A problem-centered alternative to formalistic teaching

Experimentation on a contextual teaching program of ratio and proportional reasoning in the eighth grade of the comprehensive school

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Increased attention has been given in recent times to the teacher-centered and formalistic nature of mathematics teaching in schools, and well-founded proposals have been made to change this tradition toward a problem-centered direction which takes into account pupils' previous learning experiences and out-of-school practices. After a historical scrutiny and the criticism of the current formalistic tradition – 'first theory, then practice' –, the article outlines an alternative teaching strategy which provides at least in theory better opportunities for discussive and meaningful teaching-learning processes in school mathematics. Finally, some main results from one of the teaching experiments of the project "Contextual Approach to the Teaching and Learning of Mathematics" are examined. The teaching experiment focused on developing proportional reasoning and ratio concept in the eighth grade of the comprehensive school. Although numerous problems were encountered in the implementation of the "contextual program" designed on the basis of the ideas of Freudenthal's didactical phenomenology and the ideas of the "Vygotskian school" of psychology, the learning results achieved by means of the contextual program were at least equally good and partly better than those achieved in groups taught by the textbook.

Introduction

The basic motive for the teaching of mathematics in schools can be seen in the fact that mathematics forms an essential part of the cultural historical heritage of mankind. Another fundamental motive is the fact that mathematics offers useful and partly irreplaceable tools for managing the practices of the modern technological society, for utilizing them and for solving various scientific problems connected with real life. It is therefore desirable and presumable that the school settings produce widely transferable and permanent knowledge and

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give a valid idea of both the constructive (mathematical modelling) and axiomatic-deductive nature of mathematics.

Unfortunately, many studies on every-day cognition show the lack of transfer from school contexts to the problems in out-of-school contexts. In fact, many young people and adults seem to be remarkably efficient in dealing with quantitative problems which they encounter in their everyday social and professional activities compared to the corresponding school mathematics contexts (see e.g. Carraher et al., 1985; Lave et al., 1984; Lave, 1988; Nunes et al., 1993). Resnick (1987), who studied the use of cognitive and physical tools outside the school versus the emphasis placed on "pure thought" and symbolic procedures at school, has pointed out that school stresses symbol-based learning and thinking independent of concrete objects and events. On the other hand, out-of-school activities interact with and depend on the features of the situation in which they happen. Resnick also remarks that school settings are designed around the individual. Hence these are of little value in a world of socially shared tasks. Thus, school mathematics knowledge tends to be separated too much from the real world. According to Collins et al. (1989) we are schooling young adults who are experts at memorizing inert knowledge, performing simplistic tasks and listening passively.

From the viewpoint of conceptualization and meaningful learning processes there also seems to be something wrong in school mathematics. Davidov, for instance, points out that traditional education – continuing to build on children's spontaneously constructed structures – cultivates empirical generalization, useful only for the formation of weakly transferable everyday concepts. Rather than developing "formally general" pseudoconcepts, school teaching should develop the theoretical mode of thinking required in the formation of scientific concepts (Davidov, 1982, 1990; see also Vygotsky, 1962; cf. the limits of the informal, contextually based knowledge and skills, e.g. Brown et al., 1989; Resnick, 1987). Freudenthal's criticism is similar (Freudenthal, 1983): traditional teaching of mathematics – first the formal systems, then applications – leads to empirical concepts and to rote learning. In this article this traditional and widely used strategy will be called "formalistic".

On the basis of what has been said above, there are clear grounds for outlining an alternative strategy and teaching programs to the traditional formalistic strategy of teaching mathematics. For this purpose, I will first describe briefly how the teaching of mathematics was organized in Finland before the time of "new mathematics" (in the 1940's and 50's). In fact there might be a lot to learn by the settings used in the past. In this and other scrutinies below it will be used as an example of the teaching and learning of the ratio concept and proportional reasoning. A more detailed criticism of the formalistic practices in school mathematics is followed by the outline of an alternative teaching strategy designed on the basis of the ideas of Freudenthal's 'didactical phenomenology' and the "Vygotskian school" of psychology. Finally, I will present some of the main experiences and results obtained in experiments with the developed "contextual" program for teaching and learning ratio concept and proportional reasoning in a meaningful and transferable way.

Some ideas from the late 1940's and early 1950's

An inspection of Finnish primary school (grades 1–6) mathematics in the late 1940's and early 50's shows that the teaching of mathematical knowledge and skills was based primarily on the solution of verbal problems involving situations of practical life. It is obvious that the problems were about agriculture, the cattle industry, and many other contexts that were important to maintain the functions of an agrarian society. In this way, the teaching of mathematical knowledge and skills that were intended to be learned, such as ratio and proportion, was linked from the very beginning to the pupils' own experiences and their earlier learning experiences. In general outline, the teaching of proportional reasoning and percent proceeded as follows (see Merikoski, 1952, pp. 234–254):

First, one- and two-step consumer problems were solved mentally (ibid, p. 234):

Example 1.
How much did: 5 kilos of meat cost when 1 kilo costs 100 marks?
1 kilo of meat cost when 4 kilos cost 240 marks?
Example 2.
How much did: 3 kilos of apples cost when 5 kilos cost 500 marks?

