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Promoting deep learning in mathematics through conceptual exercises

Asger Törnquist

Department of Mathematical Sciences, University of Copenhagen

Abstract

We investigate the effects of posing non-traditional "conceptual" problems to university mathematics students as a means of promoting a deeper level of learning among a class of second year undergraduate mathematics students at the University of Copenhagen.

I. Introduction

(A) Background. The traditional and still dominant mode of teaching mathematics at almost all levels of education in Denmark, and the Western world at large, divides the learning process into two separate steps:

1. Introduction of the theoretical material and main examples in a lecture format and through supporting reading material (textbooks, and, frequently at the university level, tailor-made lecture notes).
2. Consolidation of theoretical knowledge introduced in Step 1 and development of problem-solving skills by working through problems and exercises[1] (usually as homework).

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[1] Problems and exercises in mathematics often ask students to calculate a quantity or, in higher level classes, to prove (using logical reasoning) some mathematical fact.

Step 1 has the singular goal of imparting *knowledge* on the students. By contrast, Step 2 has two *distinct* goals, the first of which, *consolidation*, is often described as "checking the understanding" when mathematics teachers describe Step 2, while the second goal is to promote the learning of a *skill*, namely, solving mathematical problems and exercises.

Regarding Step 2, the traditional belief is that it (a) forces students into an active role in relation to the theoretical knowledge, and that (b) students cannot succeed at Step 2 without understanding the theoretical knowledge from the lectures and the textbook.

It is widely acknowledged by mathematics teachers that (b) above is somewhat dubious. At lower levels, here meaning up to and including the second year at university, many students 1 Problems and exercises in mathematics often ask students to calculate a quantity or, in higher level classes, to prove (using logical reasoning) some mathematical fact. take a "plug-and-chug"[2] approach to Step 2, meaning that they aim only to develop the skill of solving standardized problems, and ignore the loftier ambition of using Step 2 to develop a deeper, consolidated understanding of the theoretical material of the course (the focus of Step 1).

(B) The problem with "plug-and-chug". The discussion in (A) above suggests that the 2-step model described there suffers from the serious defect that for many students, Step 2 only will accomplish half of what it was intended to. But aside from the fact that the surface learning resulting from the plug-and-chug approach is a "no-no", frowned upon in the ivory tower of Academia, one may wonder how serious a problem this really is. Many students will tell you that they successfully made it through required math classes (even at the university) by learning to do standardized problems and reproducing this skill at the final (usually written) exam.

If the students in questions are *not* math majors, e.g., are engineering or life-science students, then the seriousness of this problem might well be limited. That being said, a real problem *does* arise when *math majors*, that is, students who are pursuing a degree in one of the mathematical sciences, take the plug-and-chug approach. The experience in Copenhagen is that the problem becomes evident when these students enter into the Master's program where they start to struggle, and it comes to a head when they must write their Master's thesis.

[2] The term plug-and-chug (or plug 'n chug) for this surface learning approach is in common use among American engineering students.

Let's be very clear about what the problem is: The very best students in mathematics invariably develop a conceptual understanding in which the mathematics comes to live in their mind as a blend of (a) knowledge of key facts (theorems) and their interdependence, (b) key examples that illustrate possibilities and limitations of the theorems, and (c) mental images (see the discussion in Hadamard (1973)) attached to the key facts. Students who have engaged in a plug-and-chug surface approach are on shaky ground with (a), and usually lack (b) and (c) entirely.

(C) Non-traditional exercises for promoting conceptual learning. The project at hand describes an attempt at giving students in a second year mathematics class ("DIS", discrete mathematics, 60 students enrolled, 7 weeks of instruction with 5 hours of lectures per week and 3 hours of discussion section per week, oral final exam, taught in the fall of 2014) at the University of Copenhagen a non-traditional kind of problems to work with in order to promote conceptual learning.

The format of these problems were often "describe in your own words, using everyday language, what — means", where — is a key concept discussed in the class. On five occasions, students were given non-traditional problems:

- On the first day of lecture, three non-traditional problems were posed on a slide for the students to work with for 10-15 minutes before they were discussed in the class.
- Twice, non-traditional problems formed part of the weekly homework assignment.
- The course had two required (for passing) hand-ins, both of which included nontraditional problems.

