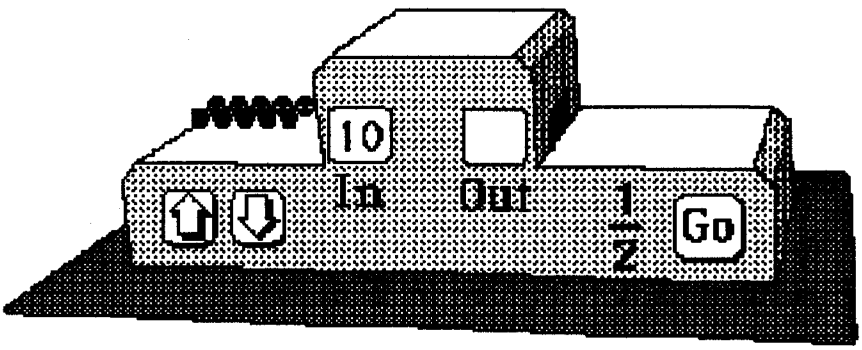

Figure 1. Graphical features of the CopyCat

The CopyCat's structural and graphical properties

On the CopyCat itself are three "buttons;" each can be activated with a mouse-click when the cursor is positioned over it (see Figure 1). The arrow buttons control the number of counters placed on the "in-tray" (left side of graphic). Counters are added or subtracted one at a time. The Go button activates a script which determines what the CopyCat will do. If the number of counters placed on the in-tray is divisible by the fraction denominator, *d*, and the Go button is clicked, an observer sees a group of *d* counters removed, one at a time, from the in-tray; simultaneously a distinctive sound is heard. Next, counters appear one at a time in the output tray until the number of counters in the tray equals the number in the numerator of the selected fraction. A new sound accompanies the appearing fraction. This process continues until all the counters in the in-tray have been used up. Finally, to the accompaniment of applause, all the counters reappear in the input tray; and the windows, above the words "In" and "Out," display the number of input and output counters. If the number of input counters is not divisible by *d*, the CopyCat "explodes" to appropriate noises. Other buttons – visible and invisible – can be used to reset the machine after each experiment and to activate a screen to cover the selected fraction. The fraction can be selected by choosing a restricted set of numerators and denominators from the menu bar at the top of the display (not visible in Figure 1). Possible numerators are numbers one through eight; possible denominators are numbers one through six, eight, and ten.

A feature of the CopyCat is that numerical inputs and outputs, in the form of countable objects, are perceptually available, and furthermore, successive experiments can be carried out rapidly to determine the function that is operating on the inputs. Success in solving these problems assumes that the common relation or rule across a class of particular instances is at least implicitly understood. Conventional approaches to teaching fractions focus on identifying specific instances of relations to wholes and parts with the formal language and symbolism. The importance of a fraction referring to a general and more abstract relation *across an infinite set of particular instances* is not addressed satisfactorily in mathematics curriculum. This tool has the potential advantage of being, in structure, an isomorphic copy of the field of rational numbers. That is, in every respect, its operations coincide with the mathematical laws governing that field. Furthermore, preliminary testing with children aged seven and eight years (Hunting, Davis & Bigelow, 1991) showed that simple versions of that tool are accessible to young children who have had no formal instruction about fractions. Indeed, it would seem that young children are capable of relatively abstract conceptions of one-half, for example, even prior to learning the associated formal symbolism. Other work with eight and nine year-old children (Davis, Hunting, & Pearn, 1993a; 1993b; in press a; in press b) has demonstrated the potential of this tool to assist in the development of powerful comparison schemes.

Note however that it is in the nature of the CopyCat *not* to operate on inputs unless they are divisible by the set denominator. Such a limitation can be a source of concern (Carraher, 1993), especially since one can easily perform simple partitions mentally which the machine will not! For example, consider one half of three items. There are other limitations too, although probably less contentious. For example, the CopyCat in its current form will not accept more than 50 input items; it will output only up to 150 items, and the range of possible numerators and denominators available when choosing a fraction is restricted. The CopyCat is not unique in this regard. Any physical instantiation used for didactic purposes has limitations.