After mental calculations, the pupils moved on to solve problems of proportionality in writing (ibid, p. 235):

Example 3. Fifteen meters of fabric is needed to make five costumes. How much fabric is needed for eight costumes?

The setting:	five costumes	—	15 m
	eight costumes	—	x m

Making use of instructions on reasoning, the pupils were directed to write the solution in the form

$$x = 8 \cdot 15/5 = 24 \text{ (m)}.$$

These "single condition reasoning problems" were followed by an introduction to problems involving inverse proportionality and an exercise. In this context, explicit mention was made for the first time of direct and inverse proportionality. These concepts were referred to as direct and inverse ratios. Brief instructions were also given (ibid. p. 236): "Whenever you start reasoning, first check if the numbers are directly or inversely related to one another. When reasoning, move from one ratio to the other through the "unit-rate" method (see "unit-rate" method, Post et al., 1988).

In a way, the teaching-learning process of the skills of proportional reasoning culminated in the solution of the so-called "biconditional reasoning problems". I quote an exemplary problem in the textbook, together with the solution (ibid, p. 245):

Example 4. Five men dig 720 meters of a ditch in 12 days. How many meters of a similar ditch do 7 men dig in 9 days?

Setting:

5 men	—	12 days	—	720 m
7 men	—	9 days	—	x m

The setting was followed by instructions on reasoning and instructions to write the operations in the form

$$x = 9 \cdot 7 \cdot 720/(12 \cdot 5) = 756 \text{ (m)}.$$

It was also required that an answer be given.

The pupils then went on to percent and interest calculations. The percent calculations were consistently based on single-condition reasoning problems, while the interest calculations were based on biconditional reasoning problems. In this way the pupils were subjected to a process of learning percent and interest calculations that was consistent both logically and psychologically (for founding percent calculations on proportional reasoning, cf. Freudenthal, 1983, Strickland & Denitto, 1989). This was all implemented in the sixth grade of the primary school, which corresponds to the sixth grade of the current basic school (age 12+).

Some critical notes about today

An inspection of Finnish mathematics textbooks in the 1980's reveals that the teaching and learning of ratio and proportionality has been arranged like many university courses in "pure" mathematics: "first theory, then practice".

The textbooks for the sixth grades of the basic school suggest, for instance, that the teaching of ratio be performed as follows: (1) identifying the ratio as the quotient of two quantities of the same kind, (2) calculating the values for the ratios of the quantities of the same quality, (3) applications (see, e.g. Matematiikkaa 6, 1988, pp. 5–57). Such a teaching-learning process appears to be meaningless. Does it make any sense not to ask until the end of the course, "How many times is the height of a camel in comparison to that of a donkey?" ?

Would it not be much more meaningful and instructive to start looking right from the beginning of the learning process for answers to questions such as: Which product is proportionally the most inexpensive to buy? Which country has the highest (lowest) proportion of people, cars, domestic animals? Which of the given figures are similar and why? Which one of the jugs contains the strongest mixture?

The teaching of proportion in the eighth-grade textbooks has not been arranged any better (see e.g. the textbooks Peruskoulun matematiikka 8, 1985, pp. 23–31; Matematiikkaa 8, 1986, pp. 14–25; Taso 8, Matematiikka, 1990, pp. 32–39). Thus, it is not surprising that, on finishing basic school, less than 30 % of Finnish ninth-grade pupils seem to be capable of correctly solving ordinary two-step purchasing and consumer problems or determining a percent, not to mention biconditional reasoning problems (Kupari, 1983, pp. 124–137).

So, the teaching of ratio and proportionality in Finland, for instance, has remained separate from those contexts and problems whose solution really requires the concepts and algorithms intended to be learned (see similar observations in Holland by Streefland (1984, 1985). According to Freudenthal, such instruction in ratio and proportion can be compared to a situation in which the carriage is put in front of the horse and not vice versa (Freudenthal, 1983, "concept attainment", cf. Cobb et al., 1992, "instructional representation approach"). In fact, there is good reason to assume like Freudenthal did that the "formalistic" teaching-learning process described above leads easily to routine learning and to lack of relevance for the issues that are meant to be learned (cf. Ausubel, 1968, rote-learning vs. meaningful learning). In consequence, what is learned is also forgotten quickly; the transfer is weak and interest in studying mathematics has decreased. In many cases the process also results in strong negative attitudes toward mathematics and studying it.

In addition to these problems, another danger of using a "formalistic" teaching strategy is that the pupils gradually get a distorted picture of the nature of mathematical activity and the origin and uses of mathematics. After all, mathematical activity is much more than just internal deduction of axiomatic systems given in a ready-made form. In fact, as Lakatos has emphasized, mathematics grows through the incessant improvements of guesses by speculation and criticism rather than through a monotonous increase in the number of indubitably established theorems (Lakatos, 1976; see also de Villiers, 1986; Kitcher, 1984; Kline, 1980; Shibata, 1979; Thom, 1973). In other words, mathematical activity is a social as well as cognitive phenomenon. Concomitantly, as Cobb, Yaeckel, and Wood have noticed (1992), mathematics learning is viewed as an active, constructive process in which there are real possibilities to discover and to explicitly negotiate mathematical meanings (see also Bauersfeld et al., 1988; Voigt, 1992).