(D) Organization. The report is organized into five sections, including the introduction. In section II, we will briefly review some of the background and literature on the subject of promoting better learning through non-traditional approaches to teaching mathematics, and in particular, we will consider a report from Høgskolen i Østfold, Norway, from 2004 by Marianne Maugesten and Per Lauvås, on a new approach taken to teaching mathematics to education students (lærerseminarstuderende). In section III, we describe in some more detail the specifics of the current project. In section IV, we will report on interviews conducted with two students (one pure math student, one statistics student) from the class DIS after the class had ended. Finally in section V, we will try to sum up our observations and conclusions about the effects of posing non-traditional problems to mathematics students.

II. Some new approaches to the teaching of mathematics

The impetus for the form of the present project comes from two sources: A study done by Maugusten and Lauvås (Maugusten & Lauvås 2004) and from work done in my pædagogikum pre-project (Holst et al. 2014).

II.1. "*Better learning of mathematics by simple means*?". Maugusten and Lauvås from Høgskolen i Østfold, Norway, reported on a project called "Bedre læring av Matematikk ved simple midler?" in 2004, see (Maugusten & Lauvås 2004).

The main problem facing Maugusten and Lauvås (Maugusten & Lauvås 2004) is that students of education (training to be schoolteachers) were failing a class on basic mathematics and teaching mathematics at alarmingly high rates (>50% fail rate). Seeking to remedy this situation, they sought inspiration in a successful restructuring of a basic math course taught at DTU (Denmark's Technical University), and restructured the math class with the high failing rates.

There are three main features of the Maugusten & Lauvås (2004) approach[3]:

- Students must through the course produce a "portfolio" (workbooks) with solutions to various problems posed in the class. (This part is lifted directly from the DTU approach).
- The students have to give each other feedback on their portfolios. This part is motivated by considerations of Gibbs (Gibbs 1999). The authors argue on this background that receiving *frequent and timely feedback* is highly motivating. It is more important, they believe, that feedback be delivered before the students move on to another topic than that it carries the stamp of authority of the teacher.
- Finally, the authors decide to at least in part focus on *open ended* questions that ask the students to explain concepts in everyday language, rather than only giving them traditional problems.

[3] It is also part of the ML approach that the students are divided into three groups based on their level in mathematics ("niveaudifferentiering"). This part is surely relevant for the success of the approach, but is only tangentially relevant to the direction of the current project.

It is the last point that is particularly relevant to the present project. Some examples of questions they give the students are:

> *"What does it mean that two triangles are congruent?"*
>
> *"What does the formula [sic] 2πr express? (Write with your own words)."*
>
> "Define with your own words: i) sin v, ii) cos v, iii) tan v."

The results are in line with what had happened at DTU: The fail rate fell from 54% to 15%, and the group of students receiving the lowest passing score fell from 66% to 20%.

Student interviews were conducted, and the students were overwhelmingly positive about all aspects of the approach. The students feel that the new approach forces them to work more, and that the peer evaluations gives them an understanding of how easy it is to think about a mathematical concept in the *wrong way*, something that they perceive as positive.

II.2. "*Bridging lessons with online quizzes*": This is the title of the author's pædagogikum pre-project from the spring of 2014, conducted and written together with Peter Holst, Linda Udby and Anja Wynns (Holst et al. 2014).

In this project, we were aiming to find a way of making lectures feel less disconnected by giving the students quick online quizzes that they had to do between two consecutive lectures. The quiz questions were to be such that their answer relied on the knowledge of the previous lecture, but also pointed forwards to topics of the next lecture.

It turned out to be very difficult to come up with traditional math problems that had this feature of pointing both to the previous and the next lecture. For this reason, I started asking conceptual questions on the online quizzes instead.

Thus the quiz questions ended up, somewhat by accident, taking a form very similar to the conceptual question of Maugusten and Lauvås. When students were interviewed later, they were overwhelmingly positive about the quizzes, but more than anything, they were pleased that the conceptual questions had helped them direct their thoughts to start *forming* the correct concepts. The questions had caused the students to spend much time pondering the key concepts of the course, and it had motivated discussion between the students outside the classroom.

III. Conceptual, untraditional problems and the course DIS

We now turn to describe the specifics of the non-traditional problems given to the students in the course DIS.