Other action-oriented instructional materials

Sticks

Steffe and his collaborators at Georgia (Biddlecomb, 1994; Steffe, 1993; Steffe & Olive, 1992; 1993; Steffe & Tzur, 1994; Steffe & Wiegel, 1994) have developed computer microworlds in their research on children's rational number learning. One such microworld is called *Sticks*. Sticks is essentially a length measurement setting in which children can manipulate graphic images of line segments. The sticks portrayed indicate hierarchies of units; and fractions are used to describe relationships between various stick lengths. Among problems posed for the children are those of determining equivalence and order of multiples of different length sticks. A limitation of Sticks is that the basic unit of length is the video screen pixel. Potential difficulties arise when comparisons are made between multiples of different stick lengths assigned by fractions where the numbers of pixels being compared do not equate.

Paper folding

Another physical setting used for ages in classrooms and recently used in a research study on rational number learning is paper folding (Kieren, 1992; Kieren, Mason, & Pirie, 1992). A fundamental operation supporting instructional experiments is the action of halving. This is reasonable since paper sheets can be manipulated using this physical action. However, problems posed are necessarily restricted to fraction denominators that are powers of two, although Kieren et al. (1992) report an ingenious method for representing 1/3 with paper using symmetry and folding. Such a method has also been reported by Hunting and Korbosky (1990), who conducted a teaching experiment with Year 5 children. Nevertheless, prime denominator fractions other than 1/2 are, as far as we know, very difficult to represent, and when achieved (as an approximation) can lead to difficulties. For example, in Hunting and Korbosky's study, a task using paper strips was used to compare two fractions. The fractions were 1/2 and 3/6. One student folded her paper strips in such a way that when one strip was placed side by side with the other, the centre folds did not match up. She concluded, based on the evidence, that 3/6 was *larger*. (She might just as reasonably have concluded 3/6 to be smaller.)

The intended effect of instruction using paper folding is often thwarted by other conceptual and procedural errors. Research with elementary and junior high school students has shown that children use a wide range of strategies on area comparison tasks, not all of them appropriate, such as comparing number of shaded blocks, or comparing length of the blocks (Armstrong, 1989). For children who do not naturally use a direct comparison method to determine equivalence, both examples in Figure 2 are troublesome. While in the first problem, the teacher might rotate the paper 90° and "suggest" that the student directly compare the shaded areas, the second example presents the additional complication that the two papers were oriented differently before the folding and coloring occurred, and interpretive discussion must be managed carefully so that the student will conclude that 2/6 is 2/6.

Other difficulties children encounter

LOGO is a computer environment designed to help children develop their conceptions of two-dimensional space by giving commands that govern the movements of a turtle and watching as those movements are recorded as a path. Kurland and Pea (1985) tested children who averaged over 50 hours of experience using the LOGO programming language and found that the students' hands-on experience did not provide meaningful learning. On their own, students had developed incorrect notions about fundamental programming con-

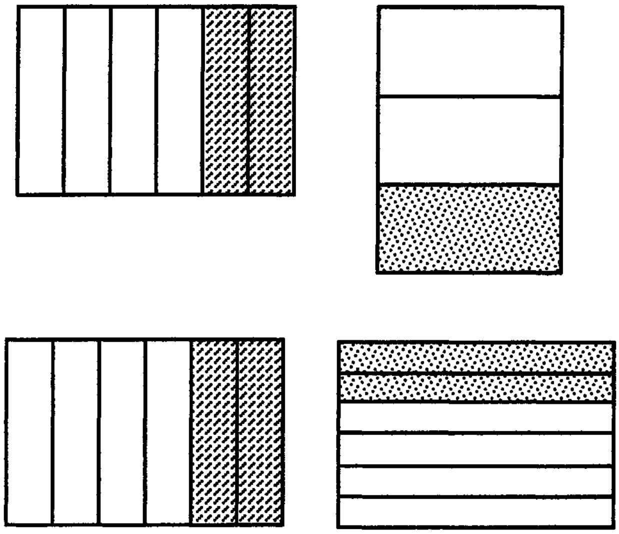

Figure 2. Equal areas through paper folding.

cepts, leading the researchers to conclude that this form of active learning "needs to be mediated within an instructional context" (p. 242). In other studies comparing college students who had hands-on experience with the BASIC computer language with others who were given direct instruction about the connections between commands and concepts, the group who had teacher-based instruction in addition to student exploration, fared better on transfer tasks and showed fewer misconceptions (Bayman & Mayer, 1983).

