What on earth is a straight line?

What can we learn from an epistemological episode in eighth grade?

Johannes Paasonen

A detailed account of a spontaneous discussion in eighth grade of the nature of geometrical objects is given and related to constructivistic learning theory. The pupils were quite eager and able to discuss such subjects as infinity and more generally the process of abstraction inherent in geometrical concepts. Discussions of this type in class are advocated in order to encourage pupils to problematize concepts and procedures often regarded as self-evident, to spot and discuss pupils' misconceptions, and thus to attempt to raise the level of abstraction in mathematical communication and concept formation in class.

A constructivist setting

Constructivist learning theory implies that all concept formation takes place as a result of an active process of construction by the learner in interaction with the environment (Noddings, 1990). In school, the environment is mostly composed of the social network of the fellow students and the teacher. The role of the teacher is to create learning environments that stimulate the pupils' constructive processes and to guide these processes (Schoenfeld, 1992). From this viewpoint, an essential factor in learning mathematics is the communication that takes place in the classroom and in the course of which the pupils share their ideas, hypotheses, and speculations with each other and the teacher (see e.g. Paasonen and Salmela, 1994; Repo, 1992).

In addition to planning suitable learning situations in advance, the mathematics teacher must constantly be on the alert to spot classroom situations which may lead to constructive activity by the pupils, situations in which the pupils can be stimulated to develop and discuss mathematical issues. Such situations may arise quite unexpectedly and evolve in directions quite different from those one has planned.

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This is often disturbing but also exciting. The teacher must be ready to enter a serious discussion with the class and risk arriving at genuine problem situations in which she or he may not have ready-made solutions at hand.

Mathematical communication

A major problem for the teacher is that the pupils' mathematical vocabulary is often limited and also that the pupils' representations of mathematical concepts and objects differ considerably from those of the teacher. Consequently much time must be used to discuss mathematics in class, to have the pupils explain mathematical concepts and relations in their own words to each other and the teacher. A very good exercise is to have the pupils explain mathematical concepts in their own words in writing and to use the pupils' work as a starting point for classroom discussions.

Often pupils must be given an intellectual shake to have them start thinking. The term cognitive conflict is sometimes used in this context to denote a situation in which the pupil's conventional thinking leads to a contradiction or a dead end and she or he must construct a new, more general concept to cope with the situation (e.g. Rowell et al., 1990).

In the following real life episode the teacher had planned a routine review lesson to start a unit in geometry in grade eight (pupils aged 14). For motivation, the history of the word geometry was reviewed and a quite common stratagem for activating the pupils was employed: The pupils were asked to list terms related to geometry and these were written on the blackboard and discussed. Next the teacher intended to review some familiar constructions but then started to wonder what the concepts the pupils had remembered really meant to them, what kind of cognitive structure they formed in the pupils' minds. He remembered a discussion with a colleague in primary school who had told that he used tales featuring Martians to arouse the interest of the pupils, and decided to employ the same idea to capture the interest of his pupils and have them problematize attributes related to a very familiar concept, the straight line. The aim of the teacher was to induce the pupils to consider and discuss abstract and difficult concepts such as infinity, straightness and dimension and thus have them develop their cognitive structure related to geometry by constructing new, more abstract attributes for familiar concepts. The object of the discussion pictured below was not so much arriving at "correct" definitions as having the pupils consider and problematize and discuss the mathematical objects in question and thus create a cognitive basis from which it would later be possible to proceed towards a more rigorous teaching-learning process.

The eighth grade group involved is somewhat above average. What follows is a reasonably accurate picture of the discussion that went on in the classroom. It was written down by the teacher immediately following the lesson.

What on earth is a straight line?

A real life drama with two epilogues

Teacher:	Now we are going to study geometry again for a while. What kind of things does geometry deal with?
Minna:	We measure and calculate areas and things.
Laura:	We draw all kinds of figures.
Teacher:	Yes in geometry we investigate figures and their characteristics. Do you remember from last year where the term geometry comes from?
Pupils:	???
Teacher:	Think of other words which begin in a similar way as geometry.
Saija:	Geology.
Elina:	What about geography?
Teacher:	(*Writes the words geometry, geology, and geography on the blackboard*) What is it that geologists do?
Markku:	They dig in the earth and look for things.
Lari:	No — those are archaeologists. Geologists investigate the earth layers.
Krista:	Does that geo then mean something like earth?
Teacher:	Geo really does mean earth in Greek. What about that part metry? What does it remind one of?
Tommi:	A meter.
Teacher:	That's right. And a meter involves measuring. In Greek metrein means measurement, so verbatim geometry means measuring of the earth. From this practical activity has evolved what we now know as geometry as a result of work done mainly by the ancient Greeks. Let's review some geometrical concepts. Each one of you will now say a word associated with geometry, and I shall write them on the board. (*Writes the pupils' suggestions*):