The criticism above resembles quit a lot the criticism of the representatives of the "Vygotskian school" of psychology (see e.g. Schmittau, 1992; Vygotsky, 1962). The everyday concepts arise from the abstraction of the similarities from collections of entities which in themselves can represent many kinds of functions and structures (e.g. "triangularity" abstracted empirically from triangular objects). The origin of many scientific concepts is in the action of their construction (e.g. the construction of the circle with a string). These concepts are generally introduced in formal settings and they function within hierarchical system of interrelationships (e.g. the definition of the circle as a set of points having an equal distance from a given point). It is important to notice that these empirical generalizations (e.g. "roundness") and theoretical generalizations (e.g. mathematical circle) are different qualitatively. According to Davidov (1990), one cannot get to the second from the first: a leap is required.

Hence, the main task of school mathematics from the viewpoint of conceptualization is to cultivate a theoretical mode of thinking useful for the formation of real mathematical concepts. How this could be done is to be focused on in the next chapter.

Outlining an alternative teaching strategy

It became obvious above that something should be done to make school mathematics more transferable and meaningful. From the Vygotskian perspective all the higher mental functions, including mathematical activity, develop similarly, following from outer action to the inner thought. For every child learning occurs twice: first and primarily at the outer, social level in collaboration with other people, and then on the individual level where learning is internalized

(Vygotsky, 1978; see also Galperin, 1957; Jones & Thornton, 1992). According to Leontyev this development ought to be produced through the internalization of socially meaningful activity. In other words, the teaching-learning process of scientific concepts should be motivated by the need and desire to understand and control one's own life practice (Leontyev, 1977; see also Engeström, 1983, pp. 139-151).

Hence Vygotskian researchers emphasize the importance of such teaching settings as presenting and posing problems, using cooperative groups and providing opportunities for significant peer interactions (Taylor, 1992). Furthermore these studies above suggest new instructional environments which are experientially real for pupils, supporting their transition or leap from informal, everyday concepts and strategies to "fully universal" scientific concepts. This is parallel for instance with Freudenthal's (1983) 'didactical phenomenology' emphasizing mathematizing or modelling in school mathematics (see also De Lange, 1993). Collins, Brown & Newman (1989) also propose the integration of realistic performance into instruction to make learning activities meaningful and purposeful.

Since 1988, I have been developing an alternative approach to the teacher-centered and formalistic tradition in school mathematics. To put it more accurately, the basic idea is to start the teaching of mathematical skills and knowledge from those real-world situations and problems, in which mathematics to be learned is really needed and can be developed. One then proceeds gradually to the discovery and internalization of general and complete models of thinking in accordance with Figure 1 (see Engeström, 1989, p. 13; Freudenthal, 1983, p. 32, "didactical solution"; Keitel, 1987, "forestage mathematics").

So, the teaching and learning of ratio and proportionality, for instance, will be based on such real-world problems (e.g. purchasing and selling, different kind of mixtures, orienteering, currency exchange, even motion, similarities) in which pupils really need the valid concepts and techniques of ratio and proportion. The process of teaching and learning would proceed stage by stage:

(a) producing cognitive conflicts by comparing the informal solution processes and results to each other,

(b) identifying and modelling the relevant relations by the mathematical models of ratio and proportion,

(c) dealing with the mathematical models – ratio and proportion – in a purely symbolic form,