Gravemeijer (1991) reported on the experiences of the Dutch researchers in the development of the Rekenen & Wiskunde textbook series in which the abacus was used as preparation for addition and subtraction of whole numbers. The Dutch researchers found that the manipulative action was not isomorphic to the mental action that was necessary to perform symbolic arithmetic.

Similarly, when using a number line to help children add and subtract whole numbers under twenty, the counting process does not correspond to one's mental work. For example, to add 5 + 4, the pupil starts at 5, counts "1, 2, 3, 4," and reads out "9," while the mental task is a case of double counting "6, 7, 8, 9," while simultaneously counting the counts (Gravemeijer, 1991).

Bishop & Goffree (1986) reported another problem concerning the relationships between representations that occurs when children use Cuisenaire rods. Children make connections between number names and colors, failing to realize the significance of the lengths of the rods.

Dienes blocks, also called multibase arithmetic blocks (MAB), are sets of wooden blocks, each embodying a different base system. A set for base b is composed of single cubes called "units," rods or "longs" composed of *b* units, squares called "flats" of area $b \times b$, and cubes , called "blocks," of dimension $b \times b \times b$. Resnick & Omanson (1987) found that children who used multibase arithmetic blocks to perform addition and subtraction operations, were unable to make a smooth transition to the use of written symbols. Whether or not the actions children have developed are a good match to the desired mental activities, the rules and algorithms children develop when working with the manipulative materials may be carried over to their operations with symbols, or, when they work with symbols not well connected to their network of knowledge, children invent flawed algorithms when they become stuck (Brown & Van Lehn, 1982).

Hiebert and Carpenter (1992) spoke to the tremendous potential for instruction involving concrete materials to go awry:

> ... it is not simply the presence of concrete materials that provides meaning for symbols, nor is it simply the juxtaposition of materials and symbols. In order for symbols to acquire meaning, learners must connect their mental representations of written symbols with their mental representations of concrete materials. The potential for these connections to create understanding is complicated by the fact that concrete materials themselves are representations of mathematical relationships and quantities. Thus, the usefulness of concrete materials as referents for symbols depends on both their embodiments of mathematical relationships and on their connections to written symbols (p. 72).

In the next section, we adopt a mathematics modeling perspective to further examine some of the problems associated with the use of concrete materials; namely, can the translation between representations induce learning or does it signify an already existing basis in understanding?

A modeling perspective

Humans use the language of mathematics to describe patterns (Steen, 1990), to collect and order experiences, and to articulate the structure of the world (Kitcher, 1984). This view of mathematics learning and problem solving relates the "doing" of mathematics to the construction of models. Because the term "model" has been devalued through broad application to a variety of ideas, to facilitate our communication, we shall use the term "model" to refer to a cognitive (internal) construction, a system of quantities, relationships, operations, and representations constructed in some subjectively meaningful way, connected to the individual's existing knowledge base, and used to make sense of one's subjective world of experience. Representations (or embodiments) such as Cuisenaire rods or Dienes blocks are concrete (external) interpretations of internal models, notation systems that facilitate the communication of our models to others.

Because of the connectedness of mathematical ideas, models develop over time and not in isolation of one another. They grow in complexity, completeness, and depth (Lesh & Lamon, 1992). They overlap and eventually unite. In short, it is not the case that understanding happens or does not happen, but rather, it can be characterized as a dynamic, multi-dimensional growth process (Kieren, Mason, & Pirie, 1992). Though we may talk about "model building" in the singular, in reality, we often use multiple models in series or in parallel, because of their differing abilities to capture relevant aspects of a given situation, and we embed these models in different notation systems, only some of which take a physical form: spoken language, written symbols, static pictures or diagrams, manipulative concrete materials, and visualization (Lesh & Lamon, 1992).