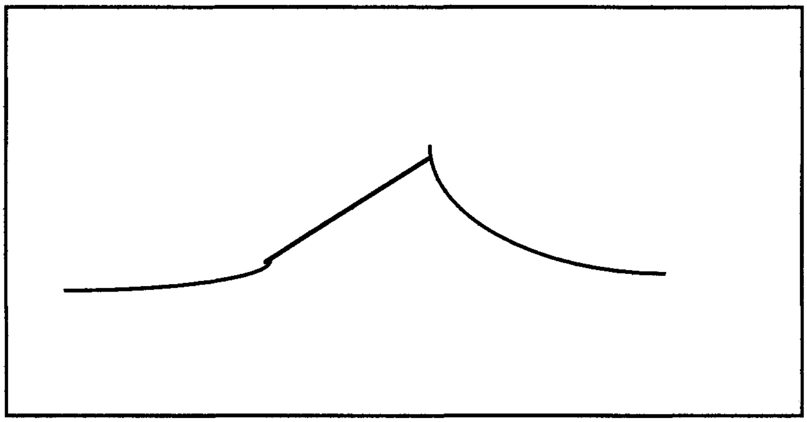
compass circle angle radius perimeter straight angle
obtuse angle ruler straight line triangle square
pentagon hexagon line segment parallel lines vertex
rectangle parallellogram isosceles triangle

Good. You really know quite a lot about geometry. But let's consider what all these words really mean. In fact, you really understand some thing when you are able to explain it in your own words.

(*Thinks for a moment*): Let's pretend now that I am a Martian who has come here and wants to find out what the concepts of the people of the Earth mean. Let's take this straight line, for example. We Martians have long been puzzled by what it really means. Please explain it to me!

Jenni: A straight line is just a line.

The teacher draws:



Is this a straight line?

Jenni: No — not like that. It must go straight.

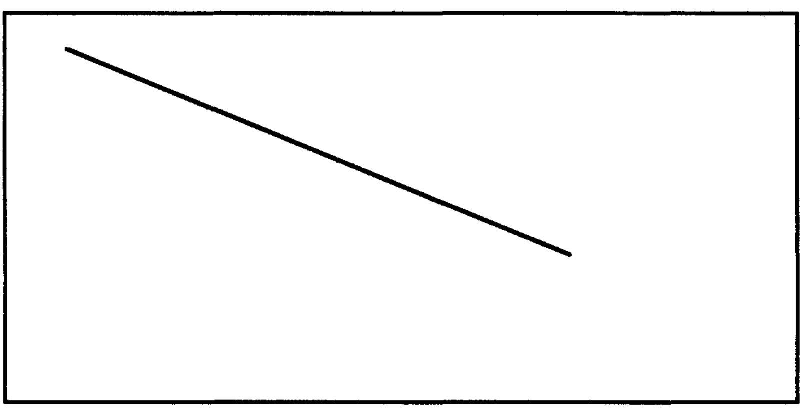
Kirsi: It is an even line.

Teacher: I don't quite understand. Could you explain to me how a straight line differs from this line?

Marianne: There are no curves in a straight line

Teacher: Aha — now I understand. (*Draws.*)

Is this a straight line?

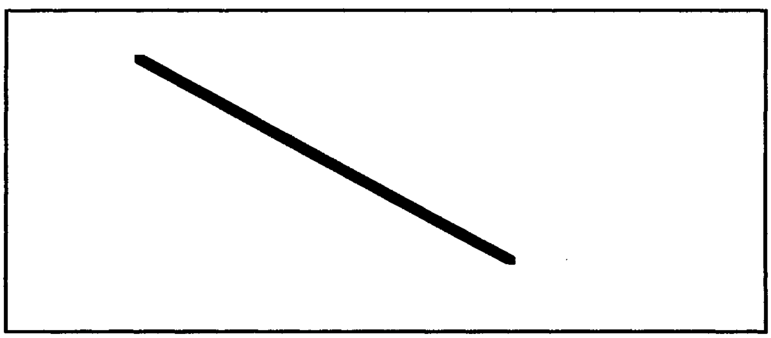


The pupils nod. The teacher doesn't seem quite satisfied.