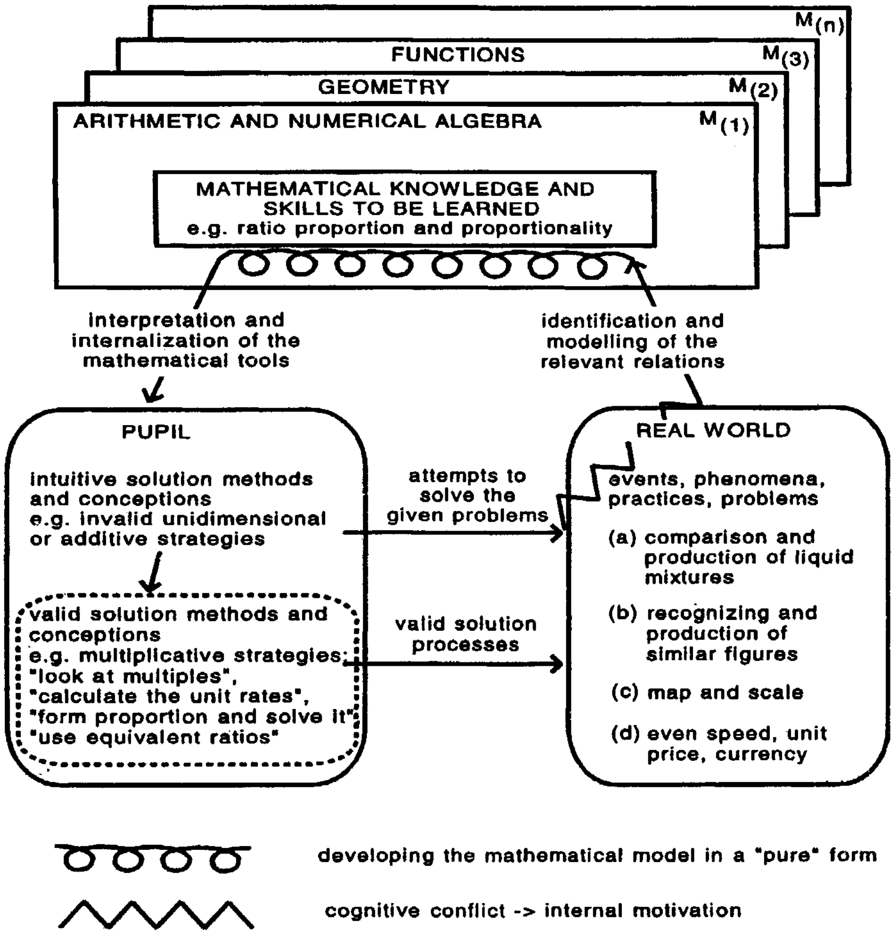

Fig. 1. Model for designing meaningful teaching-learning processes of mathematical knowledge and skills, with the learning activities of ratio and proportionality as an example.

(d) applying the learned solution models to the new contexts and discussing the validity of those models in different situations and problems.

The teaching experiment

Goals and procedures

The current teaching experiment formed a part of a larger project with the practical aim to change the formalistic and teacher-centered processes of school mathematics towards problem-centered and meaningful teaching-learning processes. The learning of ratio and proportionality was chosen as the specific target of the project, because a good deal of previous knowledge on the development of proportional reasoning was available (see e.g. Hart, 1987; Inhelder & Piaget, 1958; Karplus et al., 1983; Keranto, 1986; Noelting, 1980 a, b; Post et al., 1988; Tourniaire & Pulos, 1985; Vergnaud, 1983).

The main goal of the fourth teaching experiment was to produce systematical knowledge about the effects of the "contextual" teaching program compared to the effects of conventional instruction by the textbook (Taso 8, 1990; double pages designed according to the formalistic idea: "first theory, then applications"). Another goal – associated with the first one – was to learn more about the possibilities to develop the skills of proportional reasoning skills among eighth-grade pupils (aged 14+). The useability and reliability of the computer-based reasoning test "Juice" in the assessment of the development of proportional reasoning in the liquid mixture context was also examined. Fourthly, the obstacles and practical problems related to the implementation of the developed teaching program entitled "Learn to calculate and reason" in the ordinary classrooms were surveyed.

The "contextual" program included a teacher's guide, a problem book for the pupils, and the computer-based individual test called "Juice". The main cognitive goal of the package was to teach how to solve problems involving direct and inverse proportionalities, mentally and in writing. Another goal was to teach how to represent both direct and inverse proportional relationships in the system of coordinates and to use graphical representation in the solution of word problems.

The teaching according to the "contextual" program proceeded in the main as indicated by Figure 1. In the first activity, the pupils were given liquid mixture problems of increasing difficulty to be solved in pairs. Conflicting processes and results – as expected, some of the pupils used for instance invalid additive comparison – gave rise to lively discussions (cf. producing intra-and interpsychological conflicts, Bauersfeld et al., 1988; Voigt, 1992). In the second activity, the pupils were guided to identify relevant multiplicative relations. This was done by calling their attention to multiplicative relations between numbers (lessons 1 and 2). In the third activity, it was learned how to represent these multiplicative relations by means of the ratio and, later, the proportion (lessons 2 to 4). In the fourth activity, these concepts and the symbolical techniques involved in them were learned in a purely symbolic form (lessons 3 and 4). In the fifth activity, the techniques that had been learnt were applied in the solution of verbal missing-value and comparison type problems involving various themes (lessons 5 and 6). In the sixth activity, the knowledge and skills learnt so far were made more profound and expanded by examining direct and inverse relations both in the system of coordinates and in algebraic form (lesson 6 to 8). Finally, an effort was made to fortify the knowledge and skills that had been learnt by using them to analyze similarity and to produce similar figures in a given scale (lessons 8 to 12). The thirteenth lesson of the program was designed for pulling everything together and for repetition.

Subjects, research design and the tests

All the eighth classes of the Teacher Training School of the University of Oulu and three eighth classes of Rajakylä Comprehensive School in Oulu took part in the study (*n* = 146). Six of these nine classes were chosen to use the "contextual" program, while the last three classes (from the Teacher Training School) used a textbook called Taso 8 (double pages designed according to the formalistic idea). Each class was taught by its regular mathematics teacher during the entire teaching experiment. All these nine classes took part both in the initial and final tests of mental proportional reasoning skills ("Juice" test; used both at the beginning and in the end of the teaching experiment) and in the written final exam. So the current study would be briefly characterized as a "quasi-experimental" comparison study arranged in ecologically valid circumstances and involving nine non-randomly selected groups of subjects.