A question of ownership

Whether we ask students to examine the operator notion of fraction using a computer program such as the CopyCat, to explore the measurement notion of fraction using sticks as length models, or to attach meaning to fractions by means of paper folding, we are asking them to step into a system for describing, explaining, constructing, and manipulating some portion of *our* world of experiences. Eventually, we hope that students will be able to explore the system for its own sake so that they might use it to gain further insights, to make conjectures and generalizations about the phenomenon the system embodies, or to somehow elaborate what they know. But is this always a reasonable expectation?

In addition to environmental affordances that influence the model-building process, that is, such things as experiences, events, teacher-made interventions, and observable regularities in nature, models are shaped partly by a person's unique cognitive filtering, organizing and interpreting functions. Can we actually facilitate students' learning by offering them models designed and promoted by expert adults? We propose that model building is motivated by a certain amount of evidence, a kind of intuitive, experiential, tacit knowledge, gut feeling, or sneaky suspicion that there is something in a situation worthy of further investigation and explanation, a sense that there is something there that the person does not know, and that, in fact, this kind of "primitive knowing" (Kieren, Mason, & Pirie, 1992) provides the push that enables the personal investment of time and effort.

Unless a student already possesses this primitive knowledge, our efforts to facilitate further knowledge construction through the use of imposed didactic materials may be fruitless. In model building, "the foundation and growth of knowledge become more and more intertwined" and "knowledge appears as the cause as well as the objective of the investment and involvement" (Brousseau & Otte, 1991, pp. 26, 33). A certain amount of crude knowledge may account for the distinction between model-constructing (adopting or sharing) and merely appearing to use someone else's model; the difference between learning through the use of manipulatives and merely going through motions.

Representations simply cannot be told. If students have not yet constructed the meanings and images behind the concrete materials, then the actions on the materials are governed by adult-made rules that the student must follow, or, in many cases, guess. When children fold one paper lengthwise and one crosswise and then cannot discover equivalence; when children count MAB rods and units as if they were both tens; or when children adopt multiple strategies for area comparison during paper folding activities, they are telling us that the material re-presents someone else's meanings, not their own. As Gravemeijer (1991) warns,

> By not making a clear distinction between internal and external representation it goes unnoticed that one is mixing up the time order: the pupil needs the mental representation (model)... to be able to interpret the concrete representation (p. 67).

The rôle of materials

These questions resurrect others posed by Kieren (1971) more than 20 years ago that remain unanswered today: Where does action-based learning with physical materials fit into a sequence of experiences that contribute to mathematical understanding? For whom, for what topics, with what materials, and under what circumstances are manipulative activities valuable? Readiness, timing, degree of content-relatedness, personal preference, and other overlooked variables may be more important than we realize.

Perspectives from research

The current popularity of instructional materials is based more on opinion and personal belief than on research-based knowledge. Certain materials gain popularity because a large number of adults agree that those materials capture the essence of their own understandings about a certain piece of mathematics. Although the allure of these materials might be powerful, a close examination of the literature reveals that there has never been indisputable evidence of their effectiveness. A number of studies conducted in the 1960s and the 1970s, such as Fennema (1970), Nelson (1964), Biggs (1965), Williams (1967), Trueblood (1967), Vance & Kieren (1971), Wilkinson (1974), Suydam & Higgins (1977), Friedman (1978), and a more recent meta-analysis of 60 studies (Sowell, 1989), suggest that there is a large host of variables influencing the use of didactic materials, among these, type of material, length of time used, teacher training, age of the students, whether students or teacher chose the manipulative. It should come as no surprise that the quality of the teaching was the single most decisive variable in the successful or unsuccessful use of manipulative materials. The skill of the teacher in facilitating connections between the material and related conceptual work has a strong positive effect on achievement and understanding (Suydam & Higgins, 1977).