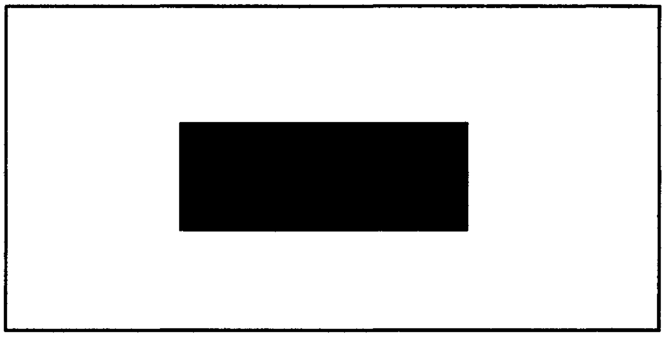
Kimmo: Well that's really a line segment.
Teacher: What's that supposed to be?
Kimmo: Well that's really quite short.
Teacher: Oh, a straight line is longer, is it? (*Continues the line to the edge of the blackboard.*) Is this better?
Kimmo: Not really. The straight line goes on for ever, infinitely.
Teacher: This is getting difficult. How can something go on forever?
Henry: It just goes on and on. Farther than the sun.
Teacher: But no one can draw that far. Even if I would draw for the rest of my life, I wouldn't get very far. And there wouldn't be enough pencils.
Saija: Well, your children and grandchildren could continue. And they would invent super pencils.
Teacher: But in the end there would be no more people...
Juulia: Come off it. You must understand.
Teacher: No, I'm serious. We in Mars want to get to the bottom of things.
Ruusu: Well go back to Mars then.
Teacher: We really must get this straight. What about when space runs out, how can the straight line go on?
Pupils: Come on and quit now — it just goes on and on.

Although one might from the dialogue get the impression that the pupils would have been quite aggressive and frustrated, the atmosphere in the classroom was relaxed and often one could really feel the pupils trying very hard to think and find good arguments to counter the Martian's "unreasonable" ideas. This impression was confirmed by the two teacher students who attended the lesson. In a way the jokes and jibes were a natural reaction to strenuous thinking, a way of summoning new energy for further argumentation.

Teacher: O.K. I'm not quite convinced that there is any sense to this, but let's for argument's sake accept that it is somehow infinitely long. But what about its thickness? (*Draws a thicker line*)
Is this a straight line too?



Jenni: Yes, it is.
Lari: No, not really. A straight line is so to say one-dimensional.
Teacher: One-dimensional? What does that mean?
Lari: Well, it just has one dimension.
Teacher: What then is two-dimensional?
Minna: Let me come and draw. *(Draws.)*



Teacher: Oh — it has length and width.

The pupils nod.

Teacher: So the straight line has no width. Then these can't be straight lines. (*Points at the figures on the board.*)

The pupils start to groan.

Krista: Look — if you take a very sharp pen and draw with it then it would be a straight line.
Teacher: But a sharp pen has some thickness too.
Juulia: Hey listen. We could think of putting all the atoms in a row like this and that would be a straight line.

Now we are getting to a new level of abstraction.

Teacher: But atoms have some thickness too. And they won't go on forever.
Pupils: Let's stop. This isn't funny.
Teacher: It really is important to clarify such matters. And I must figure these things out so I can tell my fellow Martians what a straight line is like.
Juulia: Why don't you go back to Mars now?
Teacher: But come now, seriously, straight lines are infinitely thin and go on infinitely. Where on earth can such things exist? If they are infinitely thin, then they can't exist at all.
Minna: Look at the edge of that cupboard there. Where the roof and the door join. It has no thickness.
Teacher: Yeees. That was a good demonstration. But it doesn't go on for ever. Try to really think: where can something exist that is infinitely long and infinitely thin. Such a thing **cannot** exist anywhere in this world.

A long intensive silence. The discussion has now arrived at a cognitive conflict. The teacher does not want to give the "right answer" but waits for the pupils to suggest a way of resolving it.

Noora: *(Hesitatingly)* Could you then say that what you have drawn on the blackboard are some kind of photographs of straight lines?

The teacher is speechless for a moment trying to reckon how to react to such a fine idea. Although one could say that the use of the term photograph, which is an image of an existing object, is not quite accurate, Noora seems to have grasped the difference between a concept and its physical representation.

Teacher: What do the others think — what could those be photographs of?
Silence.

To keep the pupils thinking the teacher tries not to give the impression that this is the answer he has really hopes for. Later on in the lesson he compliments Noora on her fine thinking.

Teacher: Where can such an impossible thing be, whose picture that is? What doesn't really exist can only exist in .....?

Lari: In our imagination.

Teacher: *(Forgetting his role of Martian)* Now we are really dealing with deep thoughts. Straight lines and other geometrical objects really exist only in our thoughts. Around us we see straight, uncurving objects, and from them we get the concept of an infinitely long and thin uncurving line.