The starting points for the design of the "Juice" test were provided by the paper-and-pencil tests elaborated by Noelting and the research team formed by Karplus, Pulos and Stage, as well as the multiple-choice tests developed by the author for the previous surveys and teaching experiments (Karplus et al., 1983; Keranto, 1986, 1990a, 1990b, 1990c, 1991; Noelting 1980a, 1980b). The test "Juice" contained 24 liquid mixture problems of the comparison type. In each problem the pupil's task was to reason mentally which one of the liquid mixtures, A or B, was 'stronger', or if they were 'equally strong' (pictures of two jugs containing certain amounts of water and orange juice displayed on the computer screen). Each choice was stored on a disk. These tasks were selected in such a way that it was possible to distinguish consistent use of invalid unidimensional (e.g. only the comparison of the amounts of orange juices) and additive strategies (the comparison of the differences of the corresponding amounts of waters and orange juices) from consistent use of valid multiplicative or proportional strategies (for further details, see Keranto & Kumpulainen, 1991; see also Keranto, 1991, 1992).

The test could be completed quickly (taking some 15 to 20 minutes). A performance analysis was included in it, making it possible to see at once which level of reasoning each pupil had achieved and what kind of strategy he or she had presumably used. This made both the acquisition and processing of data much easier, and also made it possible to assess the initial level of the pupils and to follow their cognitive development.

The written final exam contained seven problems, the first two of which assessed the pupils' skills in solving liquid mixture problems with paper and pencil. In the third problem three proportions represented in a symbolic form had to be solved. The fourth and fifth ones were typical word problems which could be solved using, for instance, the unit-rate method or proportions. In the sixth task the pupils had to solve a consumer problem using a graphical presentation. The seventh problem was the most difficult: Timo fills up a water tank in an hour. It takes Leena 4 hours to do the same. How long does it take for the both of them to fill up the tank together?

The written final exam was assessed by both the mathematics teachers and the author. The correlation between the author's and the teachers' assessments ranged from 0.94 to 0.99.

At the end of the teaching experiment, each pupil was also asked how interesting and pleasant they considered the teaching program to be in comparison with other mathematics studies. At this time, their general attitude toward school and mathematics studies was also inquired. A survey was also made of their own beliefs as to how much they had learned during the program.

Some main experiences and critical observations

The author did not have any chance to follow systematically the implementation of the teaching program "Learn to calculate and reason" in the "contextual" classes. Therefore, the observations of the strengths and weaknesses of the program and of the practical proceeding of each lesson were mostly based on the above-mentioned inquiry among the students, on the notes written of the lessons by the teachers participating in the experiment, on joint discussions of the teaching experiment, and on the systematic observations. The conversations were tape-recorded.[1]

Positive general traits mentioned by the teachers included, first of all, practicality of the problems and pupil assignments: "A well-

[1] The observations and recordings were made by my assistant Sinikka Kaartinen.

chosen theme. A fruitful area with a clear practical basis."; "The same pairs are still thinking over problems together."; "Finally, we have some practical problems in mathematics." The versatility and wide coverage of the teaching program was considered another positive thing. Thirdly, the problem tasks included in each phase of the teaching program were also considered to be a positive feature, and pupils like to solve them very much. The negative observations were mostly related to the small number of lessons in relation to the contents to be learned, and to the fact that the learning materials were distributed to the pupils mostly in the form of hand-outs. According to the teachers' observations, the highest degree of learning difficulties occurred in the third (ratio and the value of ratio) and seventh (inverse proportionality) lessons: "This was not a successful lesson! The m.p.h. problems and unit changes take so much time. Little time remains to practice with the issue at hand. At least two lessons should be available instead of just one."; "This was not easy for the pupils. Another lesson devoted to this theme would have been needed."

These experiences were largely parallel with earlier teaching experiments, in which the author himself acted as the teacher of the contextual class. First of all the implementation of a contextual program appears to be far more time-consuming than the widely used teacher-directed and formalistic strategies in school mathematics. It happened much too often that the process of discovery had to hastened and intensive working sessions had to be interrupted. It was also observed that much too little time was available for practising the technical skills related to the concepts of ratio and proportion (see Keranto, 1990a, 1990b, 1991).

It was also observed that much more experience would have been needed to take the pupils' own solutions into account in a valid and natural way at the various stages of the teaching process. Let us, for instance, examine the following episode which was recorded in a contextual class at the beginning of the third lesson. The lesson started with the homework as detailed below. It should be mentioned here that different kinds of liquid mixture problems had been solved and ratio had been introduced in the previous lessons.

A fence needs to be painted. Painting each four meters of fence takes 2 liters of white paint and 8 liters of red paint. You already have 8 liters of white paint in the storeroom.

(a) How many liters of red paint do you have to buy?