Bandwagons

The United States *Curriculum and evaluation standards* (NCTM, 1989) called for a more conceptually oriented curriculum, for more sense making and fewer rote activitites, for mathematics instruction that recognizes the need for students to actively participate in their own learning. The imperatives of the Standards have been popularly interpreted – without the benefit of research-based guidance – as a demand for "hands-on" learning. Although the teacher is no longer recognized as one who transmits knowledge, concrete materials have become a substitute for teaching as telling. A new (but tacit) belief is that concrete materials transmit knowledge.

A narrow interpretation of "active" learning further legitimizes the new rally cry, "learning through manipulatives." Active learning certainly goes beyond mere physical manipulation to include cognitive engagement in sense making. Because it is the cognitive engagement that is essential, early research involving instructional materials indicated that it makes little difference whether the students are doing the manipulating themselves, or whether they are watching and thoughtfully participating as the teacher demonstrated with the materials (Suydam & Higgins, 1977). As Baroody (1989) said: "The particular medium ... may be less important than the fact that the experience is meaningful to pupils and that they are actively engaged in thinking about it" (p. 5).

In the interest of pedagogically responsible use of physical materials, we propose that some unexamined assumptions underlying the use of certain materials be questioned. Because there is an adult consensus about a certain type of manipulative material, does this imply that it must be useful in helping students to achieve mathematical understandings closer to our own? Are we not giving students our models and expecting them to recognize what we already know? Do we want students to understand that they should start with experiences and model them, or do we start with models and try to relate them to our experiences? To what extent does such a classroom culture truly reflect a constructivist theory of knowledge building? Because a model is a metaphor, an inseparable mingling of a mathematical idea and a specific way of representing it (Brousseau & Otte, 1991), questions about which material to introduce and when to introduce it are, simultaneously, questions about how mathematical ideas are best sequenced for instruction. Is it possible that a particular didactic material may be ineffective for some students because for them an alternate representation or multiple representations are needed?

Further questions

Given the inadequacies of physical materials, are there guidelines for choosing the most efficacious, and what might these be? For example, there has been continuing debate about the relative merits of continuous versus discrete materials for teaching fractions and whole numbers (Hatano, 1982; Hiebert & Tonnessen, 1978; Hunting & Korbosky, 1990; Minskaya, 1975). Considerable research has been conducted on pedagogical approaches to learning whole numbers (see for example, Carpenter, Moser, & Romberg, 1982; Greer, 1992; Hiebert & Behr, 1988; Labinowicz, 1985; Leinhardt, Putnam, & Hattrup, 1992; Sowder, 1992; Steffe & Cobb, 1988). But do the seemingly greater complexities of learning rational numbers imply different strategies for teaching? Or another way, does the complexity of teaching rational numbers mean different criteria or relaxed conditions or expectations, compared to teaching whole number numeration? Do our pedagogical mathematics methods seduce us into using a traditional physics teaching approach, so that we ignore crucial distinctions between the nature of mathematical knowledge and scientific knowledge? Is there a single "best" pathway to knowledge of a mathematical concept or cluster (Ellerbruch & Payne, 1978), or are multiple pathways better (Kieren, 1976)? Mack (1990, 1993) has suggested that rather than attempting a broad-based approach to developing understanding of rational numbers a viable alternative may be to develop a particular conception, then expand that conception once students can relate mathematical procedures and their symbols to their informal knowledge. Is the distinction between concrete and abstract *thought* valid (Menchinskaya & Moro, 1975), and if so, how can it inform our pedagogy? What about the issue of visual and perceptual material actually hindering learning (Perry & Howard, 1994)?

How then do we deal with the difficulties of teaching mathematical abstractions using instructional materials?

It is clear that there is no possibility of finding physical material with the robustness needed to mirror, as it were, the precise essence of the mathematical concept we have in mind to teach. But is it necessary to assume that since a physical setting does not afford actions corresponding to operations we believe to be important, then that setting is necessarily inadequate? For example, the CopyCat won't operate on seven input items when it is set to 1/2. What if it did? Since it doesn't, does that mean it presents a serious obstacle for learning and teaching rational numbers? In any case, what settings *will* allow the possibility of infinite divisibility, other than in one's head?