(a) In the first lecture, students were introduced to problems having a non-traditional format on a slide, and asked to work on three specific untraditional problems. To give an example, one of the problems asked:

- Describe in your own words what it means for a set of numbers to be closed under the operation of subtraction? Which of the sets **N** and **Z** are closed under subtraction?

The students were given 10-15 minutes to work either alone or with the students next to them on three problems of this form. After this, the answers were discussed in class.

(b) During the second and fourth week of classes, students were given a sheet of conceptual questions to work on as part of the assignment for their weekly discussion section. The questions were then discussed by their Teaching Assistant in the discussion section. An example of one of the questions (from week 4) is:

- Explain in your own words why it is reasonable to say that two sets A and B have the same number elements if there is a bijection between them. Do you also think this is reasonable when A and B are infinite sets?

(c) The students had to hand in two mandatory assignments that they had to hand in and have marked and approved in order to be eligible to take the exam.

On the first of these assignments, due on the third week of classes, the students were asked to consider two seemingly different proofs of the same theorem from lecture. They then had to explain informally using everyday language what was going on in two proofs (what the “idea” was in each case), and try to make a drawing to illustrate. The goal was that the students discover on their own that the two seemingly different proofs actually represented the same underlying mathematical idea, just dressed up in different formal guise.

In the second mandatory assignment, due in the fifth week of classes, the students were asked to prove Ramsey's theorem (a combinatorial principle first discovered in the 1930s). The task was broken down into steps for the students in the exercise. The first step asked the students to give an informal argument for the following statement: "If at least six people are present at a party, there must be at least three people at the party who either have never met before the party, or three people who had all met at some time before the party". The next step presented the formal statement of Ramsey's theorem, and asked the students to explain informally why the "Party Problem" is a special case of Ramsey's theorem. Finally the students are asked to draw on their informal argument for the Party Problem to give a formal proof of Ramsey's theorem.

IV. Interview with two students

To investigate the student's perception of the non-traditional problems posed in the course, two students were interviewed after the course ended. An interview was scheduled with a third student, but this student didn't show up for the interview, and did not subsequently respond to emails.

The students interviewed are:

- Student M, a statistics major[4]
- Student N, a pure math major.

Student M and N both successfully passed the course: M received the mark 7 at the oral exam, N received the mark 12. They were interviewed separately four days apart.

The interview opens with some brief background discussion of which activities they feel they get the most out of when learning. Student N emphasizes going to lecture and taking good notes, in particular of "the things that are said but not written down anywhere". Student M says reading the book very carefully is the best way to learn.

Q: What about exercises and going to discussion section?

Both students agree that the usefulness of discussion section varies. They both try to read the exercises before going to discussion section, but rarely

[4] Student M requested anonymity in order to participate in the interview.

have time to work through them. Student N emphasizes that "doing exercises helps with the understanding", and adds "you use it [the theoretical knowledge]". Student M finds that the Teaching Assistants are very skilled, but feels intimidated by the difficulty of the problems and exercises: "the two discussion sections are too similar. Both have a very high level. One should be on a lower level and the other on a higher level for those students who want that".

Student M adds: "I take a sloppy approach to solving the [traditional] exercises. First I read what it says, and then if I don't understand every word, I have to go back and read again how it is all connected. You have to have an overall grasp before you can approach the exercises [...] I am good at getting ideas, but when I finally get an idea down, it takes too much time to work out the details. I would rather wait until I can ask someone else [how to do it]".

Q: During the course, I gave you some exercises that were a little bit different from the usual. What do you think of this kind of exercise?

Student N: "I remember those problems very well. The first one we got for the mandatory assignment was very difficult, and I was very tired of that problem. But it was a really, really good problem if you wanted to understand what really goes on. The reason it was so irritating was that your notes were so difficult to read".

Student M: "The attitude most people had to this problem [the non-traditional problem on the first mandatory assignment] was 'this is something we haven't tried before'. It was difficult to find out what you wanted us to do... we talked about it, and about which drawing you could make, but we couldn't figure it out."

"The other [second] assignment was better, because you could easily make a drawing, and then afterwards you could use it to make an argument."

Q: There were other untraditional exercises...