(b) How many meters of fence can you paint with the mixture?"

The joint examination of this problem proceeded as follows
(t = teacher; p = pupil; comments by author)

t: How many liters of red paint do you need?
p(1): 14 liters
t: How did you calculate that?
p(1): I used subtraction [a typical erroneous strategy]
t: p(2) [teacher neither corrects the previous solution nor continues from it; why?]
p(2): 32 liters
t: How did you calculate it? [teacher is still using the term 'calculate']
p(2): There is four times more red paint than there is white paint. So I multiplied the amount of white paint by four. [p(2) is using a faulty expression, mathematically; he should say "four times as much as"; note the mixed usage even in the mass media!]
t: Are there any other solutions? [teacher does not initiate a discussion of the conflicting expression]
p(3): There's four times the amount of white paint here (p(3) has compared the amounts of white paint to each other: further specification would be needed)
t: There's four times the amount of white paint in here. So we also have four times the amount of red paint. [teacher specifies the answer provided by p(3) but does not ask him to do it himself]
t: How many meters of fence can be painted?
p(4): 16 meters
t: A fourfold amount [teacher does not inquire about the solution process and is content with giving himself a vague hint of the process]
t: How many of you did get the right answer?

This is a highly typical episode. It is not easy to transfer from formalistic and teacher-centered solutions to discussive and problem-centered solutions no matter how much one would like to do so, like this teacher!

Some main results

The research results from the fourth teaching experiment were consistent and similar to those from the first three experiments (Keranto, 1990a, 1990b, 1991). Firstly, it was established that the subjects' use of the various strategies corresponded for the most part to the expected levels of proportional reasoning indicated by the analysis of the students' results on the "Juice" test. In other words the computer-based test was proved to be structurally valid. In addition to this, the items in the test formed empirical hierarchies as expected (Table 1), thus making it possible to determine the pupils' levels of reasoning reliably.

	Levels of proportional reasoning					
	I	IIA	IIB-	IIB	IIIA	IIIB
Items	IE	E	WB&W	B	WX&BX	NX
Initial test *f*	128	128	87	81	36	15
(*n* = 136) %	94	94	64	45	26	9
Final test *f*	131	131	107	97	86	50
(*n* = 141) %	93	93	76	69	61	35

Table 1. Pupils' performance on the initial and final test, measured by the "Juice" test and categorized in terms of levels of proportional reasoning and related item types. The figures describe the number of students on each level of reasoning.

The levels are: I, "intuitive"; IIA, "lower concrete"; IIB-, "transition"; IIB, "higher concrete"; IIIA, "lower formal"; IIIB, "higher formal". (For further details, see Noelting, 1980a, b.)

The types of items are: IE (n:n vs. p:q); E (n:n vs. p:p); WB & W (within composition ratio integer); B (between composition ratio integer); WX & BX (unequal ratios); NX (no ratio integer). (For further details, see Karplus et al., 1983.)

Initial test	CR = 0.95	MMR = 0.78	PPR = 0,77
Final test	CR = 0.95	MMR = 0.76	PPR = 0,79
Recommendations for "complete" empirical hierarchy	CR ≥ 0.90	MMR ≤ 0.80	PPR ≥ 0,70

Table 2. Values of hierarchy on both tests in terms of the constants CR, MMR and PPR.

The order of difficulty of the items was as we expected. In order to measure the values of hierarcy, we used the constants CR (Guttman, 1944), MMR and PPR (White & Saltz, 1957), Table 2. As we can see, the values of hierarchy were high and exceed the values for "complete" empirical hierarchy.

Secondly, it was observed similarly to the previous experiments in the project that the possibilities to develop the mental proportional reasoning in the liquid mixture context are clearly related to general study achievement in mathematics. This can be seen in more detail in Figure 2.