It may well be that particular physical settings are to be preferred. Even so, we still have a problem. Cobb (1992) has argued the inadequacies of a learning theory which assumes students modify their internal mental representations to construct mathematical relationships or structures that mirror those embodied in external instructional representations. Fundamentally, sources of meaning are assumed *not* to be inherent in external representations but located in "students' purposeful, socially and culturally situated mathematical activity" (p. 6). A trap mathematics educators can fall into is to imbue selected physical material with the meanings of experts (us, as well as other teachers) which, to those experts, are self-evident. Being self-evident, we assume that our interpretation "is shared with everyone else who knows mathematics" (Cobb, 1992, p. 9). In the process the expert "smooths over" or overlooks serious deficiencies inherent in the material – the noise. How can we hope that students will focus on just those relationships we "see" in the materials as structured, and not on any of a host of others? The teacher's saving grace is the necessity and significance of interactive communications between students, and students and teacher, as they together develop "taken-as-shared" mathematical ways of knowing. As Cobb (1992) argued: "Such a view emphasizes that the learning-teaching process is interactive in nature and involves the implicit and explicit negotiation of mathematical meanings" (p.10). Thompson (1992) drew two conclusions from his research into the relationships between concrete materials, and the signs and conventions of whole number numeration:

> The first is that before students can make productive use of concrete materials, they must first be committed to making sense of their activities and be committed to expressing their sense in meaningful ways. The second is that for concrete embodiments of a mathematical concept to be used effectively in relation to learning some notational method, students must come to see each as a reflection of the other – constraints and all. They must end up feeling just as constrained in their notational actions as they do with those actions' counterparts in a concrete setting (p. 146).

Having rejected the idea that mathematical meanings lie in physical embodiments used as teaching devices, the burden is somewhat lifted from our shoulders. It is interesting to note that Cobb, Yackel, and Wood (1992) prefer to call materials such as *The Candy Factory* – used to teach whole number numeration and place value – "pedagogical symbol systems" (p. 22), rather than instructional representations, in order to emphasise the symbolizing role they play in individual and collective mathematical activity. As he goes on to explain:

> ... we might say that materials typically characterized as instructional representations are of educational value to the extent that they facilitate students' individual and collective constructive activities and thus their increasing participation in the mathematical practices of wider society. In this view, correctness does not mean conforming to the dictates of an authority who spells out his or her own interpretation. Instead, it means making mathematical constructions that have clout in that they enable students to increasingly participate in socio-historically evolving mathematical practices (Bruner, 1986). Moreover the notion of instructional materials as a means of delivering mathematical knowledge to students is displaced by the view of teachers initiating and guiding emerging systems of mathematical meanings and practices in their classrooms (p. 25).

Other material upon which students operate

It is as well to remember that not all children's mathematical thinking is based on physical material, models, or devices. In fact we argue that the stuff of mathematical thinking is intricately bound up with material *other than* the physical or perceptual. For example, we are only beginning to explore the role of the imagination in the learning process (Johnson, 1985). Mathematics is a curious body of knowledge, in that its invention and application may, for extended periods, be an almost entirely cognitive experience. Consider this question and where it might lead: Is there a fraction between one half and one third? What might it be? Children in the elementary school need to consider questions such as these. Physical materials and experience serve as intuitive bases for thought and reflection. Reflection on objects and relationships apprehended or constructed, in the context of interactive communications with other students and adults, promotes the development of general mental structures and operations which will transcend specific instances and examples.

Mathematics learning may be viewed as a cybernetic process. Existing cognitive "material" allows a learner to observe, interpret, and verbalize about actions on physical material. Relationships in problematic task settings involving physical material can stimulate the development of cognitive material when the phenomena bound by the physical material cannot be fully "assimilated" into the learner's mental schemes. Constructivists advance the hard core device *disequilibrium* with its affective (discomfort, anxiety, frustration) and cognitive (accommodation, reorganisation) consequences to explain this learning process.