Marianne: There's no sense in that at all. How can something exist only in my thoughts and anyway we can talk about it. I think those there are straight lines and that's that.

Marianne's answer actually deals with one of the real problems in radical constructivist philosophy: If everybody constructs concepts on his own, how come we can talk of them and know that we are thinking of the same thing!

Teacher: You don't have to believe, but if you stop to think carefully ... oh well.

The teacher decides that it is better not to try to force abstraction on pupils who are perhaps not ready for it. Instead he decides to furnish the pupils with more material for thought.

Teacher: Now I'm a Martian again. Now I understand that a straight line is the image we have in our minds of an infinitely long and infinitely straight and infinitely thin object. So now I understand what a straight line is. But what then is a circle?
Try and explain, please.

The lesson continued in the same vein. Next we considered a circle and finally we discussed what a natural number such as three really means. Even though the pupils groaned and gave vent to their frustration in the face of difficult questions they evidently enjoyed the situation and many were really intrigued by the challenge of having to consider familiar objects in a new way. For homework the pupils were given the following instructions:

Teacher: O.K. We've spent an hour discussing really deep things. For home work you will.
(*Writes on the board*)

Explain to a Martian, what is a) a square b) an angle.

The pupils: We don't have to write, do we?
Teacher: Of course. Remember, that when you can explain it in your own words you really understand what something is.

The bell.

Epilogue 1.
(In the physics class following the class described above):

Riikka: Listen Mr. H! Our math teacher has gone quite mad. All last lesson he pretended that he was a Martian and we had to explain all kinds of things to him. It was terrible.

Epilogue 2.
Written explanations by the pupils on the subject:
Explain to a Martian what an angle is.

Pia:
An angle is a figure, whose vertex can be obtuse or oblique. Let's put it like this: it is a straight line that is folded at some point. The folding point is called the vertex. A straight line can be folded so that the sides are close to each other, then the angle is small. As the sides draw farther and farther apart, the angle gets bigger. It can also come back to a straight line. When it goes over a straight line, it is no longer an angle.

Jenni:
Two straight lines that meet each other make an angle.

Henri:
Angle is a word that is made of five letters. Note: There is no life on Mars so there don't exist any Martians.

Lari:
When two straight lines intersect, an angle is formed. An angle can be oblique, right, obtuse, reflex or straight. The angle is the area between the straight lines.

Laura:
An angle is a part of a straight line that has been slightly bent. But what we really do is draw pictures of angles, even though an angle is something that exists and is not infinite, it is something that is in our minds or then it isn't.

Noora:
When two straight lines intersect and they stop at the meeting point, the area of the plane lying between them forms an angle.

Discussion

From the point of view of concept learning the lesson showed that many pupils have quite primitive and concrete (mis)conceptions of geometrical concepts, as is shown by the "definitions" some pupils wrote down for angles. On the other hand, the pupils were surprisingly ready to become involved in sophisticated arguments about abstract concepts like dimension and infinity and to accept the problematization of familiar concepts. It seems natural that many pupils did not accept the necessity to modify their conception of the straight line and did not see the discrepancy between the attributes of the concrete line on the blackboard and the concept of straight line. Conceptual change is a slow process, and, to quote a recent article by S. Caravita and O. Halldén (1994):

> Consciousness of a contradiction between schemata is not produced until at the level at which the subject becomes capable of transcending it has been reached, but this requires a higher theoretical level in which one can integrate (assimilate) the negative fact into a richer conceptual schema.

From the class discussion and the pupils' definitions related above, one can infer that the levels of the pupils' cognitive structures varied considerably. Especially the pupils who had reached higher levels of abstraction seemed to have gained new material for thought, an impression that was strengthened by spontaneous questions and discussions initiated by them in subsequent classes.

I feel that encouraging such discussions is one way of creating a learning environment in which the pupils actively engage in discussing concepts, spotting and resolving cognitive conflicts, and, on the long run hopefully raising the level of abstraction of their cognitive structures.

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Vad i all världen är en rät linje?

Sammanfattning

Artikeln beskriver, utifrån en konstruktivistisk inlärningsteori, en spontan diskussion i en åttonde klass om vad som menas med geometriska objekt. Eleverna diskuterar engagerat och ingående saker som oändlighet och, mera allmänt, den abstraktionsprocess som ligger bakom bildandet av geometriska begrepp. Sådana diskussioner behövs för att få eleverna att problematisera begrepp och procedurer, som kan verka självklara. De är också värdefulla för att komma åt och diskutera elevernas missuppfattningar och deras tänkande över det egna tänkandet. Därmed får man möjlighet att höja abstraktionsnivån i de matematiska samtalen och i begreppsbildningsprocessen i klassen.

Författare

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