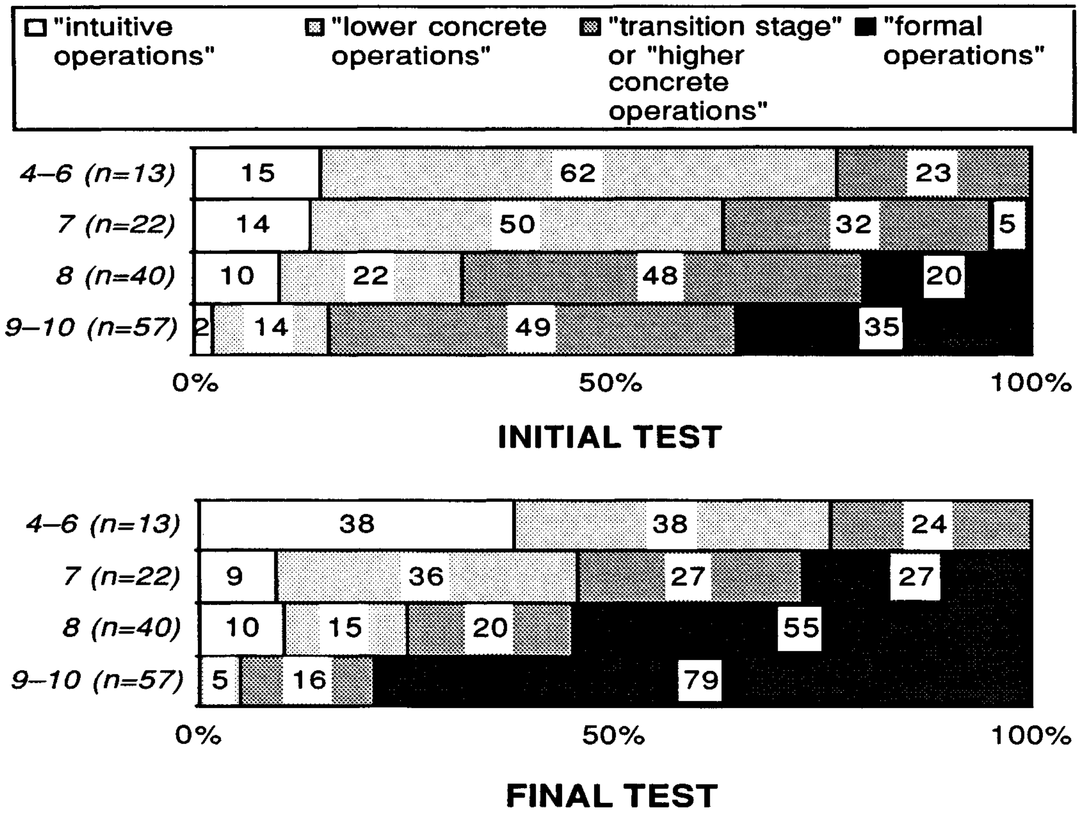

Figure 2. The percent distributions of the achieved levels by grade category in the initial and final tests.

As expected, the development of mental reasoning skills in liquid mixture problems was strongest in the highest grade category (9-10). The level of reasoning could be improved or it was already on the level of formal operations mainly for those pupils who had performed well (8) or excellently (9-10) in their studies of mathematics (55 % and 79 % in the final test). On the other hand, not a single pupil in the grade category 4-6 reached that level even in the final test. Similar results were also observed in the written final exam, although they were not quite so marked.

Thirdly, the learning results produced by the contextual program in comparison to the formalistic method were analyzed. Because the original teaching groups were rather small (n = 12 to 20), they were joined to form three larger groups as follows: Con I (Teacher Training School; 3 contextual groups, n = 45); Con II (Rajakylä Comprehensive School; 3 contextual groups, n = 53); Book (Teacher Training School; 3 textbook groups, n = 48).

group	Con I (n=42)			Con II (n=52)			Book (n=38)		
marks in math	4–7	8	9–10	4–7	8	9–10	4–7	8	9–10
ProRea (initial)	2.1	3.5	4.0	2.3	3.6	4.4	2.0	2.5	3.3
ProRea (final)	2.2	3.8	5.4	3.7	5.1	5.2	1.9	2.9	4.7
Difference	0.1	0.3	1.4	1.4	1.5	0.8	-0.1	0.4	1.4

Table 3. Changes in mental proportional reasoning (ProRea) both in the "contextual" groups Con I & Con II and in the "formalistic" group Book.

The levels of ProRea: 0, "zero"; I, "intuitive"; IIA, "lower concrete"; IIB-, "transition"; IIB, "higher concrete"; IIIA, "lower formal"; IIIB "higher formal operations" have been converted into corresponding scores 0 - 6 and a mean score has been calculated for each cell.

This decision was also advocated by the fact that the performance of the original groups in each group (Con I, Con II and Book) was highly similar. This also made the presentation more compact and economic. The results related to the mental reasoning in liquid mixture problems are presented in a concise form in Table 3. Only those pupils are included in the analyses who also took part in the final test ($n = 132$).

According to Table 3 the development of the mental reasoning skills in the liquid mixture context was just about similar in the contextual group Con I and in the "formalistic" group Book. On the contrary, the development of proportional reasoning in the group Con II in the grade categories 4-7 and 8 was clearly better than in corresponding categories of the groups Con I and Book (see columns 1 and 2 in Con II). According to the t-tests of repeated measurements, the observed changes in the groups Con I, Con II and Book were significant ($p < 0.01$), while the mutual magnitudes of the changes between the groups were not even close to statistical significance.

In the written final exam, the pupils in the contextual groups Con I and Con II did all parts of the exam at least as well as those of the textbook group. Especially in the liquid mixture problems the groups Con I and Con II succeeded much better than the group Book; with the correct answers amounting to 78 % and 75 % in the contextual groups, respectively, while numbering only 30 % in the Book group. In the other liquid mixture problem the percentages were 47 %, 40 % and 22 %, respectively ($p < 0.001$).