The physical and perceptual materials mathematics teachers use with students are only one source of the total storehouse of material upon which children's minds work. Other material includes language and vocabulary used to communicate about experiences, problem situations, and other printed literature, as well as recall of static images and re-presentation of dynamic experiences stored in memory.

Summary and discussion

The popularity of manipulative materials is understandable. They may be very motivational; they are often colorful and engaging, like play things (Steffe & Wiegel, 1994). They provide an alternative to pure expository teaching. In a sense, they give instruction a certain grounding in reality. Children like to manipulate and there is broad consensus that active lessons with concrete embodiments improve students' attitudes about mathematics class. To use instructional materials responsibly, however, requires attention to their shortcomings as well as their strengths and to some of the assumptions underlying their use.

We have seen that representations often highlight certain aspects of mathematical interest at the expense of others. It is important to recognize precisely which ideas they instantiate and which they neglect. We have also noted that materials do not always demonstrate their own validity. Sometimes movements performed with concrete objects do not correspond with the mental activity needed to perform certain operations; in such cases, they may be creating obstacles for, rather than facilitating, the jump to mental and symbolic operations. Finally, adults who find certain embodiments attractive, often find it difficult to put themselves into the minds of children. For the uninitiated, concepts and connections are not as easily deduced from the materials, and when children operate according to the teacher's rules without any subjective meaning, we run the risk that they will memorize teacher actions or invent their own (sometimes "buggy" (Brown & van Lehn, 1982)) algorithms to get them through the task.

One useful way in which to accommodate these issues in classroom instruction is to clearly distinguish "teaching mathematics through manipulatives" or "hands-on" teaching as a form of instruction, from the use of concrete manipulatable materials as part of a broader instructional environment. When viewing concrete materials as a teaching strategy, one often assumes that activity necessarily produces learning, and the inadequacies of the materials may be overlooked. If, however, materials are viewed as tools, and the teaching *strategy* is negotiation of meaning, students are given further opportunity to build meaning through reflection, communication, cognitive conflict and its resolution, reality testing of subjective impressions, and other community-based experiences.

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Undervisningsmaterial i matematikundervisningen

– en översyn

Sammanfattning
Två samtidiga fenomen, tillgången till datorbaserade inlärningsmiljöer och förnyat intresse för "hands on"- inlärning är en populär reaktion på rekommendationer i nationella kursplanedokument som NCTM Standards. Detta ger anledning till en översyn av frågor kring bruket av undervisningsmaterial. Effektiviteten hos ett flertal manipulativa material i undervisningssituationer diskuteras. Styrka och begränsningar analyseras.

Kritiska skillnader mellan inre modeller och yttre konkreta former samt mellan den av läraren avsedda innebörden och den subjektiva naturen hos elevens tolkning lyfts fram genom att anlägga ett modell-perspektiv på användningen av undervisningsmaterial vid matematikinlärning. Konsekvenserna av en snäv tolkning av aktivt lärande studeras och det varnas för att sätta sin lit till ett enda manipulativt material för att fånga rikedomen hos och sambandet mellan matematiska idéer. Som ett alternativ till att låta elever upptäcka begrepp och idéer, så som de uppfattas av vuxna, föreslås att material används i ett större pedagogiskt sammanhang i vilket elever individuellt och kollektivt underhandlar om den matematiska innebörden. Återstående obesvarade frågor framställs i form av en agenda för framtida forskning.

Författare
Robert P. Hunting är Senior Lecturer i Mathematics Education vid La Trobe University, Victoria, Australia.
Susan J. Lamon är Associate Professor vid Kansas State University, Kansas, USA.

Adresser
Robert P. Hunting, Graduate School of Education,
La Trobe University, Bundoora Victoria 3083, Australia.
Susan J. Lamon, Department of Mathematics, 22 Cardwell Hall,
Kansas State University, Manhattan, KS 66506, USA.