Student N: "Yes. They were really good, because it becomes informal, and if you don't understand the formal mathematics, then it is probably because you don't understand the informal. [...] I like the informal things a lot, the side remarks in lecture and discussion section, because it gives you a feeling of what you're working with instead of just definitions and theorems."

Student M doesn't initially remember there being other non-traditional exercises in the course, so to refresh the memory the ones from week 2 and 4 are shown.

"They force you to think along different lines than what we are taught to do in this place. So it is a challenge. But I don't like them, because my perception is that a theorem is the justification of some maneuver. The proof is just something we learn so that we know that the theorem is true. It is very unpleasant when you have just learned a new theorem to have to relate to it in this way [of the non-traditional exercises]."

Q: The non-traditional exercises focus more on concepts than on doing formal mathematics. Do you think learning to do this kind of exercise is important going forward, as you start taking more advanced classes?

Student N: "For sure. Even without this type of exercises, if I see a definition I don't really understand, I want to find something more concrete, to put it in a box, so I will remember it. But it is nice to get some help with this process, because it takes so much time to do it alone. But the more you do it, the better you get at thinking in this way".

Student M: "It is probably something that becomes more important later. But it is very different in statistics and math classes. In statistics, you just want to see [in lecture] how the analysis is done, and which words they are using in lecture. In the math classes, you need to take everything apart to learn it. I can't always cope with that, so this is what I need more help with."

The students are asked if they would like more conceptual exercises. Student M says "no", and adds that "they would be better to get at the end of the course". Student N says "yes", and adds: "for the discussion section, preferably every week. We don't need it in lecture where other things are more important. But they are nice because the [traditional] exercises are like 'prove this'. Then I get stuck. But if you get 'explain this', then it is much easier to get started."

Finally, I ask if the non-traditional exercises help them remember the material from the course better. Student M feels that a review of the material at the end of the course would be more helpful. Student N says "Completely!

It gives you a sense of having a perspective of what really goes on, and then the formal mathematics is just a certain way we have to work, but to have a perspective which isn't just about formulas and symbols."

V. So what?

The problem posed in the introduction was if a non-traditional question format would promote conceptual learning among students in the class DIS, and steer students away from taking a surface learning approach. The motivation for doing so is a rather difficult to measure long term goal: To make the students better prepared for Master's level study.

Some facts: Of the 60 students enrolled, 54 took the oral final exam, and 53 passed. This is a vastly higher success rate than the course has had before (previously, around 15% of students have failed at the exam), but it is impossible to know what the reasons for this is[5]. Many students used what they had learned from the non-traditional exercises in their presentation during the oral exam.

Having just two student interviews to go on, drawing any sort firm conclusion about the effectiveness of the approach from what has been described above would certainly be a fallacy. All the same, the experience suggests some key points:

- The effectiveness of assigning traditional exercises may be severely restricted because the students find them too hard, and don't have the time to actually do them.
- The non-traditional exercises seems to have the advantage that they are easier to approach.
- The students feel unsure about what is expected of their answers when doing nontraditional exercises. This seems to be a cultural issue: They have never met this form of exercise before. It also suggests the importance of giving the students timely feedback on the non-traditional exercises.
- The apparently stronger student is far more enthusiastic about what is gained from thinking informally about the material (in general), whereas the apparently weaker student is far less enthusiastic.

[5] The author feels that 3 or 4 more students should have failed, but could not reach agreement with censor (the external examiner) to fail more students. Censor later conceded that he was too lax.

- The students acknowledge that in the future, their may be a value in having learned to do conceptual exercises.

Based on the experience, my recommendations are:

- Non-traditional exercises are a useful learning tool that challenges the students to approach the material differently, but the students are not used to this type of exercises, and will need to get clear feedback in order to learn how to do them, and feel confident about their answers.
- Incorporating them into the weekly exercises seems the most productive way for the students to benefit from doing these exercises (and gaining routine doing them).
- Finally, by paying close attention to how the students deal with the non-traditional exercises, the teacher gets a new and valuable source of information about how the students are doing in the class, and which concepts are causing the most difficulty for the students.

All contributions to this volume can be found at:

http://www.ind.ku.dk/publikationer/up_projekter/2015-8/

The bibliography can be found at:

http://www.ind.ku.dk/publikationer/up_projekter/kapitler/2015_vol8_nr1-2_bibliography.pdf/