When the teaching program called "Learn to calculate and reason" was designed, one of the goals was that the lessons included in this program should really interest the pupils and that they should be felt to be pleasant. As a brief comment on the results obtained, it was discovered that the majority of the pupils in Con I in particular felt that these lessons were more pleasant and interesting than other mathematics lessons (62 % and 45 %), while the experiences in groups Con II and Book were not quite so positive (40 % and 29 %; 39 % and 23 %). These differences in the experiences are not surprising, considering the fact that various circumstances can have an effect on learning experiences, such as the teacher's attitude toward the pupils, the materials used, style of teaching, the general atmosphere in the classroom, and the pupils' own attitude toward school and, specifically, toward mathematics. Experiences of the difficulty of mathematics and ideas of their own standard of learning may also have had some influence on the differences which were observed between the groups. In fact, the correlative analyses which were performed indicated that the experiences of interest and satisfaction were closely associated with the pupils' general interest in mathematics and with their attitudes towards studying mathematics, as well as their experiences of having learned something during the teaching program (correlations 0.31–0.29, $p < 0.001$) (for more details, see Keranto, 1992, pp. 89–92).

Conclusions

It is a known fact that the teaching of mathematical knowledge and skills can be designed and realized in a number of ways. In the current research, a teaching model was tested in which the teaching of the knowledge and skills meant to be learnt – ratio, proportion, proportionality – proceeds roughly in accordance with the following scheme: practice – theory – practice. This scheme, developed on the basis of the ideas of the Vygotskian school of psychology and the ideas of Freudenthal's 'didactical phenomenology', resembles in many ways the kind of teaching of ratio and proportionality which was applied in the sixth classes (age 12+) of primary schools in the 1940's and 50's as detailed. Problems were first learned to solve mentally – emphasizing the unit-rate method – and then in writing using proportions. In this way, the pupils' real-world experiences and spontaneous models of solution were utilized naturally in teaching, at least in principle. At the same time, a valid conceptual basis was laid for the teaching and learning of percent and interest calculations (see page 38; also see Post et al., 1988; Strickland & Denitto, 1989).

On the other hand, in the light of the teaching experiment described above, treatment of more demanding problems involving proportionality – inverse proportionality and biconditional reasoning problems – in these classes would appear to be too demanding on the basis of the results which were obtained. After all, obvious difficulties were observed even among pupils who were two years older in learning to solve fundamental single-condition reasoning problems. However, in the light of the experiences yielded by this research, this does not mean that it is not worthwhile to start developing proportional reasoning by the sixth grade. Quite on the contrary, various proportional situations provide good opportunities for problem-centered and discussive teaching.

Based on the discussions and experiences in connection with the experiment, the following order would be worth testing:

(1) unit-rate is known;

(2) unit-rate has to be determined;

(3) solution of two-step problems (such as "How much do 3 kg of apples cost, if 5 kg of apples cost 20 crowns?")

Such problems would be first solved mentally, and later in writing by means of proportions. In this way, the symbolic representation and practising with connected mathematical operations – which was experienced difficult – could be linked in a meaningful way to the preceding mental reasonings (see p. 42.). The comparison, missing-value and percent problems would not be discussed until after this stage.

Of course, this alone will not guarantee – as was seen above – that the teaching-learning processes become meaningful and problem-centered, making use of the pupils' earlier learning experiences. In fact, the experiences yielded by the teaching experiment show that the teachers and pupils have become accustomed to working within the limits of the system of 45-minutes lessons. This system clearly directs the teaching and learning process toward the use of teacher-centered working methods and ready-made solutions. So, although many teachers and pupils would be quite willing to act according to the new problem-centered teaching programs and ideas, there seem to be many kinds of obstacles – physical, social and economical – which in many cases limit desirable changes in school mathematics.

Anyway, I dare to argue on the basis of the experiences obtained in the teaching experiment that it is possible to break the teacher-centered tradition in school mathematics by developing competitive alternatives to formalistic solutions in school mathematics together with the teacher students and teachers. It is obvious that such changes do not take place in a minute. In my opinion, we need both basic research and developmental work. To put it in more specific terms, we now need follow-up studies related to different mathematical contents, in which the activities of the teachers and the development of the pupils is monitored over a relatively long span of time (see e.g. Cobb et al., 1991). This would provide more experiences and ideas for the planning and implementation of problem-centered and meaningful teaching programs for the entire school year.

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Ett problemcentrerat alternativ till formell undervisning

Försök med kontextinriktad undervisning om förhållande och proportionalitet i årskurs 8

Sammanfattning

Ökad uppmärksamhet har på senaste tiden riktats mot den lärarcentrerade undervisningen i skolan. Välgrundade förslag har getts för att förändra denna tradition i en riktning mot problemorientering och hänsynstagande till elevernas tidigare erfarenheter av lärande. Med aktivitetsteorin som grund analyseras i denna artikel en undervisningsmodell som utvecklats som ett alternativ till den nuvarande formella undervisningstraditionen om förhållande och proportionalitet. Även om talrika problem upptäcktes vid implementationen av programmet så blev de uppnådda resultaten minst lika bra eller bättre än de som uppnåddes med lärobokens metod. Det datorbaserade individuella test som utvecklats och använts visade god strukturell validitet och var lätt att använda.

Författare

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